Anisotropic, adaptive finite elements

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Collaboration with

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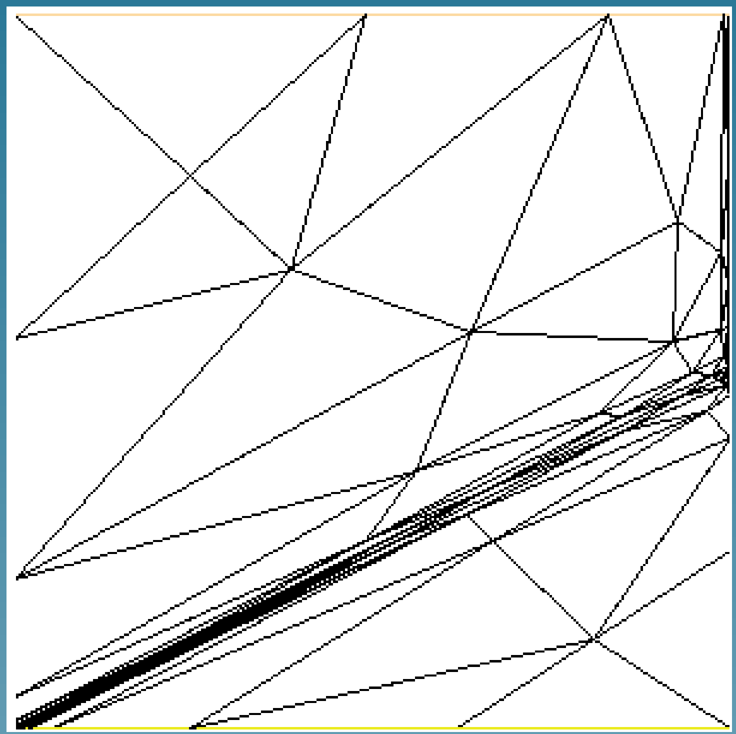
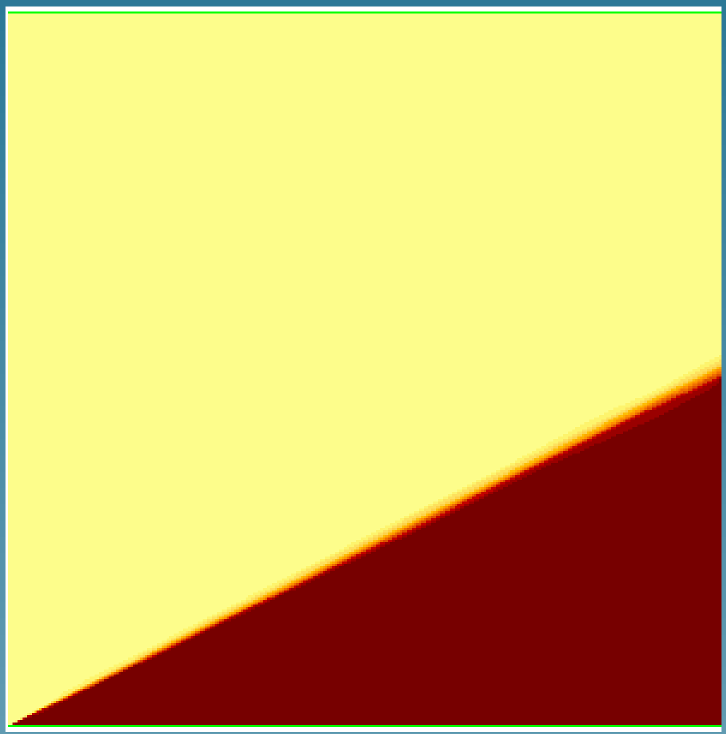
Example of strongly anisotropic mesh

- Boundary layer 0.0001, 241 vertices, asp. ratio $\geq 10^4$.



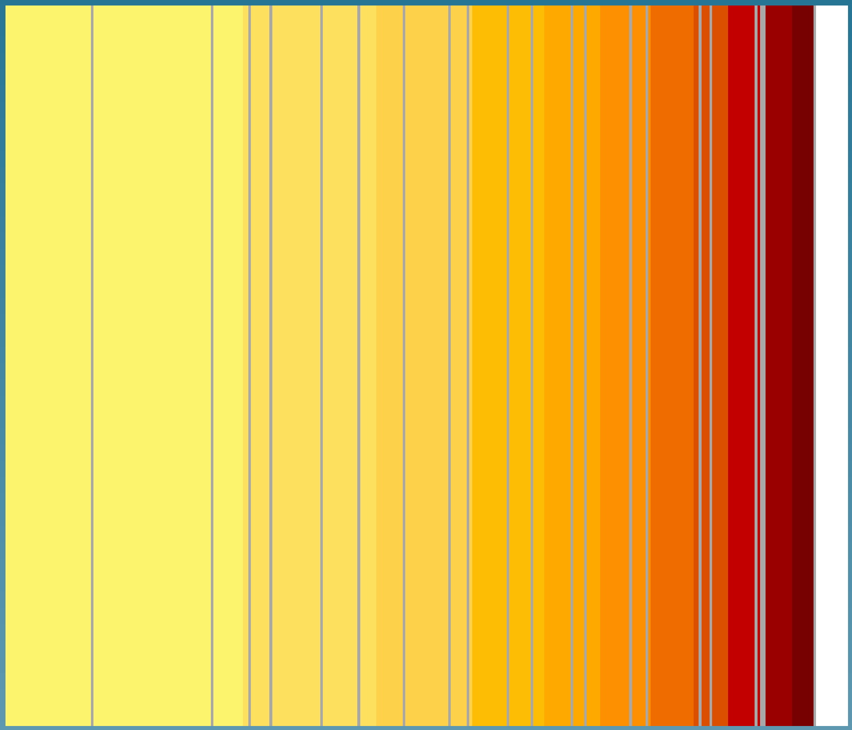
Mesh



Isovalues

- Same accuracy with isotropic, adaptive finite elements : $O(10\ 000)$ vertices !

Example of strongly anisotropic mesh



Zoom 10 000x

Anisotropic a posteriori error estimates and anisotropic, adaptive finite elements

- Laplace problem
- Advection-diffusion
- The heat equation
- Strongly nonlinear parabolic problems arising from solidification of binary alloys (dendrites, cristal growth)

Laplace problem

- Find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

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- Let $\mathcal{T}_h$ be a mesh of Ω into triangles with diameter less than h.
- Find $u_h \in V_h$ (continuous, piecewise linears) such that, forall $v_h \in V_h$

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h.$$

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- A posteriori error estimates : $\exists C(\text{aspect ratio})$, $\forall h$

$$\int_\Omega |\nabla e|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K^2(u_h, K, data)$$

A posteriori error estimates : terminology

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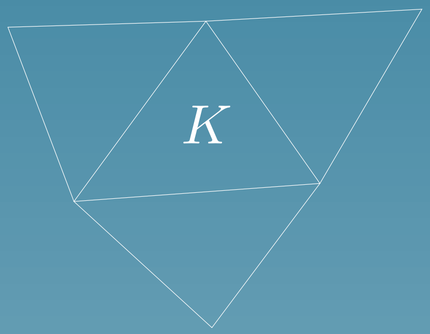
- Asymptotically equivalent error estimator : $ei \underset{h\to 0}{\longrightarrow} 1.$

A posteriori error estimators : example 1

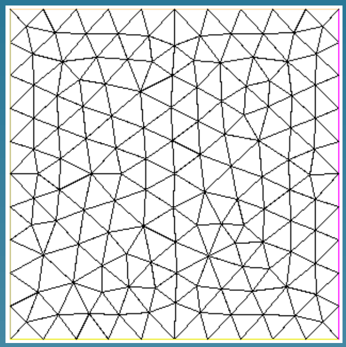
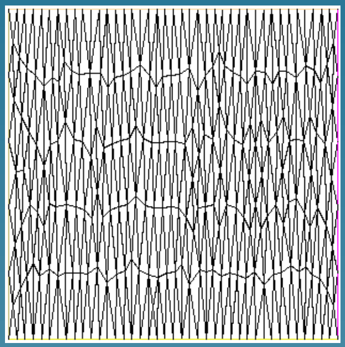
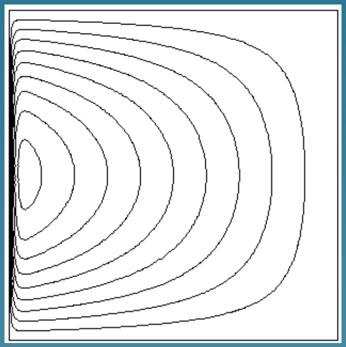
- Isotropic, residual-based, explicit a posteriori error estimator (Baranger El-Amri M2AN 1991, Babuska Duran Rodriguez SIAM Numer. Anal. 1992)

$$\int_\Omega |\nabla e|^2 \leq C \sum_{K \in \mathcal{T}_h} \left(h_K^2 \int_K (f + \Delta u_h)^2 + \frac{1}{2} h_K \int_{\partial K} \left[\frac{\partial u_h}{\partial n} \right]^2 \right).$$



- Equivalent to the true error (the effectivity index depends on the mesh aspect ratio).

Numerical results (isotropic, residual-based, explicit err. est.)

-   

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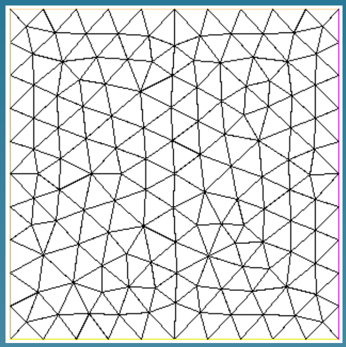
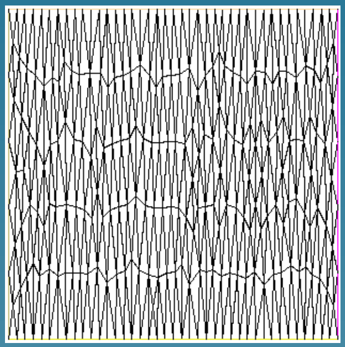
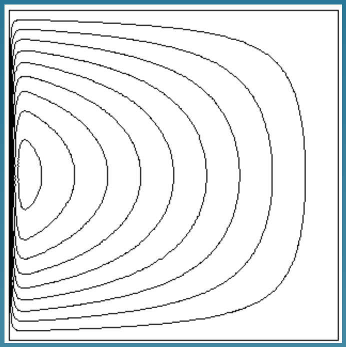
-   

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$h1 - h2$	error	ei
$0.01 - 0.01$	1.36	4.71
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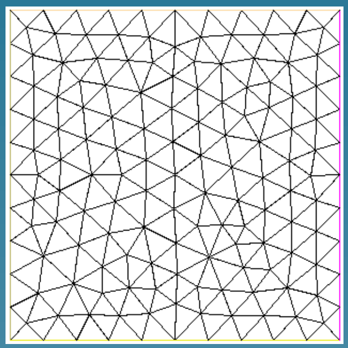
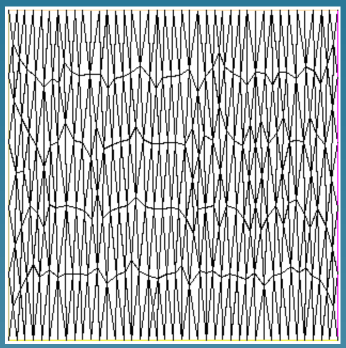
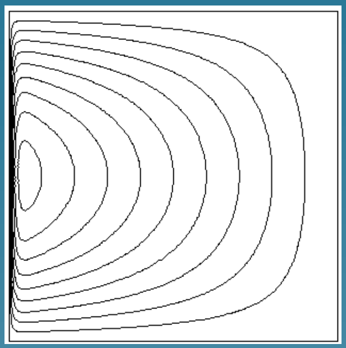
-   

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$0.0025 - 0.02$	0.33	13.4
$0.00125 - 0.01$	0.16	13.6

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- Then $\boldsymbol{\eta}^{ZZ} = \left(\int_\Omega |\nabla u_h - G u_h|^2 \right)^{1/2}$ is asymptotically exact on parallel meshes (Rodriguez NMPDE 1994, Ainsworth Oden CMAME 1997) .

