

**COUPLED CONTINUOUS AND DISCONTINUOUS
GALERKIN METHODS FOR CONVECTION-DOMINATED
FLOW**

SEMINAIRE DE CALCUL SCIENTIFIQUE DU CERMICS

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Acknowledgements

- Current work
- Numerical results
- Strategy: shallow water equations
- Strategy: model transport problem
- Literature review
- Motivation

Outline

Figure 2: Southern Louisiana coast line

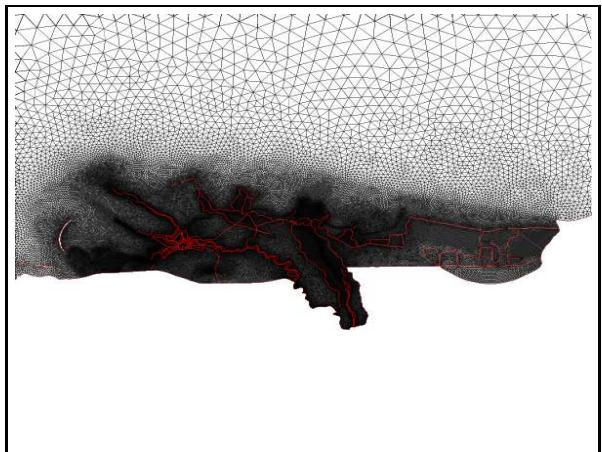
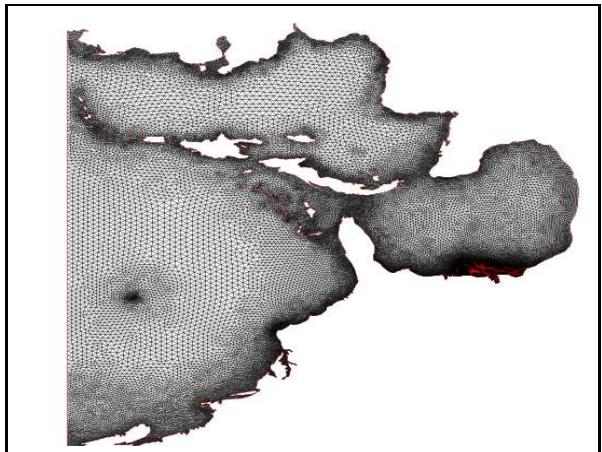
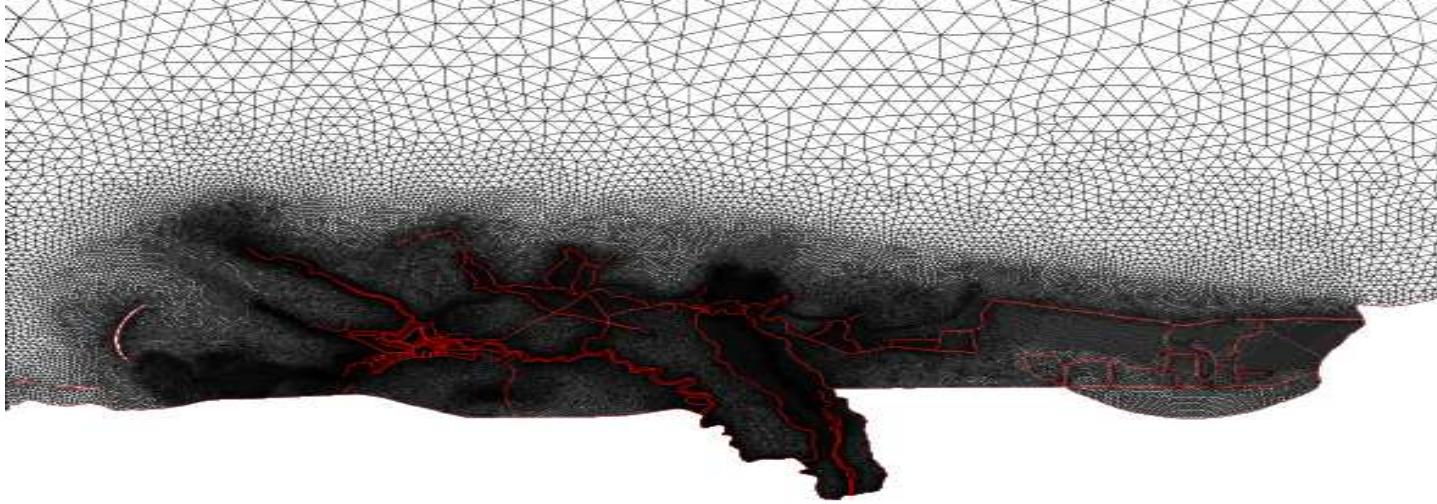


Figure 1: Western north Atlantic ocean



Motivation

- Application: Modeling of hydrodynamic flow regimes and transport processes in shallow water
- Environmental, economic and public health implications
- Effective numerical models are needed to manage coastal environments
 - Modeling of river flows and flood plains
 - Prediction of hurricanes, storm surges
 - Salinity and contaminant transport
 - Simulation of tidal fluctuations
- How to obtain most accurate results at minimal computational cost?



Current simulators handle some but not all of these difficulties!

- Large computational domains
- Complex domains with irregular boundaries
 - multiple flow regimes
- Rapidly changing and complicated solution behavior
 - steep gradients, numerical oscillations
- Nonlinearities and forcing terms in shallow water equations

Challenges

- appropriate features by strategically coupling methods
- across subdomains in region of interest: "domain-decomposition"
- across variables in model: "coupling by equations"
- combination of both strategies: "multi-coupled"

*Develop and implement new multi-algorithmic methods based on discontinuous Galerkin (DG) and continuous Galerkin (CFEM)
finite element methods.*

Approach

- handles diffusion dominated flow regimes efficiently
- degrees of freedom usually associated with nodes

CONTINUOUS GALERKIN FEM

- handles advection dominated flow regimes efficiently
- degrees of freedom associated with elements
- suppresses non-physical oscillations
- easily handles complex geometry and nonmatching grids
- locally conservative and accurate
- easily incorporates h- and p-adaptivity

DISCONTINUOUS GALERKIN FEM

Features

Literature Review

CFEM on SWE

DG on hyperbolic-type

LDG on second-order

NIPG on second-order

on non-matching grids

LDG/CFEM

Wave Continuity Model
Lynch, Gray (1979)
Luetje, Westerink, Scheffner (1991)

Cockburn, Shu (1989)
Bey, Oden (1994)

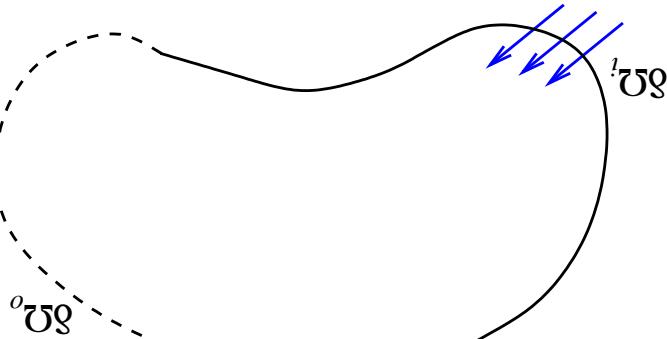
Local Discontinuous Galerkin Method

Aizinger, Dawson (2000)
Cockburn, Shu (1998),
Bassi, Rebay (1997),

Baumann, Babuska, Oden (1998)
Riviere, Wheeler, Girault (1999-)