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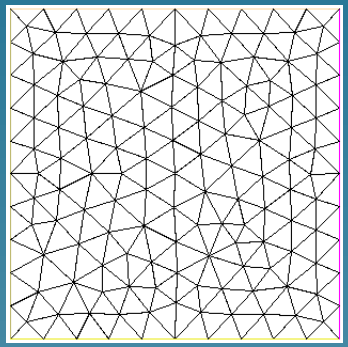
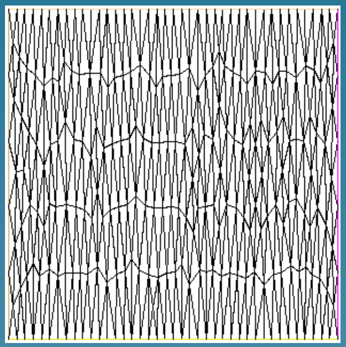
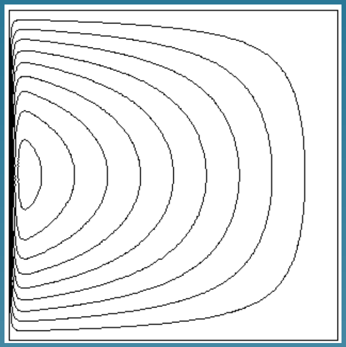
-   

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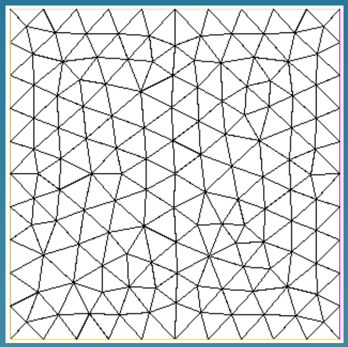
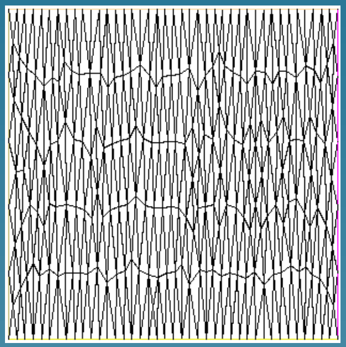
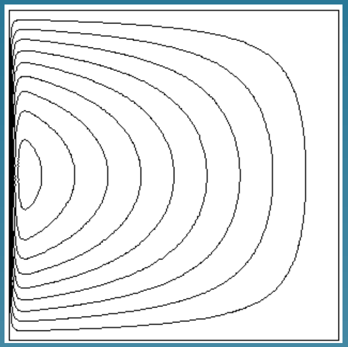
-   

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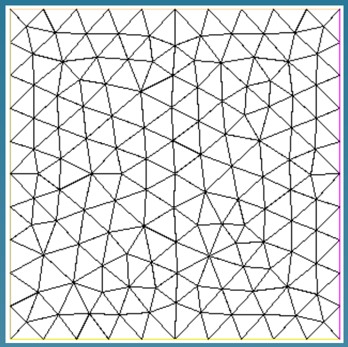
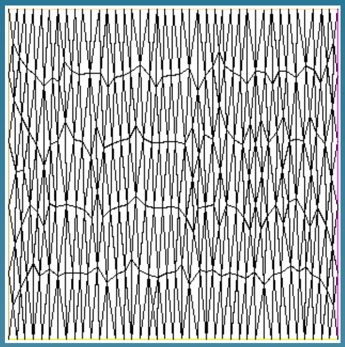
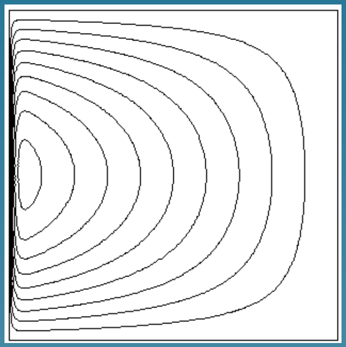
-   

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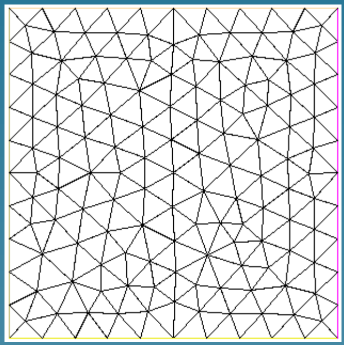
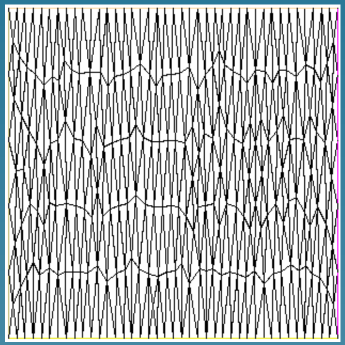
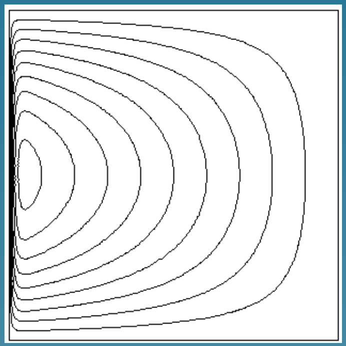
-   

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- Open question : why is ZZ asymptotically exact on some non-parallel meshes ?

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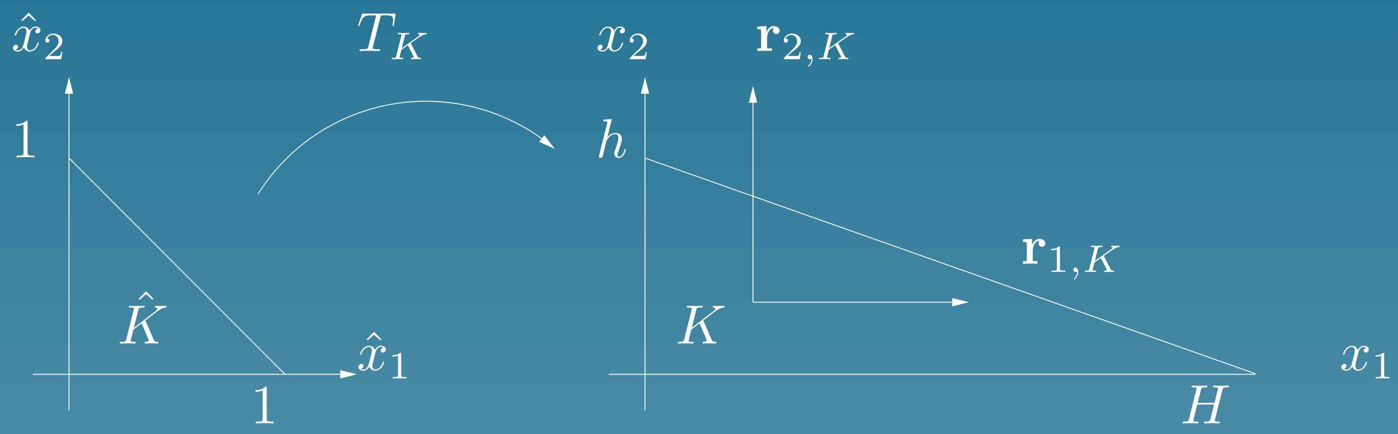
- $\lambda_{1,K}$? $\lambda_{2,K}$? $\mathbf{r}_{1,K}$? $\mathbf{r}_{2,K}$? How to approach $G_K(e)$?

Anisotropic interpolation estimates (Formaggia Perotto, Numer. Math. 2001)



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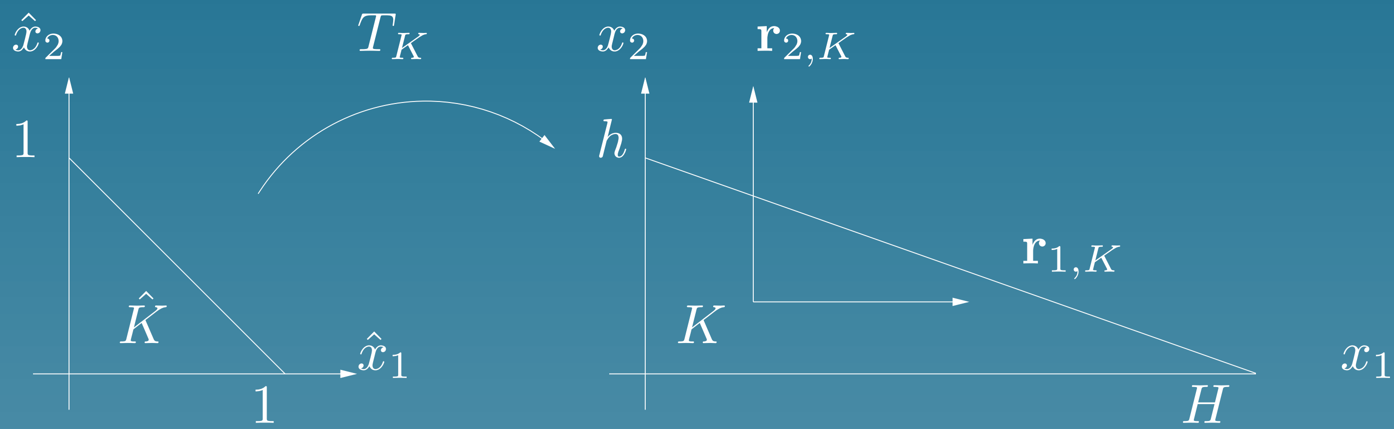

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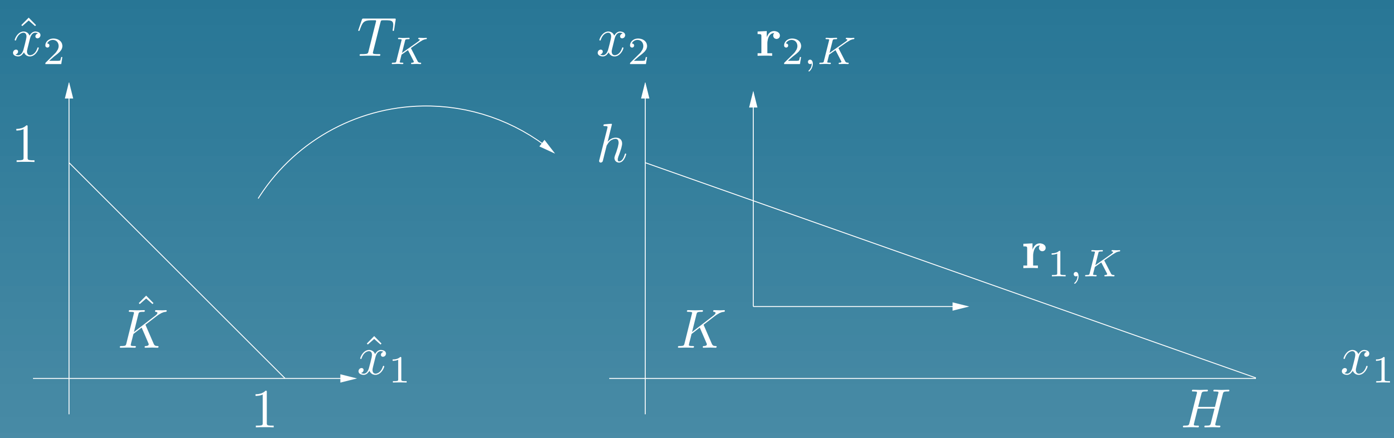

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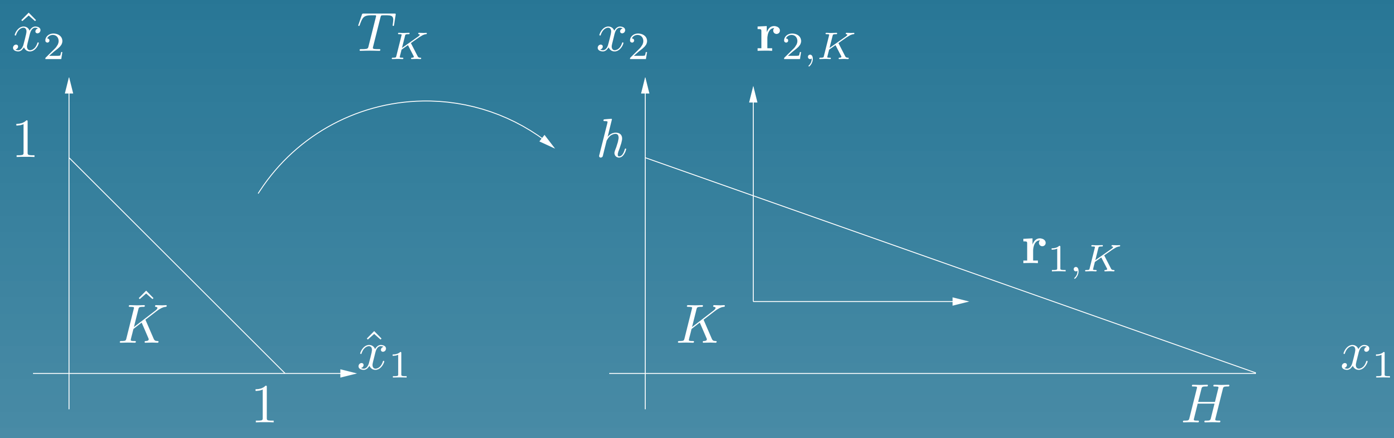

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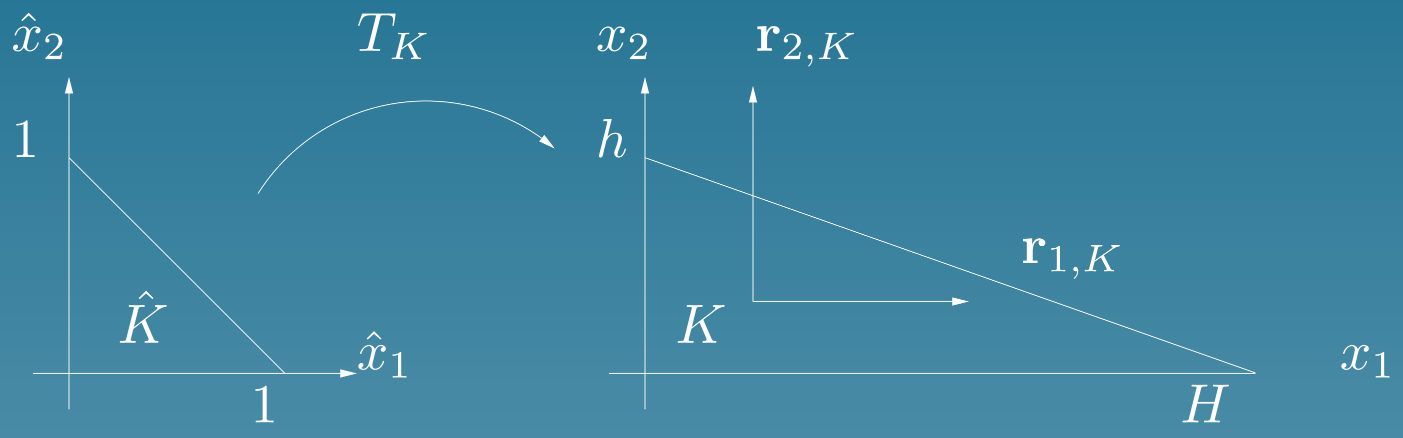

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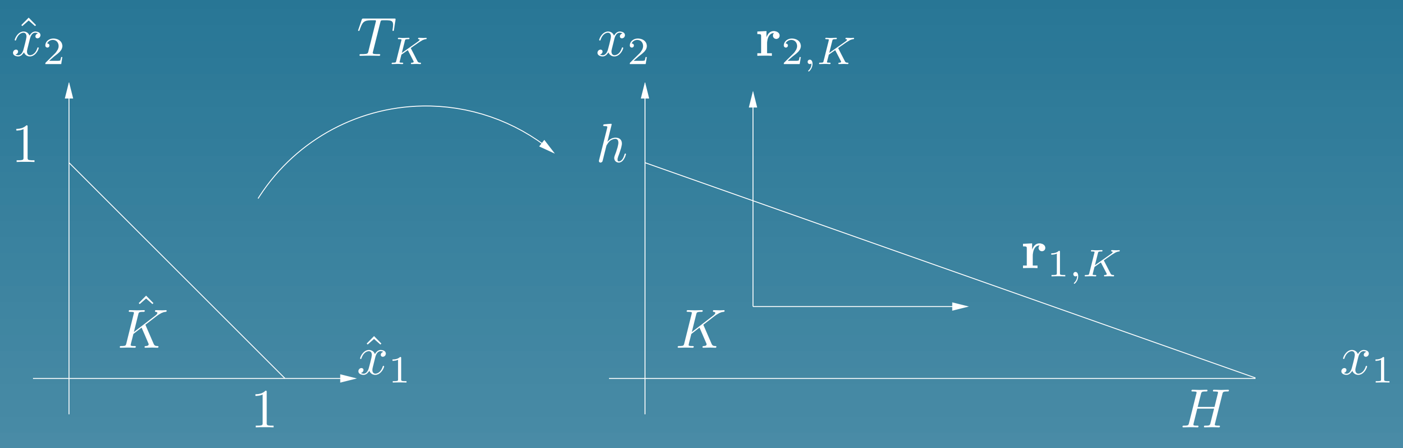

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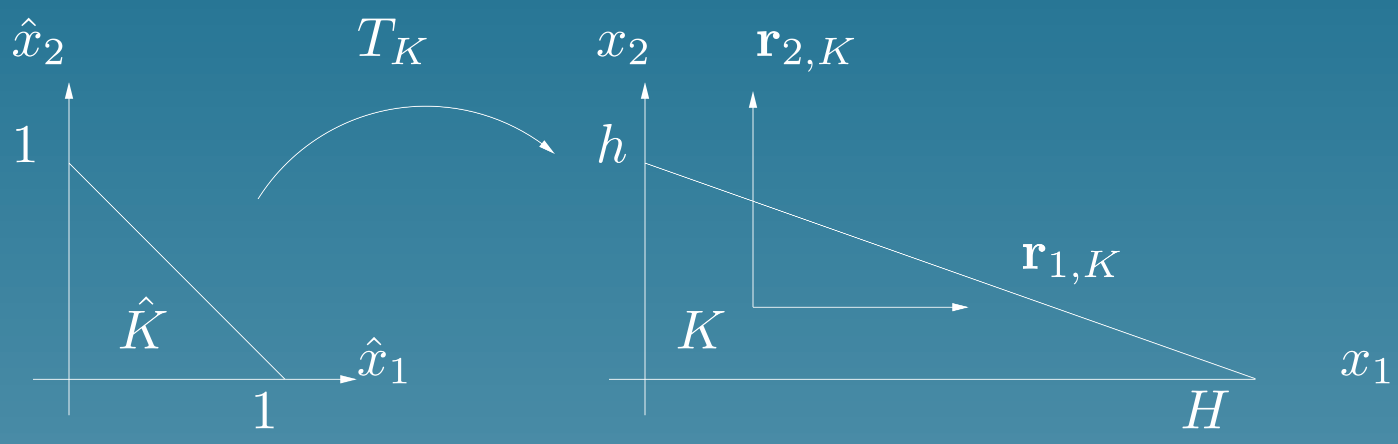

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Anisotropic, a posteriori error estimates

- Recall that

$$\int_\Omega |\nabla e|^2 \le C \sum_{K\in\mathcal{T}_h} \left(\|f + \Delta u_h\|_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{L^2(\partial K)} \right)$$
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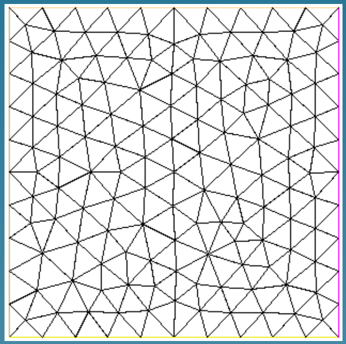
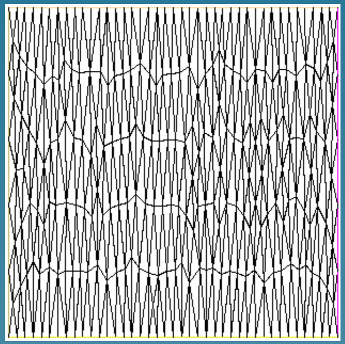
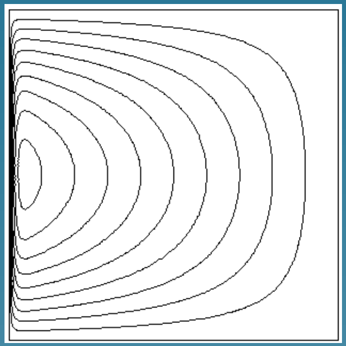
- Zienkiewicz-Zhu (ZZ) error estimator $\int_K \left(\frac{\partial e}{\partial x_1}\right)^2 \to \int_K \left(\frac{\partial u_h}{\partial x_1} - (Gu_h)_1\right)^2.$