Non-symmetric Interior Penalty Galerkin Method

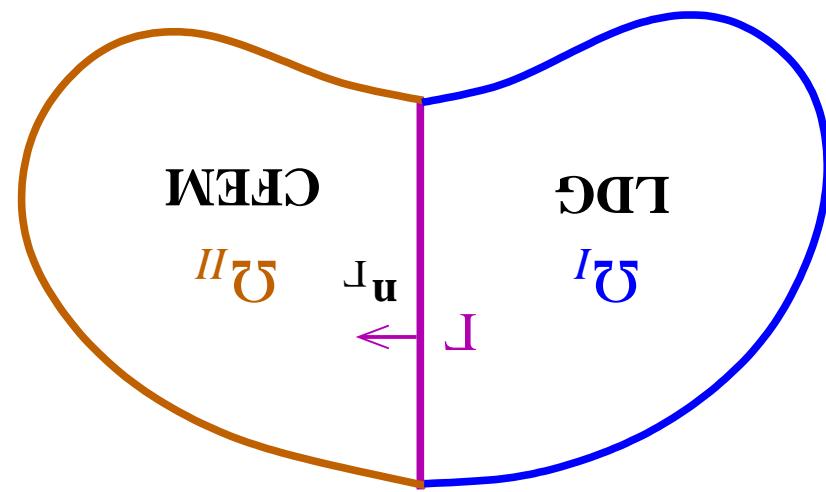
Alotto, Berthoni, Perugia, Schotzau (2000)
Perugia, Schotzau (2000)



$$\begin{aligned} \{0 \leq \mathbf{u} \cdot \mathbf{n} : \mathbf{u} \in \mathcal{Q}\} &= \mathcal{Q}_o^+ \\ \{0 > \mathbf{u} \cdot \mathbf{n} : \mathbf{u} \in \mathcal{Q}\} &= \mathcal{Q}_i^- \end{aligned}$$

- Let $\mathcal{U} \in \mathbb{W}^p$ have Lipschitz boundary $\partial\mathcal{U}$ decomposed into positive semi-definite and bounded. Could be advection-dominated.
- smooth to admit a unique smooth solution c . Assume D is symmetric, assume coefficients, initial and boundary data, and domain are sufficiently

$$\begin{aligned} c(\mathbf{x}, 0) &= c_0 && \text{on } \mathcal{U}. \\ (-D\Delta c) \cdot \mathbf{n} &= 0 && \text{on } \partial\mathcal{U}, t < 0. \\ (\mathbf{u}_c - D\Delta c) \cdot \mathbf{n} &= \mathbf{u}_c \cdot \mathbf{n} && \text{on } \partial\mathcal{U}, t < 0, \\ Q_c + \Delta \cdot (\mathbf{u}_c - D\Delta c) + ac &= \mathbf{f} && \text{in } \mathcal{U}, t < 0, \end{aligned}$$



“Domain Decomposition” Coupling Strategy

$$\left. \begin{aligned} & 0 > {}^e\mathbf{u} \cdot \mathbf{n} & c_+, \\ & 0 \leq {}^e\mathbf{u} \cdot \mathbf{n} < 0 & c_-, \end{aligned} \right\} = c_\downarrow$$

- Define the “upwind value” on an element edge

$${}_+m - {}_-m = [m] \quad ({}_-m + {}_+m) = \underline{m}$$

with

$$({}^e\mathbf{u}s + \mathbf{x})^k u^o \underset{\substack{(\infty \leftarrow k \\ +0 \leftarrow s)}}{\lim} = (\mathbf{x})_+ m \quad , \quad ({}^e\mathbf{u}s + \mathbf{x})^k u^o \underset{\substack{(\infty \leftarrow k \\ s \leftarrow -)}}{\lim} = (\mathbf{x})_- m$$

- Define trace operator $\gamma_o : H_{1/2}(\partial\mathcal{U}^e)$, let $u^e \in C_1(\underline{\mathcal{U}})$, then

$$\sum_{h \in \mathcal{T}_h} (\cdot, \cdot)_h \text{ all elements, } \sum_{i \in \mathcal{E}_{int}} \langle \cdot, \cdot \rangle_i \text{ interior edges}$$

surfaces or lines, with

- $(\cdot, \cdot)_h$ the $L^2(R)$ inner product for $R \in \mathbb{R}^d$, and $\langle \cdot, \cdot \rangle_i$ integration over

(quasi-uniform in $\mathcal{U}^{(i)}$)

- $\{\mathcal{T}_h\}_{h>0}$ denotes a uniform family of FE partitions of \mathcal{U}

$$\begin{aligned} \cdot \nabla \varphi & \cup \partial \varphi \text{ on } \partial \Omega^o \\ \cdot \nabla \varphi & \cup \partial \varphi \text{ on } \partial \Omega^i \quad \boldsymbol{u} \cdot \boldsymbol{n}_{\partial \Omega^i} = \boldsymbol{u} \cdot (\mathbf{z} + \boldsymbol{n}_{\partial \Omega^i}) \end{aligned}$$

with boundary conditions

$$\mathbf{z} = D\tilde{\mathbf{z}}$$

$$\Delta \mathbf{z} = \mathbf{z}$$

$$\mathbf{j} = \boldsymbol{\alpha} + (\mathbf{z} + \boldsymbol{n}_{\partial \Omega}) \cdot \Delta + \partial \Omega$$

Rewrite model problem in mixed form

$$A^I \mathbf{LDG}$$

$$\cdot 0 = {}^e\mathcal{U}(\underline{\Lambda}, \mathbf{z}) \sum_{\mathcal{L} \in \mathcal{T}_h} - {}^e\mathcal{U}(\underline{\Lambda}, \mathbf{z}D) \sum_{\mathcal{L} \in \mathcal{T}_h}$$

$$\begin{aligned} 0 = & \textcolor{blue}{\langle} {}^e\mathbf{u} \cdot \underline{\Lambda}, \mathbf{I} \textcolor{blue}{\rangle} + \textcolor{blue}{\langle} \mathbf{I} / \mathbf{I} \mathcal{U} \varrho, {}^e\mathbf{u} \cdot [\underline{\Lambda}] \textcolor{blue}{\rangle} + \\ & \textcolor{blue}{\langle} {}^e\mathbf{u} \cdot \underline{\Lambda}, \mathbf{I} \textcolor{blue}{\rangle} \sum_{\mathcal{L} \in \mathcal{T}_h} + {}^e\mathcal{U}(\underline{\Lambda} \cdot \Delta) \sum_{\mathcal{L} \in \mathcal{T}_h} - {}^e\mathcal{U}(\underline{\Lambda}, \mathbf{z}) \sum_{\mathcal{L} \in \mathcal{T}_h} \end{aligned}$$

$$\begin{aligned} & {}^e\mathcal{U} \cup {}^e\mathcal{U} \varrho \langle m, b(\mathbf{u} \cdot \mathbf{n}) \rangle - {}^e\mathcal{U}(m, \mathbf{j}) \sum_{\mathcal{L} \in \mathcal{T}_h} = \\ & {}^e\mathcal{U}(m, \mathbf{a} \cdot \mathbf{c}) \sum_{\mathcal{L} \in \mathcal{T}_h} + \textcolor{blue}{\langle} \mathbf{I} \cup {}^e\mathcal{U} \varrho, m, \mathbf{u} \cdot (\mathbf{z} + \mathbf{I} \mathbf{c} \mathbf{n}) \textcolor{blue}{\rangle} + {}^e\mathcal{U} \cup {}^e\mathcal{U} \varrho \langle m, \mathbf{u} \cdot \mathbf{c} \mathbf{n} \rangle + \\ & \textcolor{blue}{\langle} [m], {}^e\mathbf{u} \cdot (\mathbf{z} + \mathbf{I} \mathbf{c} \mathbf{n}) \textcolor{blue}{\rangle} \sum_{\mathcal{L} \in \mathcal{T}_h} + {}^e\mathcal{U}(m \Delta, \mathbf{z} + \mathbf{I} \mathbf{c} \mathbf{n}) \sum_{\mathcal{L} \in \mathcal{T}_h} - {}^e\mathcal{U}(m, \mathbf{u} \cdot \mathbf{c}) \sum_{\mathcal{L} \in \mathcal{T}_h} \end{aligned}$$