An anisotropic error indicator based on ZZ error estimator

-   

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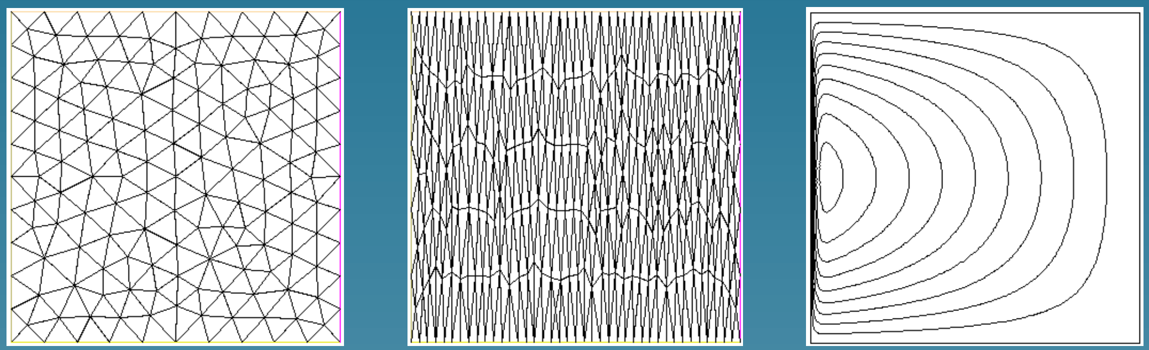
- 

$h1 - h2$	error	ei
$0.01 - 0.01$	1.36	2.22
$0.005 - 0.005$	0.69	2.42
$0.0025 - 0.0025$	0.35	2.54

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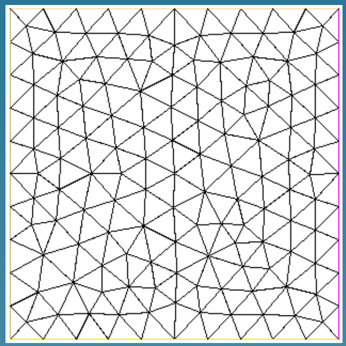
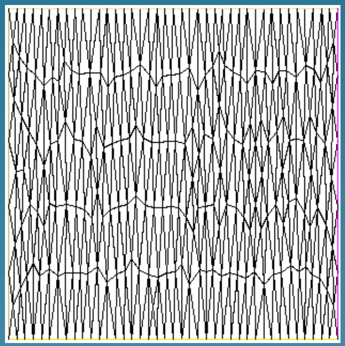
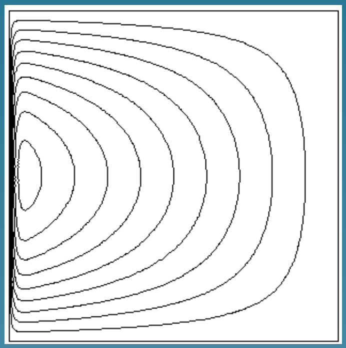
-   

-

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$0.005 - 0.005$	0.69	2.42
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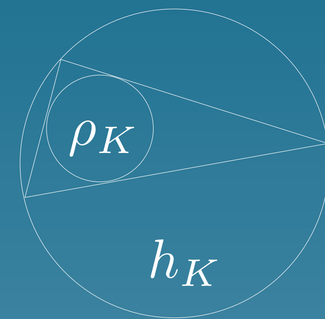
$h1 - h2$	error	ei
$0.005 - 0.04$	0.65	2.43
$0.0025 - 0.02$	0.33	2.62
$0.00125 - 0.01$	0.16	2.68

Anisotropic interpolation estimates (Formaggia Perotto, Numer. Math. 2001)

- Isotropic Clément

$$\|v - R_h v\|^2_{L^2(K)} \leq C(\frac{h_K}{\rho_K}) h_K^2 \|\nabla v\|^2_{L^2(\Delta K)}$$



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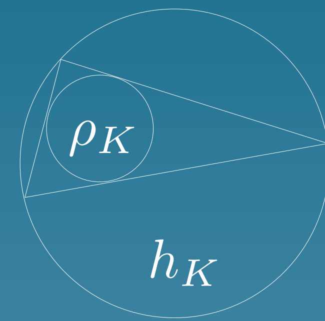
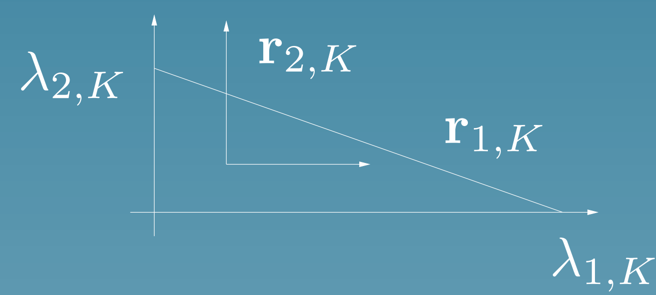


- Anisotropic Clément

$$\|v - R_h v\|_{L^2(K)} \leq C(\hat{K}) \Bigg(\lambda^2_{1,K} \left(\mathbf{r}^T_{1,K} G_K(v) \mathbf{r}_{1,K} \right) + \lambda^2_{2,K} \left(\mathbf{r}^T_{2,K} G_K(v) \mathbf{r}_{2,K} \right) \Bigg),$$



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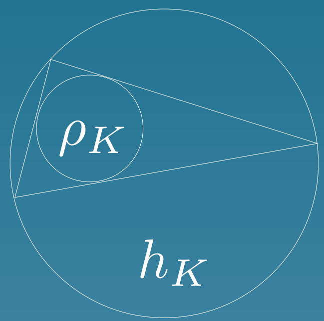

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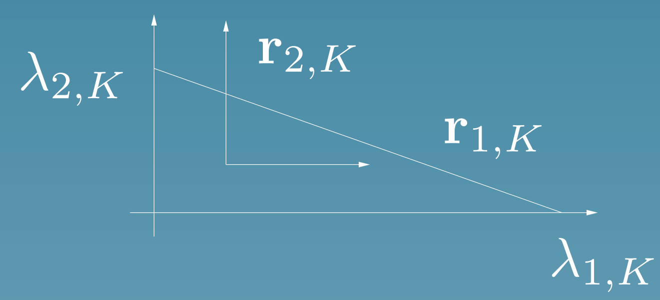

- Optimality : $v(x_2)$, $\mathbf{r}_{1,K} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{r}_{2,K} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\lambda_{1,K} = 1$, $\lambda_{2,K} = h$.

Anisotropic, adaptive finite elements

- Goal : find $\mathcal{T}_h$ s.t. $0.75\ TOL \leq \dfrac{\left(\sum_{K\in\mathcal{T}_h} \eta_K^2\right)^{1/2}}{\left(\int_\Omega |\nabla u_h|^2\right)^{1/2}} \leq 1.25\ TOL.$

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- with $\eta_K^4 = \left(\|f + \Delta u_h\|_{L^2(K)} + \dfrac{1}{2\lambda_{2,K}^{1/2}} \left\| \left[\dfrac{\partial u_h}{\partial n} \right] \right\|_{L^2(\partial K)} \right)^2$
$\times \left(\lambda_{1,K}^2 \left(\mathbf{r}_{1,K}^T G_K(e) \mathbf{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\mathbf{r}_{2,K}^T G_K(e) \mathbf{r}_{2,K} \right) \right).$

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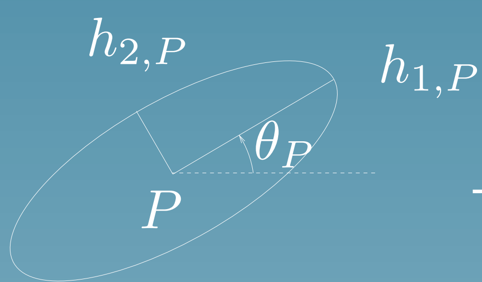
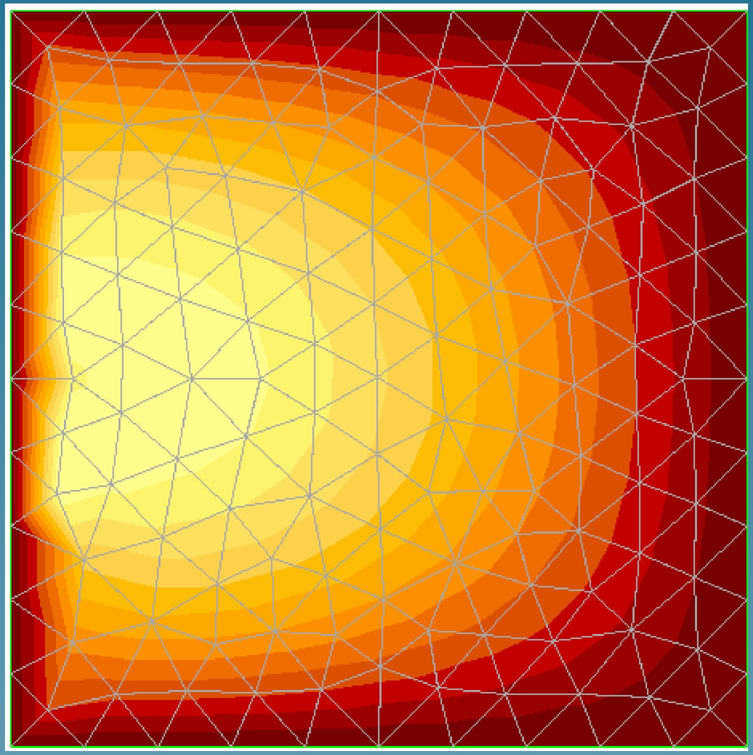
- 