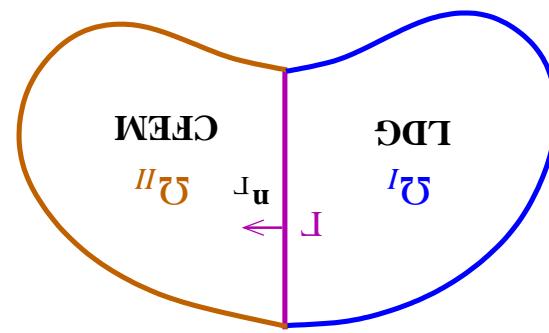
For arbitrary test functions $u \in H_1(\mathcal{U}), \underline{\Lambda} \in \mathcal{L}(\mathcal{U})$,

$$\begin{aligned}
 & \nabla \cdot \mathbf{u}^{\text{II}} \nabla \varphi \langle \mathbf{u}, \mathbf{b}(\mathbf{u} \cdot \mathbf{n}) \rangle - \nabla \cdot (\mathbf{J}) = \\
 & \quad \nabla \cdot (\mathbf{u}_{\text{CII}}^{\text{II}} - D_{\text{CII}} \Delta_{\text{CII}} \mathbf{u}_{\text{CII}}) + \langle \mathbf{u}_{\text{CII}}, \nabla \cdot \mathbf{u} \rangle + \\
 & \quad \langle (\mathbf{u}_{\text{CII}}^{\text{II}} - D_{\text{CII}} \Delta_{\text{CII}} \mathbf{u}_{\text{CII}}) \cdot \mathbf{n}_{\text{CII}}, \nabla \varphi \rangle - \langle \mathbf{u}_{\text{CII}}, \nabla \cdot (\mathbf{u}_{\text{CII}}^{\text{II}} - D_{\text{CII}} \Delta_{\text{CII}} \mathbf{u}_{\text{CII}}) \rangle
 \end{aligned}$$

For test function $\varphi \in H_1^{\text{II}}$,

CFEM II

Define $c_T = \begin{cases} c_{II}, & \mathbf{u} \cdot \mathbf{n}_T < 0 \\ c_I, & \mathbf{u} \cdot \mathbf{n}_T \geq 0 \end{cases}$ on Γ .



$$\Delta c_I \cdot \mathbf{u}_T = \Delta c_{II} \cdot \mathbf{u}_T \text{ at } \Gamma.$$

$$c_I = c_{II} \quad \text{at } \Gamma,$$

We enforce the following transmission continuity conditions between the two domains:

Transmission Conditions on Γ

$$\begin{aligned}
 D(C_{\text{II}}, C_{\text{I}}^{\downarrow}, Z, r) &= F(r) \quad \forall r \in R_h. \\
 {}_p^*(W_h) \ni \underline{\lambda} &= (\underline{\lambda}, \tilde{Z}, Z) \\
 {}_p^*(W_h) \ni \Lambda &= (\Lambda, \tilde{Z}, Z) \\
 A(C_{\text{I}}, C_{\text{II}}^{\downarrow}, Z, u) &= (u) \mathcal{J} = (u) J(u)
 \end{aligned}$$

For each $t < 0$, we seek approximating solutions $(C_{\text{I}}, C_{\text{II}}, Z, \tilde{Z}) \in W_h(\mathcal{U}_I) \times R_h(\mathcal{U}_{\text{II}}) \times W_h(\mathcal{U}_{\text{II}}) \times W_h(\mathcal{U}_{\text{II}})$ satisfying

On $\mathcal{U}_e \subset \mathcal{U}_I$, let finite dimensional approximating space $R_h \subset H_1^0(\mathcal{U}_{\text{II}})$.
 On $\mathcal{U}_e \subset \mathcal{U}_{\text{II}}$, let finite dimensional approximating space $W_h = P_k(\mathcal{U}_e)$.

$$\boxed{\begin{array}{c} D(C_{\mathrm{I}},C_{\mathrm{II}},Z,r)=\mathcal{F}(r) \quad \text{A } r \in R^h. \\ \mathcal{C}(Z,\tilde{Z},M) \ni \Lambda \wedge \emptyset = (\Lambda,Z,M) \\ B(C_{\mathrm{I}},C_{\mathrm{II}},Z,0)=(\Lambda,Z,0) \\ A(C_{\mathrm{I}},C_{\mathrm{II}},Z,w)=(w,Z,M) \end{array}}$$

$${}^{\prime }\left[{}^{\vartheta }\upsilon (\Lambda ,{}^{\prime }ZD)+{}^{\vartheta }\upsilon (\Lambda ,{}^{\prime }Z-) \right] \sum _{{}_{\mathcal{L}\ni \mathcal{J}}^{\mathcal{U}}}={}(\Lambda ,{}^{\prime }Z,\tilde{Z})\mathcal{C}$$

$$\begin{aligned} & {}^{\mathcal{U}}\langle {}^i\boldsymbol{u}\cdot[\Lambda]{}^{\mathrm{I}}\underline{\mathcal{C}}\rangle \sum _{{}_{\mathcal{J}\ni \mathcal{I}}^{\mathcal{U}}}+ \left[{}^{\vartheta }\upsilon (\Lambda \cdot \Delta ,{}^{\mathrm{I}}\mathcal{C})-{}^{\vartheta }\upsilon (\Lambda ,{}^{\prime }Z) \right] \sum _{{}_{\mathcal{L}\ni \mathcal{J}}^{\mathcal{U}}}={}(\Lambda ,{}^{\prime }Z,\tilde{Z})B(C_{\mathrm{I}},C_{\mathrm{II}},Z) \\ & {}^{\mathcal{U}}\langle {}^i\boldsymbol{u}\cdot n ,{}^{\mathrm{I}}\boldsymbol{u}\cdot \boldsymbol{n} \rangle +{}^{\vartheta }\upsilon (n ,{}^{\prime }f)\sum _{{}_{\mathcal{L}\ni \mathcal{J}}^{\mathcal{U}}}={}(\mathfrak{n})\mathcal{J} \\ & {}^o\upsilon \varrho \cup {}^{\mathrm{I}}\upsilon \varrho \langle \mathfrak{n} ,{}^{\mathrm{I}}\boldsymbol{u}\cdot \mathfrak{I}\boldsymbol{n} \rangle +{}^{\mathcal{U}}\langle [\mathfrak{n}]{}^{\mathcal{U}}\boldsymbol{u}\cdot (\underline{Z}+{}^{\mathrm{I}}\mathcal{C}\boldsymbol{n}) \rangle \sum _{{}_{\mathcal{J}\ni \mathcal{I}}^{\mathcal{U}}}+ \\ & \left[{}^{\vartheta }\upsilon (n \Delta ,{}^{\prime }Z+{}^{\mathrm{I}}\mathcal{C}\boldsymbol{n})-{}^{\vartheta }\upsilon (n C_{\mathrm{I}},\mathfrak{n}) \right] \sum _{{}_{\mathcal{L}\ni \mathcal{J}}^{\mathcal{U}}}={}(\mathfrak{n},Z,\tilde{Z})A(C_{\mathrm{I}},C_{\mathrm{II}},Z) \end{aligned}$$