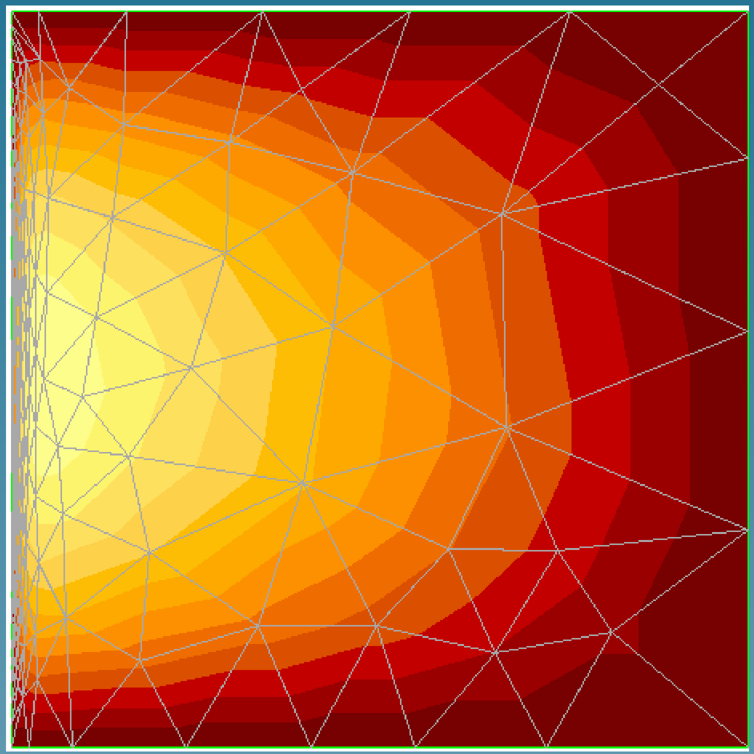
The BL2D mesh generator (INRIA, Borouchaki, Laug)

Adaptive meshes for the Laplace problem



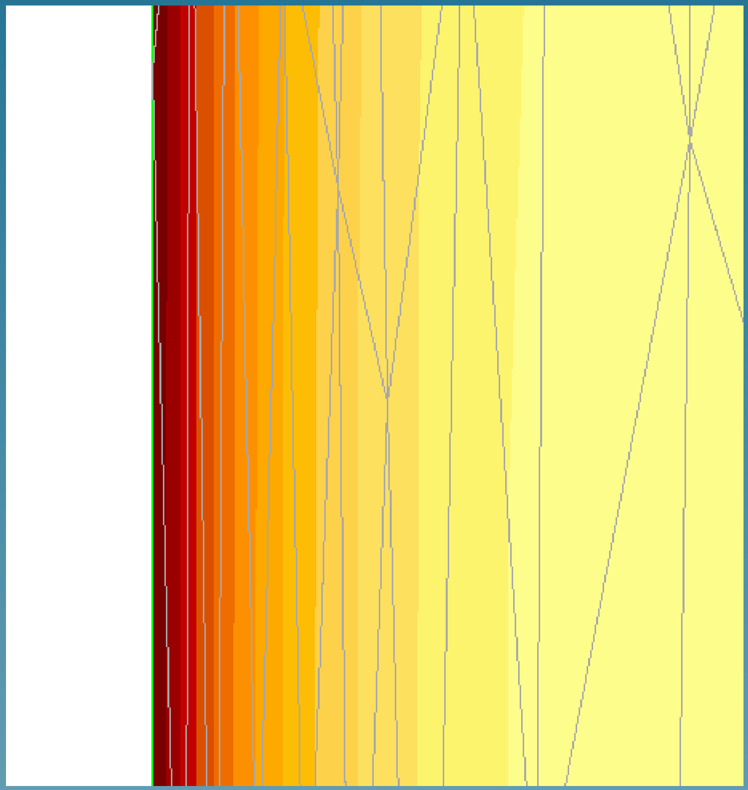
Initial 10×10 mesh

Adaptive meshes for the Laplace problem



$TOL = 0.25$: adapted mesh after 30 mesh generations, 145 vertices

Adaptive meshes for the Laplace problem



$TOL = 0.25$: adapted mesh after 30 mesh generations, 145 vertices (zoom)

Advection-diffusion

- Find $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} -\epsilon \Delta u + \mathbf{a} \cdot \nabla u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

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- Question : what is the stabilization coefficient on strongly anisotropic meshes ?

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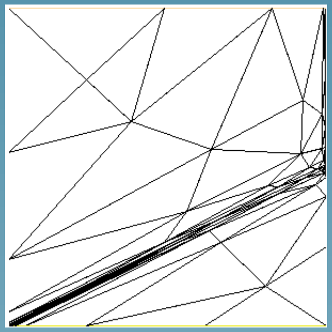
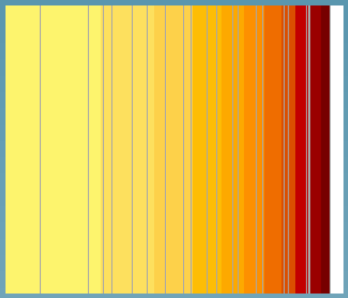
- Continuous, piecewise linear, stabilized finite elements
- Question : what is the stabilization coefficient on strongly anisotropic meshes ?
- $\Omega = (0,1)^2$, $\epsilon = 0.0001$, $a = (2,1)^T$, $f = 0$, 241 vertices, asp. ratio $\geq 10^4$.



Mesh

Isovalues



Zoom

The heat equation

- Find $u : \Omega \times (0,T) \longrightarrow \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} - \Delta u = f \qquad \text{in } \Omega \times (0,T),$$

plus initial and boundary conditions.

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- For $n = 1, ..., N$, find $u_h^n \in V_h$ such that, for all $v \in V_h$

$$\frac{1}{\tau}\int_\Omega (u_h^n - u_h^{n-1}) v dx + \int_\Omega \nabla u_h^n \cdot \nabla v dx = \int_\Omega f^n v dx.$$

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- $u_{h\tau}(x, t) = \dfrac{t - t^{n-1}}{\tau} u_h^n(x) + \dfrac{t^n - t}{\tau} u_h^{n-1}(x)$ for all $t^{n-1} \leq t \leq t^n$.

The heat equation

- Error : $e = u - u_{h\tau}$, $\int_0^T \int_\Omega |\nabla e|^2$.

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$$\left(\lambda_{1,K}^2 \left(\mathbf{r}_{1,K}^T G_K(e) \mathbf{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\mathbf{r}_{2,K}^T G_K(e) \mathbf{r}_{2,K} \right) \right)^{1/2}.$$

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-

h	τ	error	ei
0.05	0.1	3.48	5.25
0.025	0.025	1.68	2.09
0.0125	0.00625	0.82	2.57
0.00625	0.0015625	0.42	2.66

The heat equation

- Adaptive finite elements : $0.75\ TOL \le \dfrac{\left(\displaystyle\sum_{n=1}^{N} \sum_{K \in \mathcal{T}_h} \eta_{n,K}^2\right)^{1/2}}{\left(\int_0^T \int_\Omega |\nabla u_{h\tau}|^2\right)^{1/2}} \le 1.25\ TOL.$

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- Sufficient condition :

$$0.75^2 TOL^2 \int_{t^{n-1}}^{t^n} \int_K |\nabla u_{h\tau}|^2 \leq \eta_{n,K}^2 \leq 1.25^2 TOL^2 \int_{t^{n-1}}^{t^n} \int_K |\nabla u_{h\tau}|^2$$

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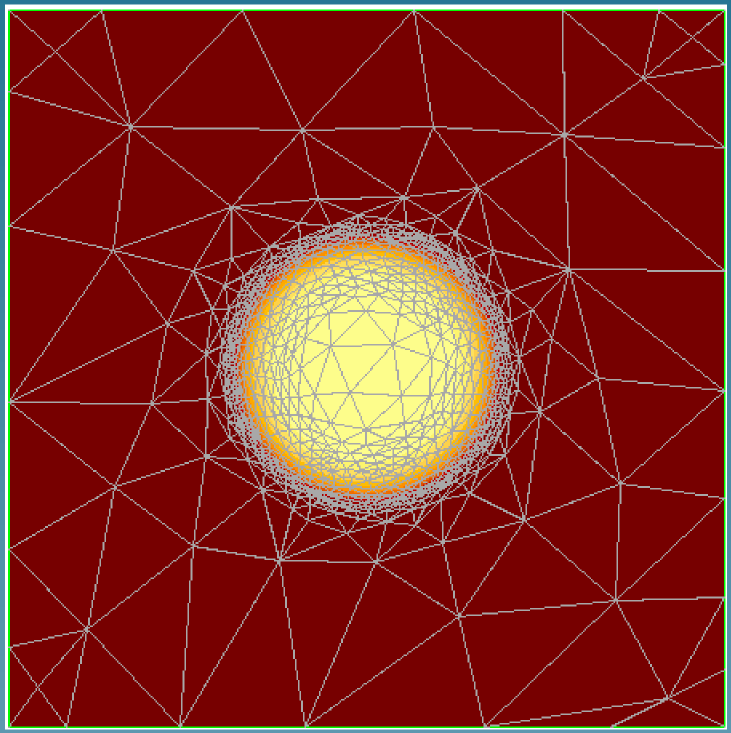
-

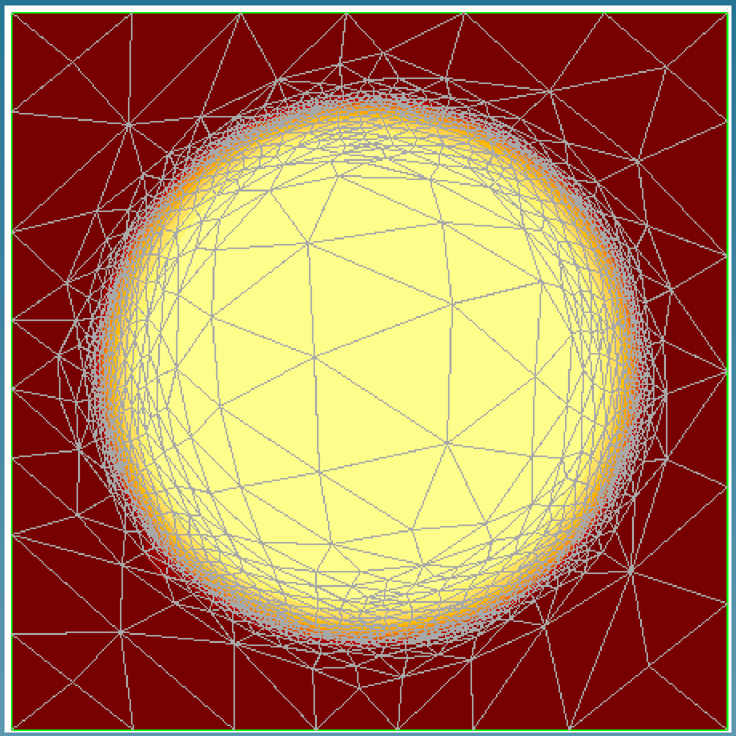
TOL	τ	error	ei
0.25	0.025	0.55	2.56
0.125	0.00625	0.26	2.86
0.0625	0.0015625	0.13	2.88