$$\begin{aligned}
 D(C_{\downarrow}^{\text{II}}, C_{\downarrow}^{\text{I}}, Z, r) &= f(r) \quad \forall r \in R^h. \\
 C_p(W^h) &\ni \Delta \wedge 0 = (\Delta, Z, Z) \\
 C_p(W^h) &\ni \Lambda \wedge 0 = (\Lambda, Z^{\text{II}}, \Lambda) \\
 n \in W^h &\wedge (n)J = (n, Z^{\text{I}}, n)
 \end{aligned}$$

$$\vartheta^{\text{II}} \vartheta^{\text{II}} \langle r, u \cdot n b \rangle - \vartheta^{\text{II}}(r, f) = (r)F$$

$$\begin{aligned}
 &- n C_{\downarrow}^{\text{I}} \cdot u_{\text{I}} \cdot Z \rangle - \langle r_{\text{I}} \cdot u_{\text{I}}, r_{\text{I}} \rangle \\
 &+ (a C_{\text{II}}, r) \vartheta^{\text{II}} \vartheta^{\text{II}} \langle n C_{\text{II}} \cdot u, r_{\text{II}} \rangle + \vartheta^{\text{II}}(r, C_{\text{II}}) - \vartheta^{\text{II}}(r, \Delta C_{\text{II}}) + \vartheta^{\text{II}}(\Delta, C_{\text{II}}, r) = (r)D
 \end{aligned}$$

$$\| (C_I, C_{II}, Z) \|_2^2 = \| C_I(T) \|_2^{q_I} + \| C_{II}(T) \|_2^{q_{II}} + \| Z(T) \|_2^{q_Z}$$

$$+ 2 \int_T^0 \| \alpha_{1/2} C_I \|_2^{q_I} dt + 2 \int_T^0 \| \alpha_{1/2} C_{II} \|_2^{q_{II}} dt$$

$$+ 2 \int_T^0 \| D_{1/2} Z \|_2^{q_{II}} dt + 2 \int_T^0 \| D_{1/2} \Delta C_{II} \|_2^{q_{II}} dt$$

$$+ 2 \int_L^0 \| D_{1/2} C_I \|_2^{q_I} dt + 2 \int_L^0 \| D_{1/2} C_{II} \|_2^{q_{II}} dt$$

$$+ \int_L^0 \| u \cdot n \| \langle |u|, (C_I)_2 \rangle dt + \int_L^0 \| u \cdot n \| \langle |u|, (C_{II})_2 \rangle dt$$

$$+ \sum_{i \in I^{out}} \int_L^0 \| u_i \| \langle |u|, (C_I)_2 \rangle dt + \int_L^0 \| u_i \| \langle |u|, (C_{II})_2 \rangle dt$$

$$+ \int_L^0 \| u \cdot n \| \langle |u|, (C_I)_2 \rangle dt + \int_L^0 \| u \cdot n \| \langle |u|, (C_{II})_2 \rangle dt$$

$$+ \int_L^0 \| u \cdot n \| \langle |u|, (C_I)_2 \rangle dt + \int_L^0 \| u \cdot n \| \langle |u|, (C_{II})_2 \rangle dt$$

WHERE

$$\|f\|_T^0 + \left\langle g_2' e^{g_2 t} dt \right\rangle_{1/2}^0 \int_T^0 \|c_0\|_2^{\frac{q}{2}} + \|c_0\|_2^{\frac{q}{2}} \geq (C_I, C_{II}, Z)$$

Proposition. The scheme satisfies the stability result

This result only assumes tensor D to be positive semi-definite.

$$\| (C_I - \tilde{C}_I, C_{II} - \tilde{C}_{II}, \mathbf{z} - \tilde{\mathbf{z}}) \| \leq K h^k.$$

Theorem. For K a generic positive constant, h the maximal element diameter, and k the degree of the approximating space \mathcal{P}_k , the scheme satisfies the error estimate

Kh_k	Cockburn & Shu, 1998	$L^\infty(L^2)$ norm	On convection-diffusion equations:
Kh_k	Cockburn & Dawson, 1999	generalization	
Kh_{k+1}	Castillo, 1999	one dimensional result	
$Kh_{k+1/2}$	Dawson & Proft, 2000	penalty term, $L_2(T_2)$ norm	

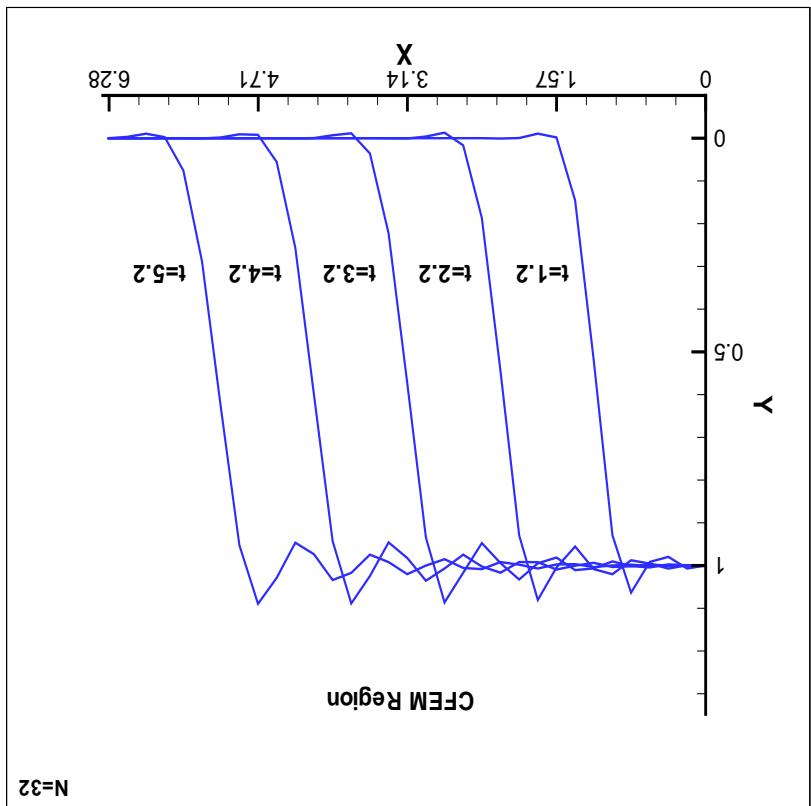
- slope-limiting for LDG solution
- TVD Runge-Kutta time discreteizatior
- P_1 basis functions in CEM region
- P_1 Legendre polynomials in LDG region
- equally spaced elements in one dimension on $[0, 2\pi]$

$$\begin{aligned} \cdot f &= (u_c - Dc^x)(0, t) \\ c &= (0, x, c) \\ \partial_t c + u \partial_x c - D \partial_{xx} c &= 0 \end{aligned}$$

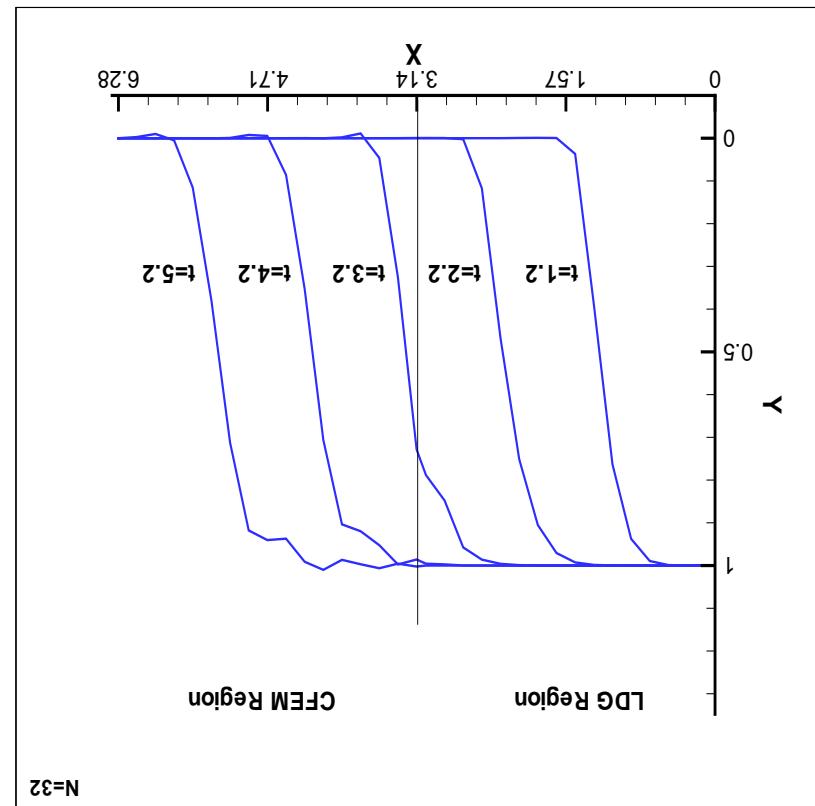
Test Problems

Numerical Results

(b) Continuous FEM N=32

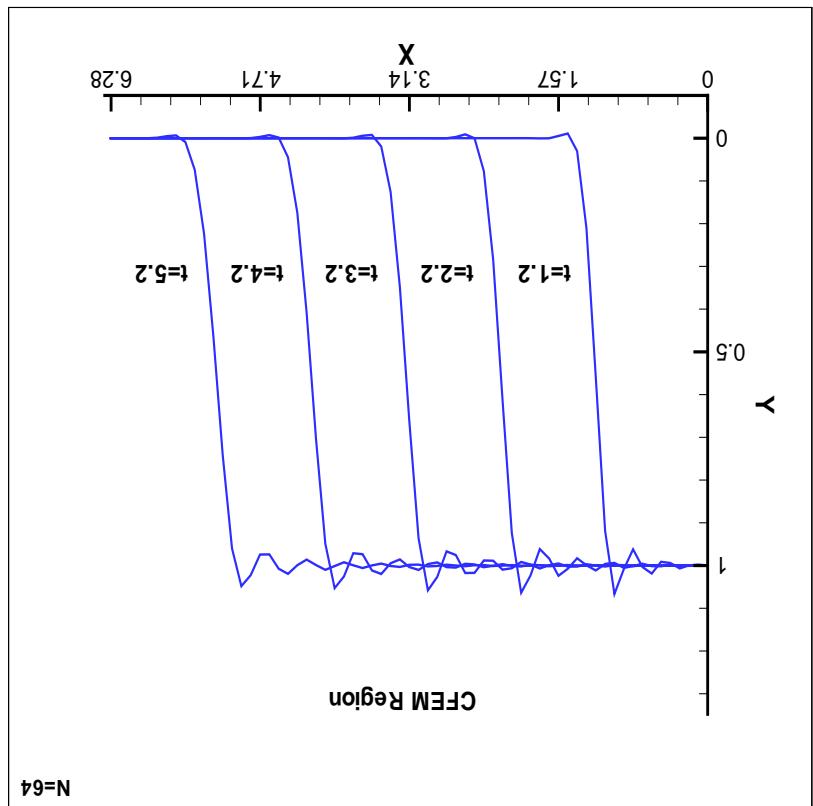


(a) Coupled FEM N=32

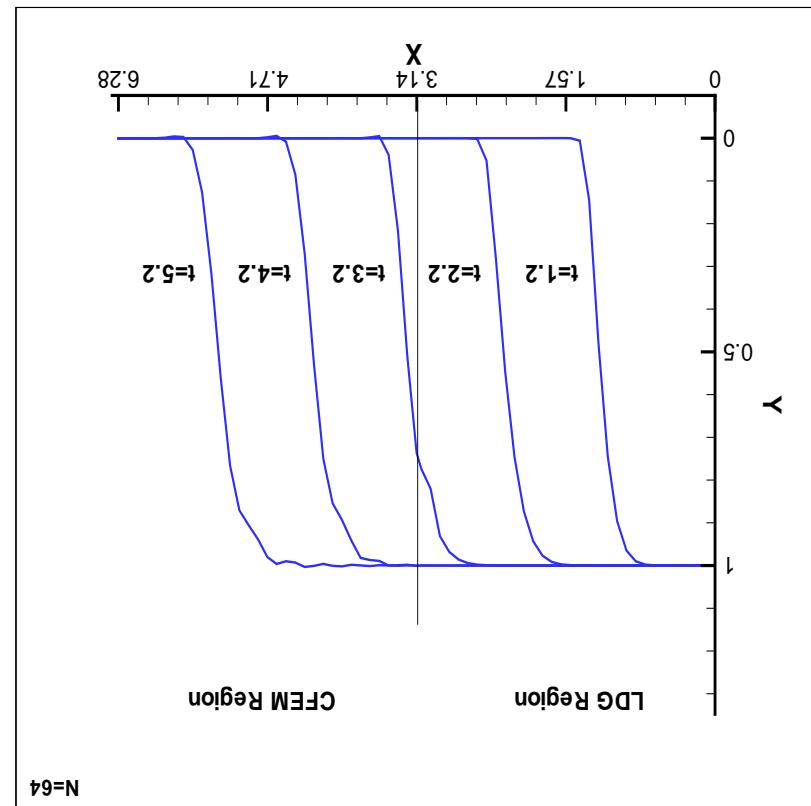


Advection Dominated: $\alpha=1.0, D=.001$

(b) Continuous FEM N=64

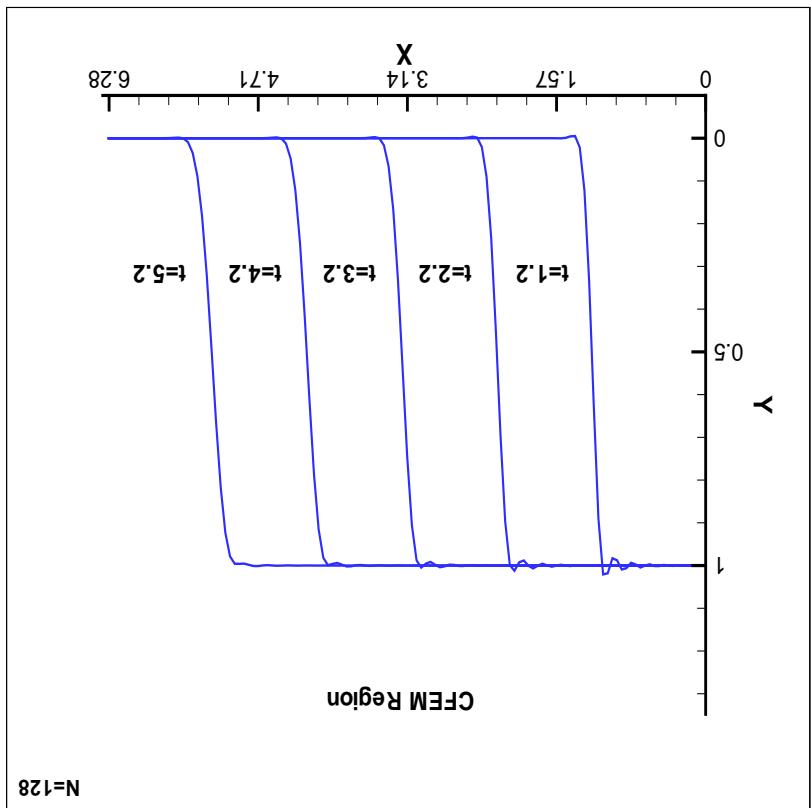