Adaptive meshes for the heat problem



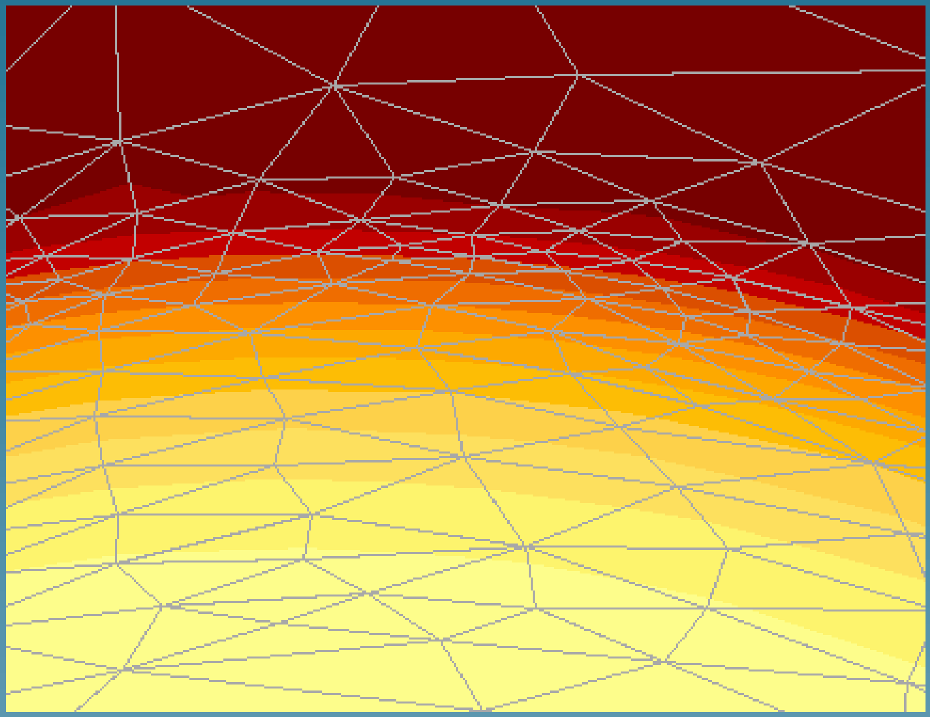
$TOL = 0.25$, 40 times steps : first time step

Adaptive meshes for the heat problem



$TOL = 0.25$, 40 times steps : last time step

Adaptive meshes for the heat problem

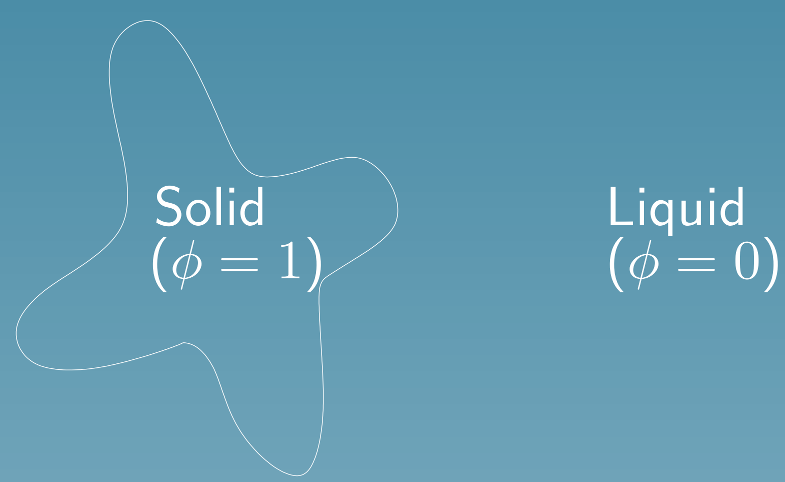


$TOL = 0.25$, 40 times steps : last time step (zoom)

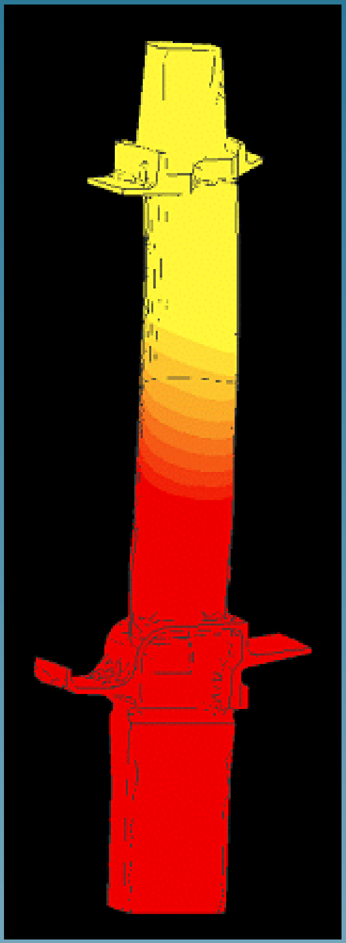
Strongly nonlinear parabolic problems : Solidification of a binary alloy

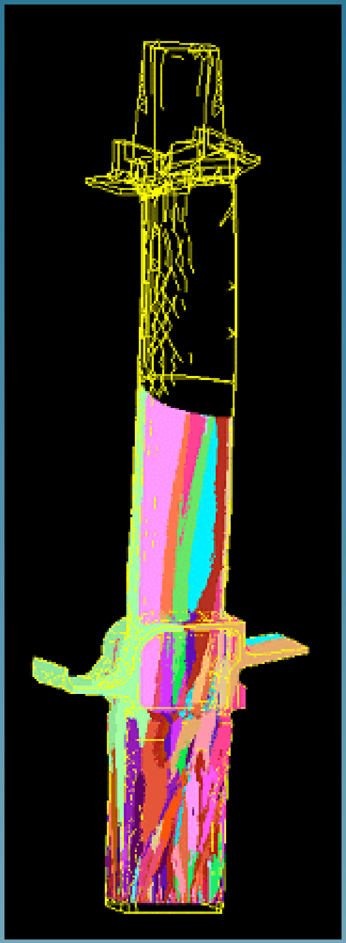
- with Lab. Métallurgie Physique, M. Rappaz A. Jacot.
- Phase field model. Find $c, \phi : \Omega \times (0,T) \rightarrow \mathbb{R}$ such that

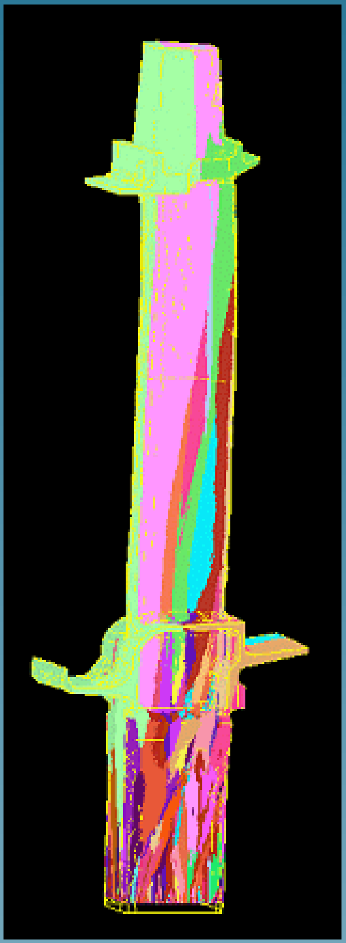
$$\begin{aligned} &\frac{1}{M}\frac{\partial \phi}{\partial t} - \text{div}\ (A(\nabla\phi)\nabla\phi)) \ - S(c,\phi) = 0 && \text{in } \Omega \times (0,T), \\ &\frac{\partial c}{\partial t} - \text{div}\ \Big(D_1(\phi)\nabla c + D_2(c,\phi)\nabla\phi\Big) = 0 && \text{in } \Omega \times (0,T). \end{aligned}$$