(a) Coupled FEM N=64

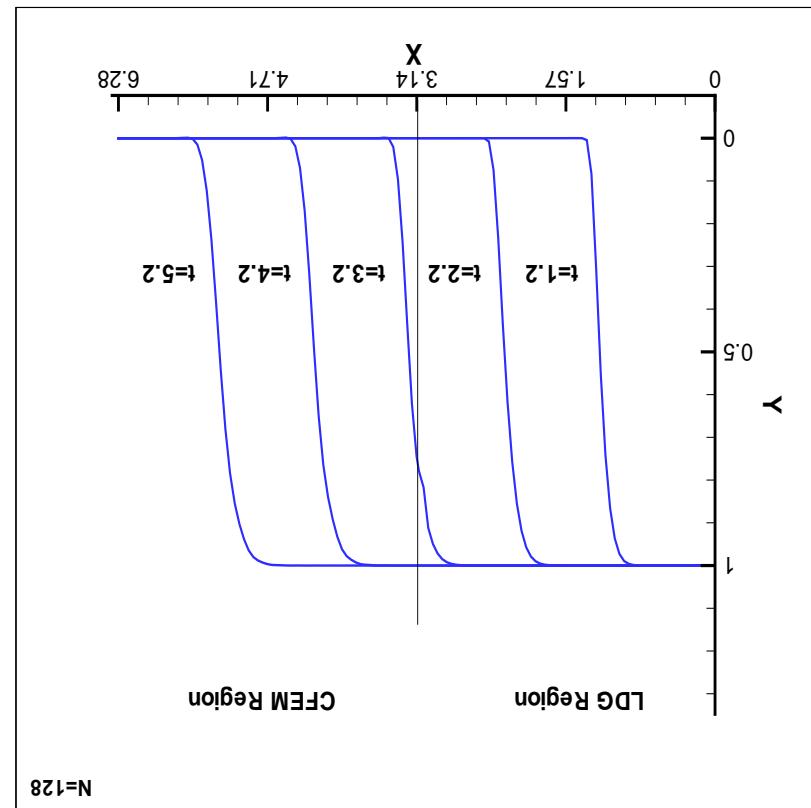


Advection Dominated: $\alpha=1.0, D=.001$

(b) Continuous FEM N=128

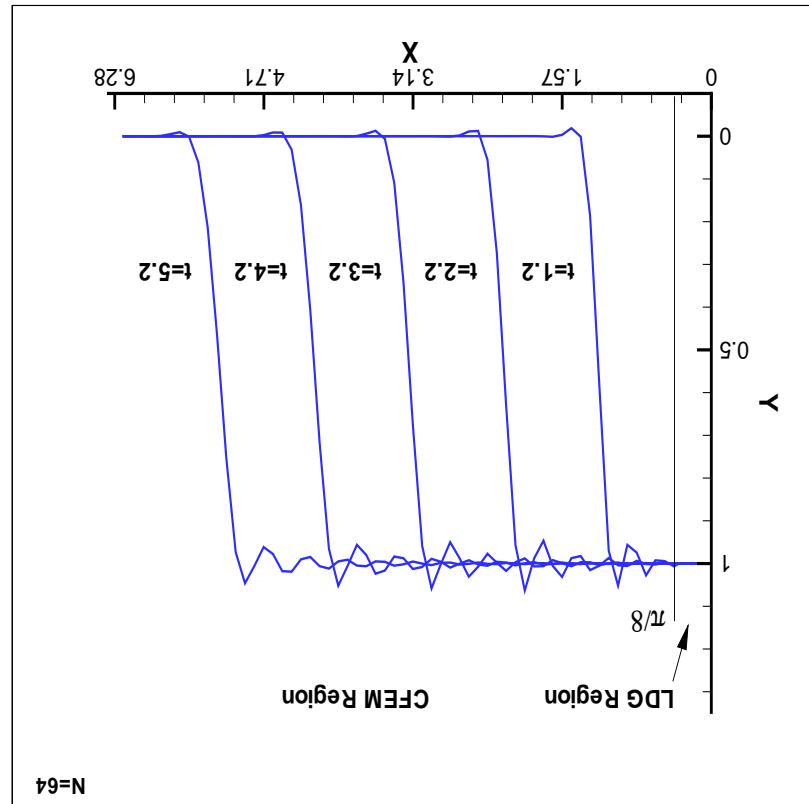


(a) Coupled FEM N=128



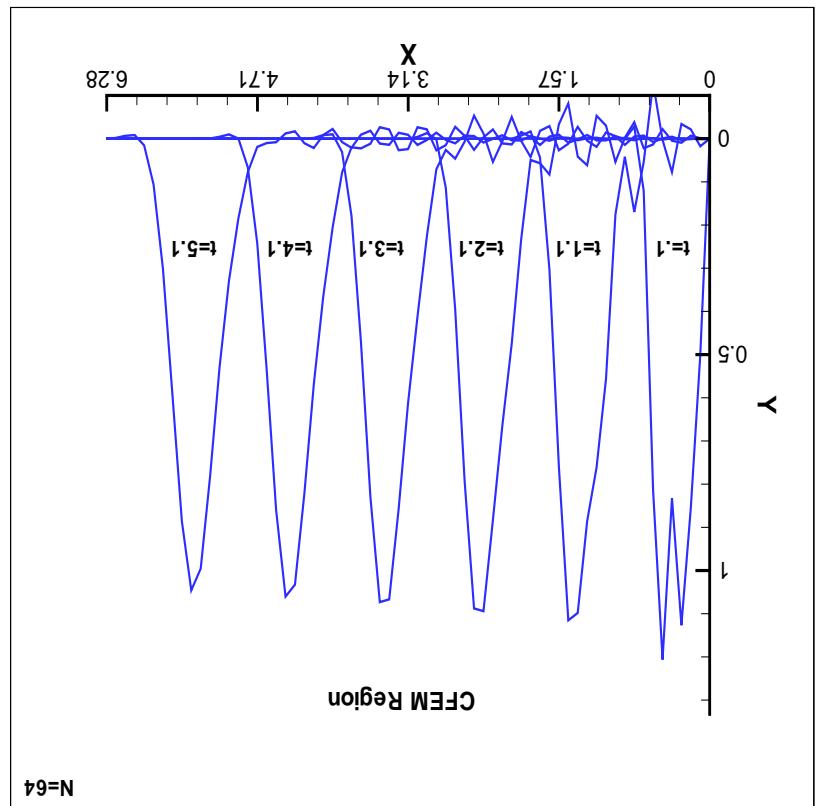
Advection Dominated: $\alpha=1.0, D=.001$