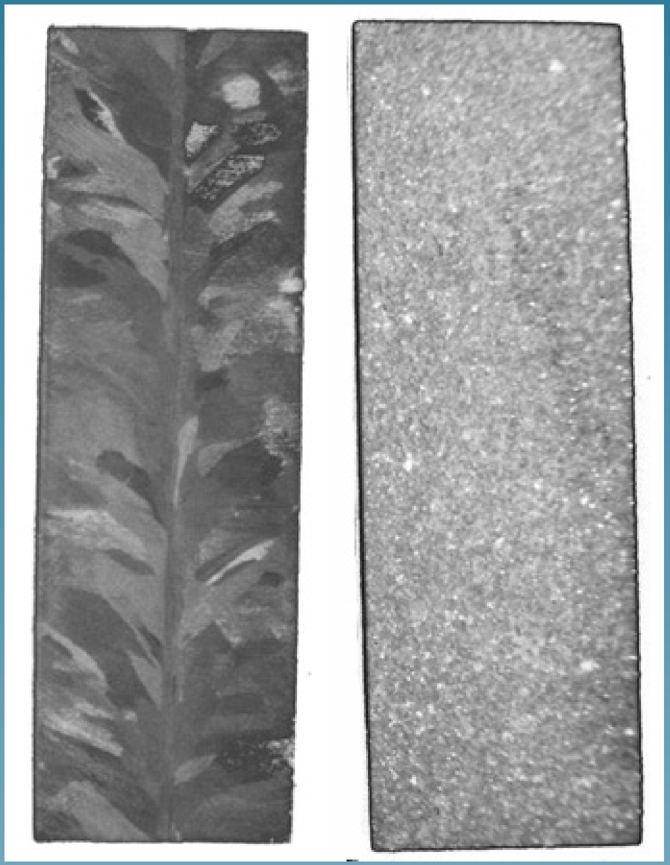


Solidification : from macro to meso scale

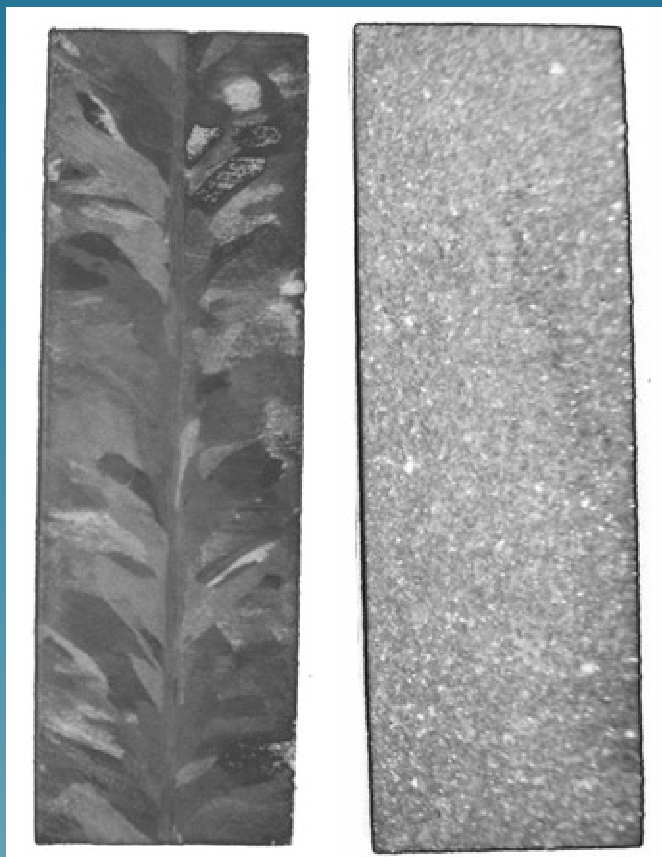


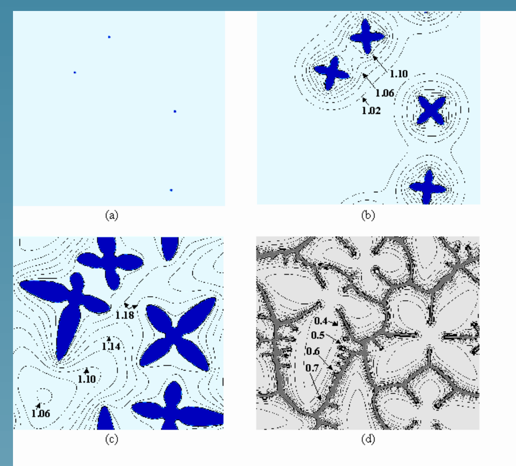


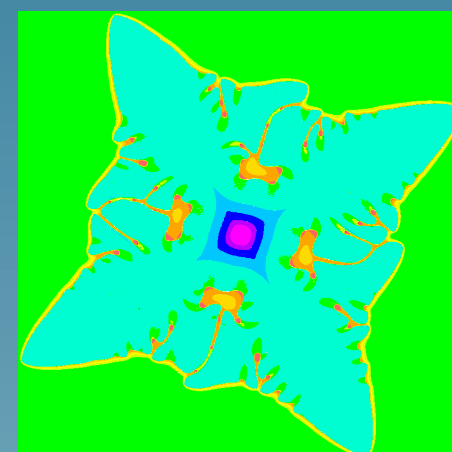




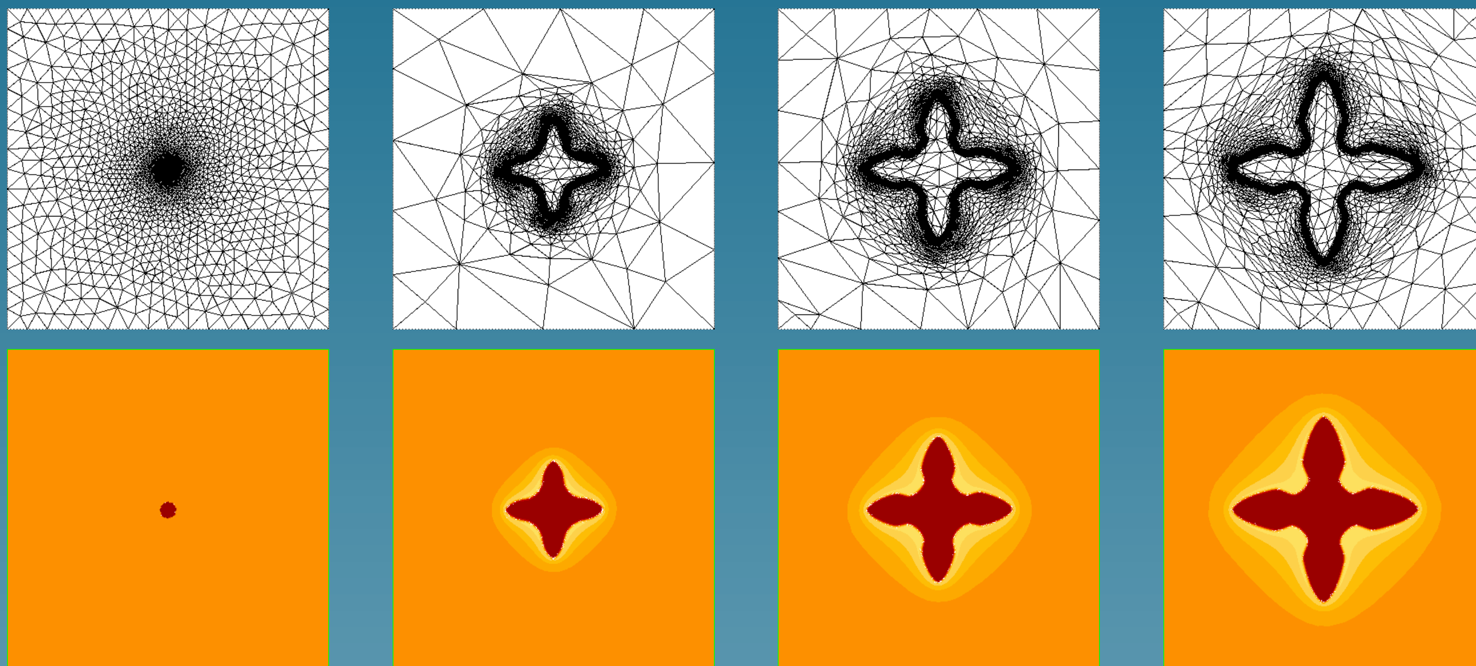
Solidification : from meso to micro scale