Coupled solution with interface location at $\pi/8$

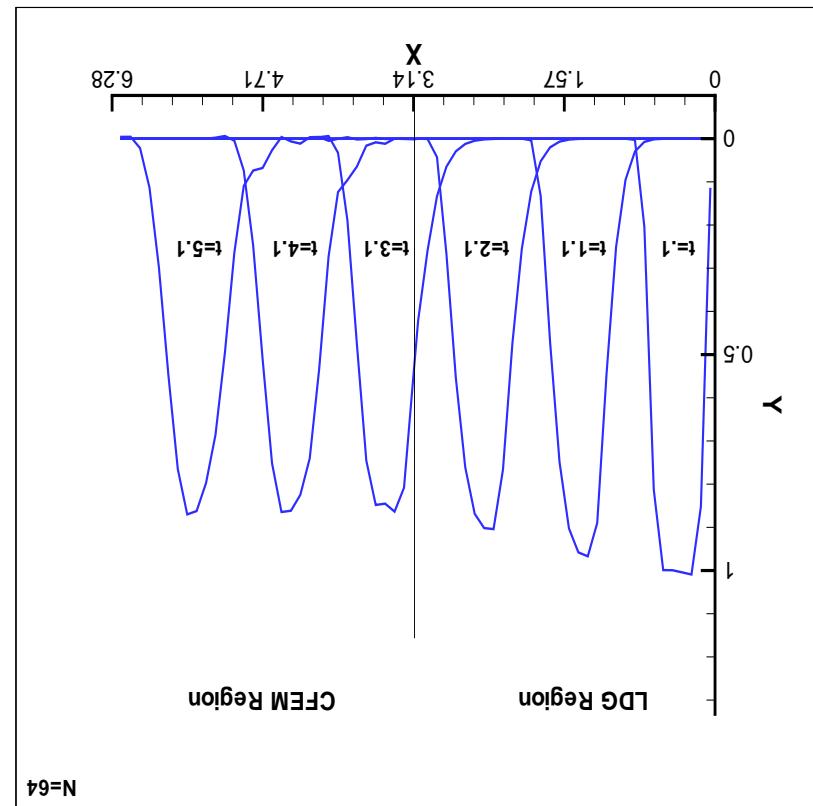


Dependency on Interface Location

(b) Continuous FEM N=64

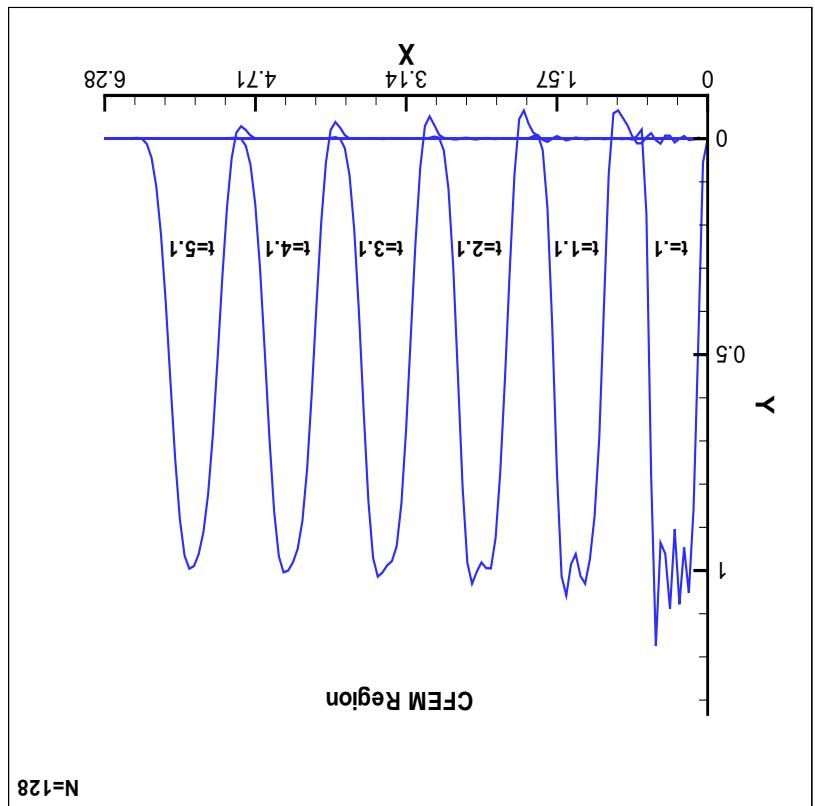


(a) Coupled FEM N=64

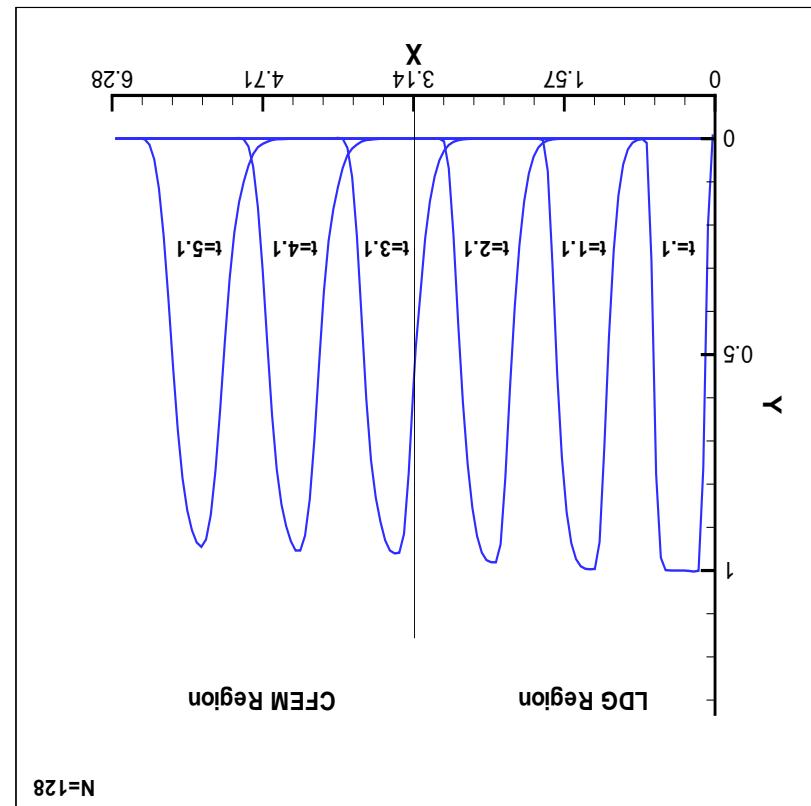


Advection Dominated: $\alpha=1.0, D=.001$

(b) Continuous FEM N=128

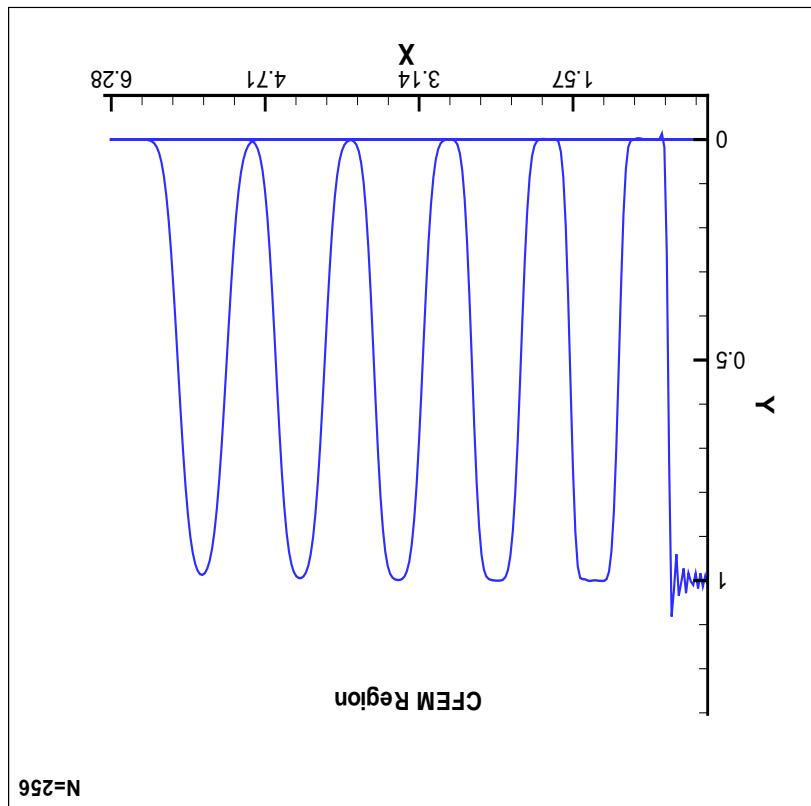


(a) Coupled FEM N=128

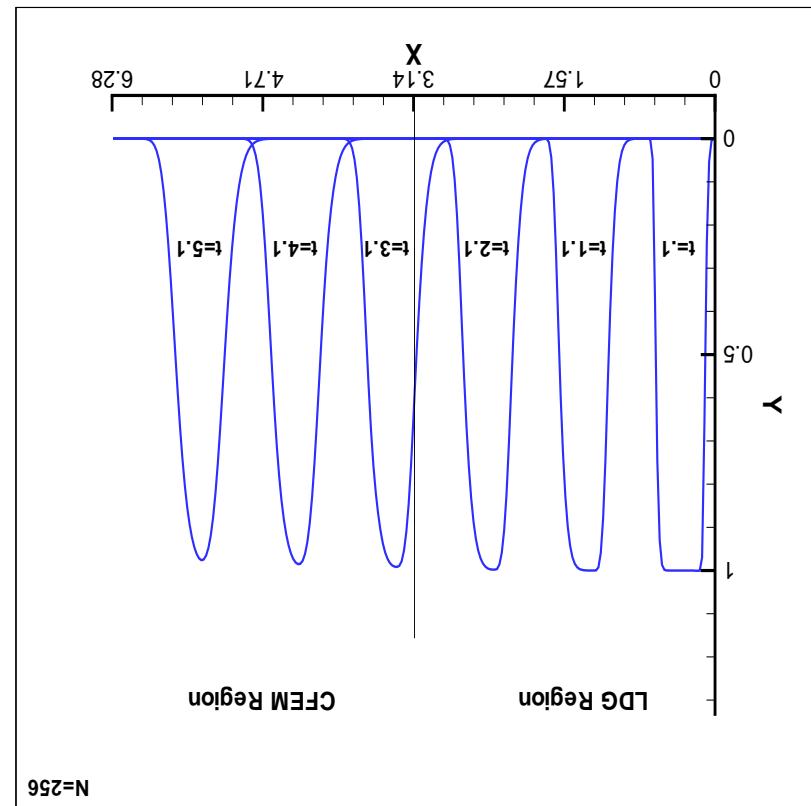


Advection Dominated: $\alpha=1.0, D=.001$

(b) Continuous FEM N=256



(a) Coupled FEM N=256



Advection Dominated: $\alpha=1.0, D=.001$

- CEM region $[\pi, 2\pi]$
- LDG region $[0, \pi]$
- L^2 error measured in $[0, 2\pi]$

$$c(x, t) = e^{-Dt} \sin(x - ut)$$

exact solution

$$\begin{aligned} c(x, 0) &= \sin(x) \\ 0 &= \partial_t c + u \partial_x c - D \partial_{xx}^2 c \end{aligned}$$

Test Problem

Convergence Results

Numerical Results

Convection equation

$D = 0.0$	N	L^2 error	$k=1$
32	.005299	2.00	
64	.001322	2.00	
128	.000328	2.01	
256	.000080	2.04	

Heat equation

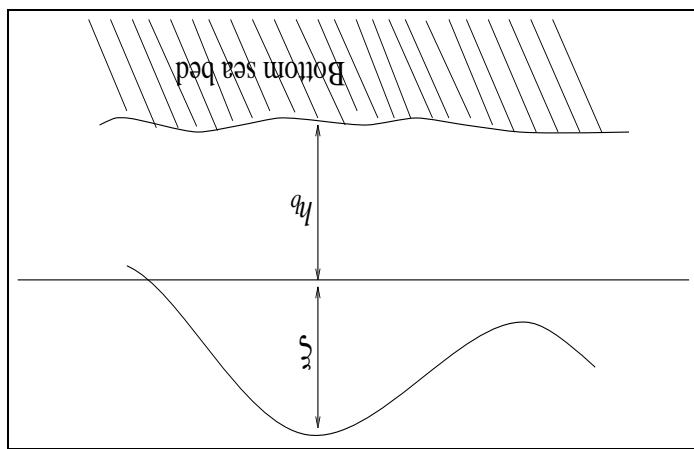
$D = 0.0$	N	L^2 error	$k=1$
32	.004227	2.00	
64	.001054	2.00	
128	.000263	2.00	
256	.000066	2.00	

Convection dominated diffusion equation

$D = 1.0$	N	L^2 error	$k=1$
32	.005076	2.02	
64	.001251	2.02	
128	.000309	2.02	
256	.000075	2.04	

$D = 1.0$	N	L^2 error	$k=1$
32	.004152	1.99	
64	.001049	1.99	
128	.000264	1.99	
256	.000066	2.00	