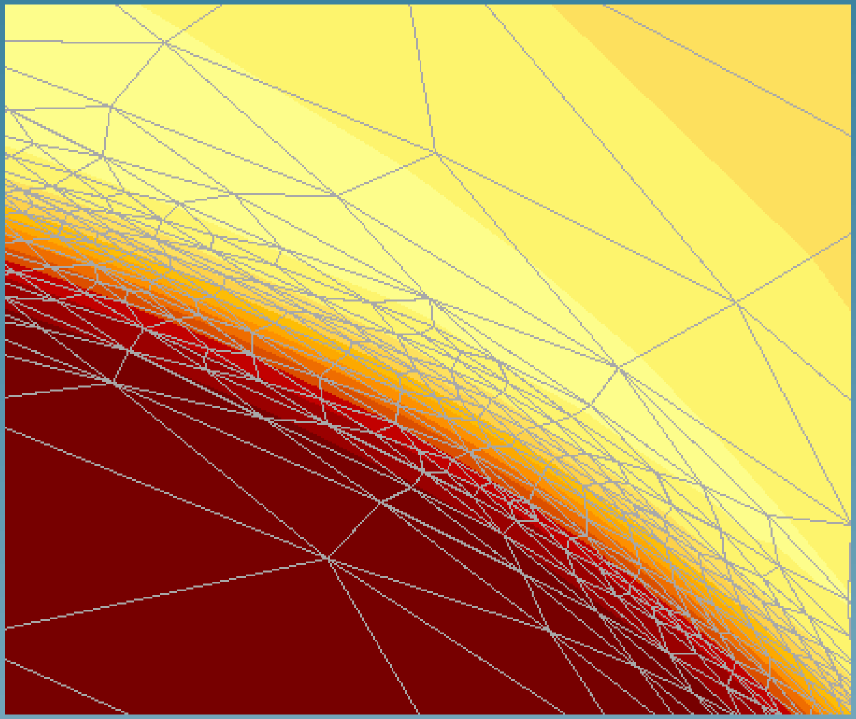


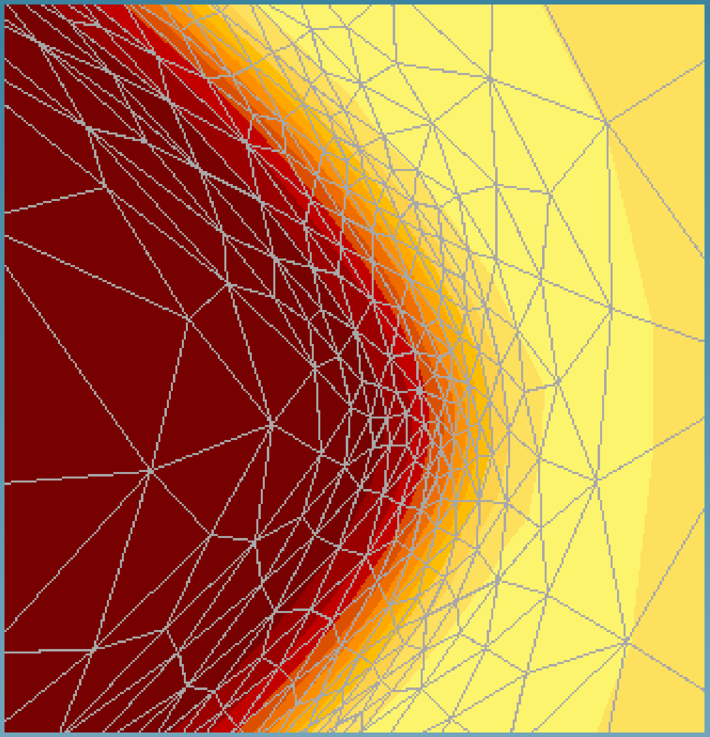


Phase field with low anisotropy

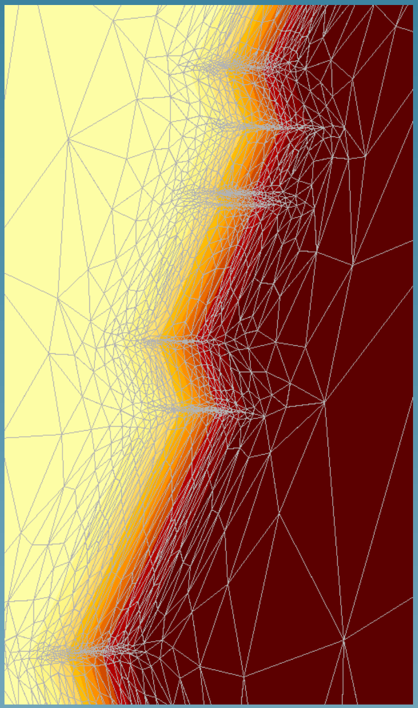


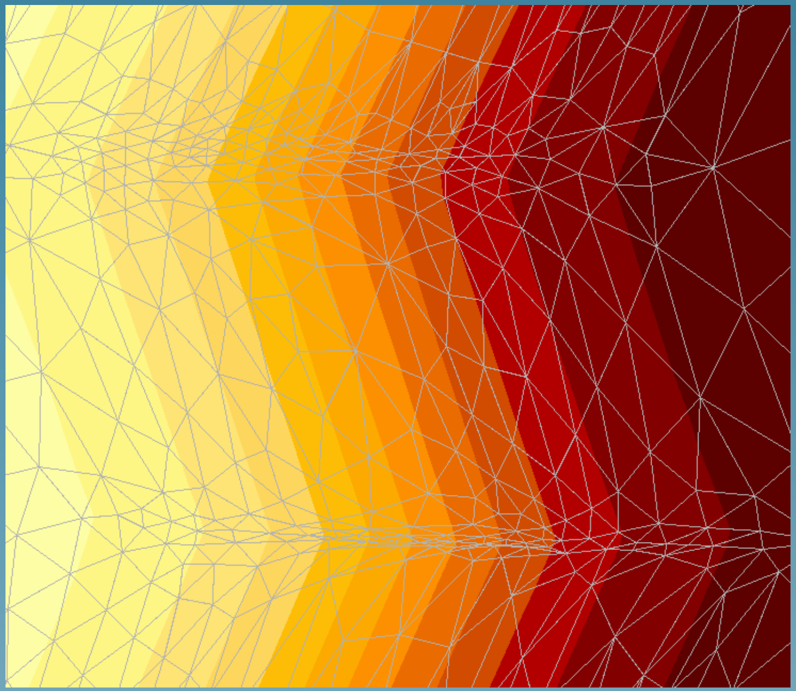
Phase field with low anisotropy



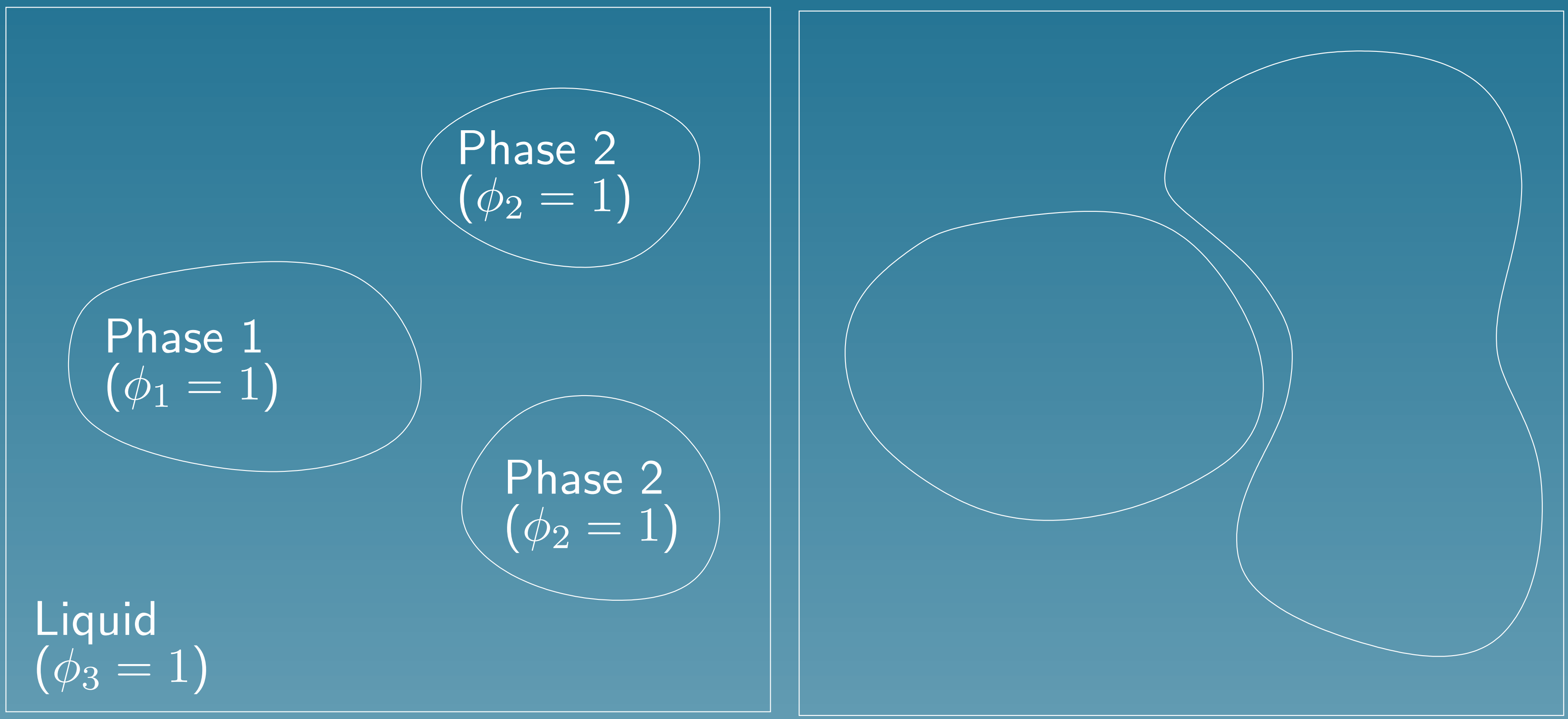


Phase field with strong anisotropy



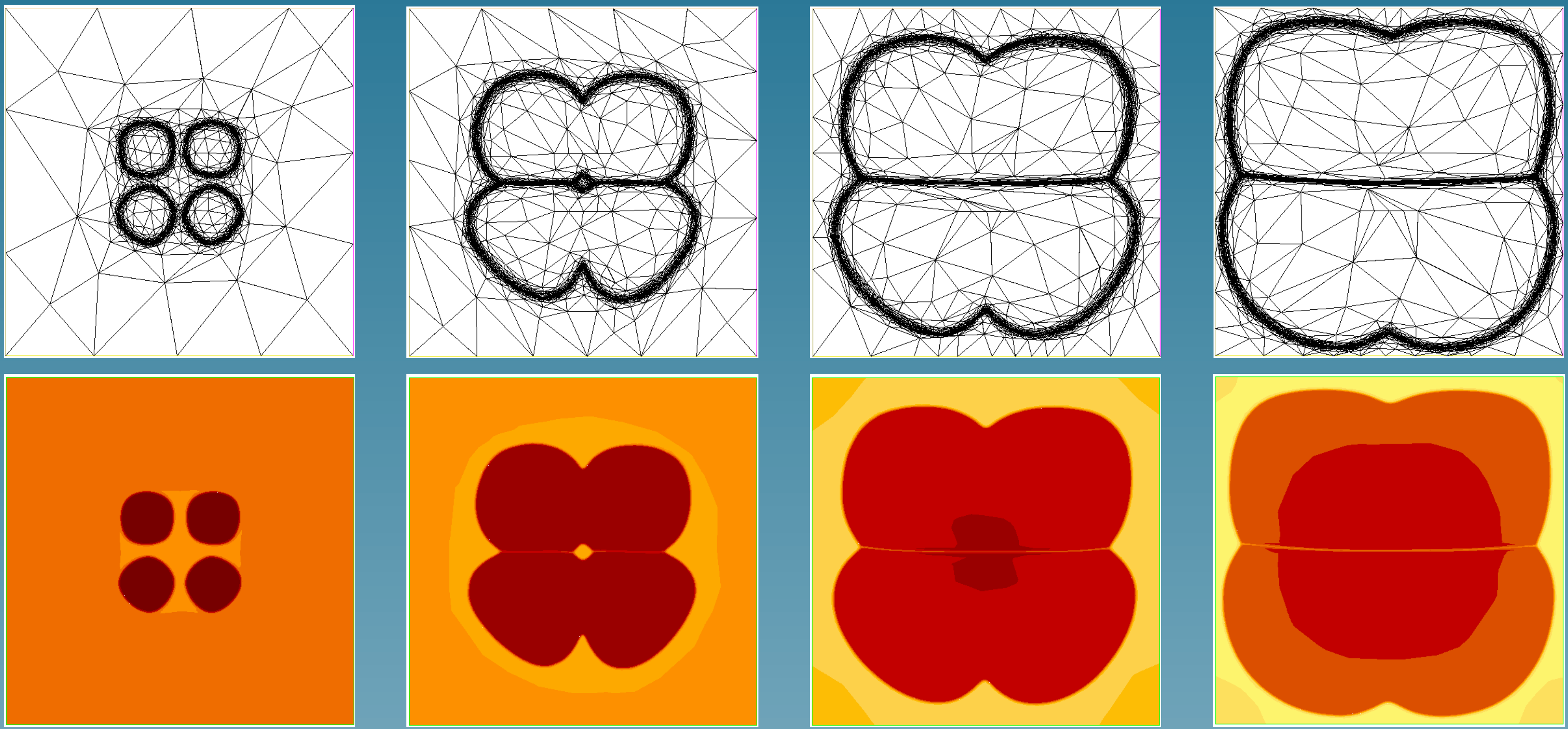


The multiphase field model

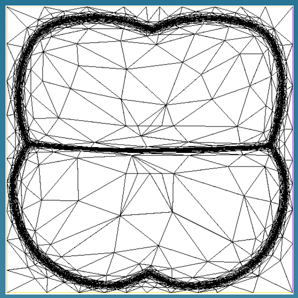


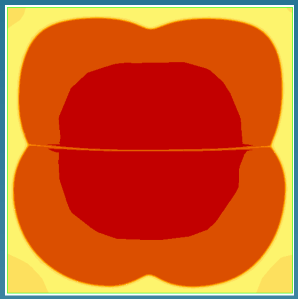
- Unknowns : ϕ_1, ϕ_2, ϕ_3, λ (Lagrange multiplier $\phi_1 + \phi_2 + \phi_3 = 1$) and c.

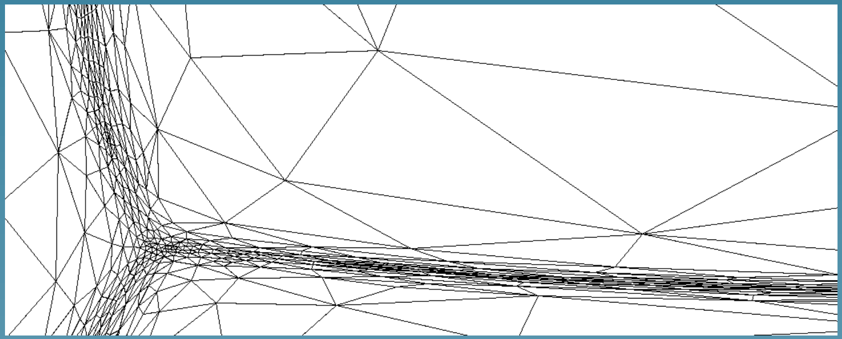
The multiphase field model

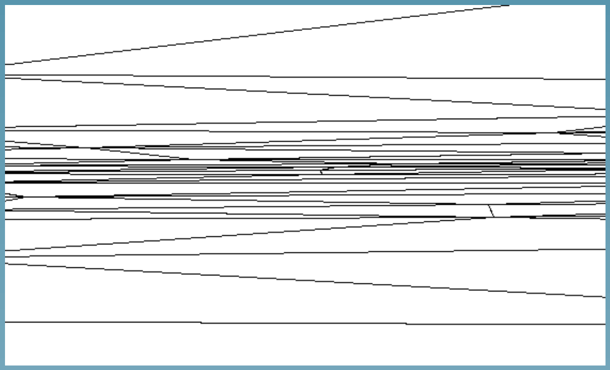


The multiphase field model









Conclusions and perpectives

- Use of anisotropic, adaptive grids : same accuracy with fewer vertices.
- Robustness ? Lower bound ? ZZ ?
- Systems of p.d.e ? (Stokes)
- Optimal control
- Anisotropic meshes in 3D ?