- Solve for variables ζ and \mathbf{u} or \mathbf{b} where $\mathbf{b} \equiv \mathbf{u}_h$
- Applicable eddy viscosity closure model
- Hydrostatic pressure and Boussinesq approximations
- Vertical wavelength is much smaller than horizontal wavelength
- Derived from depth-averaged 3d incompressible Navier-Stokes equations



ζ free surface elevation
 h_b bathymetric depth
 h = $\zeta + h_b$, total water column
 g gravity constant
 T_{bf} bottom friction
 f_c coriolis force
 \mathbf{u} depth-avg. horizontal velocities

2D Shallow Water Equations

$$\frac{\eta}{E} = (\eta \mathbf{n}) \nabla \frac{\eta}{L} - \zeta \Delta \theta + (\mathbf{n}^f \times \mathbf{k}) + \mathbf{n}^f q_L + \mathbf{n}_\Delta \cdot \mathbf{n} + \frac{\theta}{n}$$

Non-Conservative Momentum Equation

$$E = b \nabla L - \zeta \Delta \theta + (\mathbf{b} \times \mathbf{k}^f) + \mathbf{b}^f q_L + (\mathbf{b} \mathbf{b}^f \frac{\eta}{L}) \cdot \Delta + \frac{\theta}{b}$$

Conservative Momentum Equation

$$0 = \mathbf{b} \cdot \Delta + \frac{\theta}{\zeta}$$

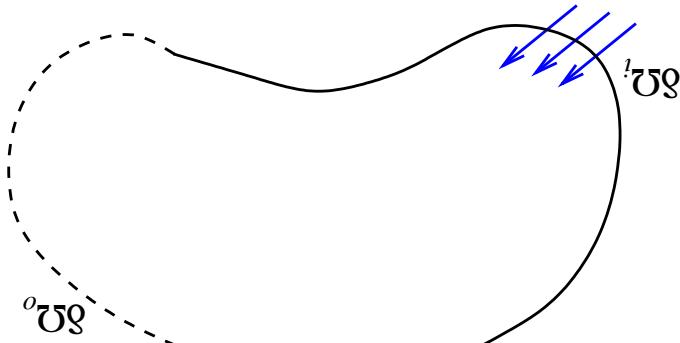
Primitive Continuity Equation

Shallow Water Equations

$$0 = \left[\mathbf{F} - \mathbf{b} \nabla^T \boldsymbol{\nu} - \zeta \Delta \boldsymbol{\psi} + \right. \\ \left. \mathbf{b} (\boldsymbol{\tau}_o \boldsymbol{\nu} - \boldsymbol{\tau}_f \boldsymbol{\psi}) + (\mathbf{b} \times \mathbf{k}) f + (\mathbf{b} \mathbf{b}^T \frac{\boldsymbol{\psi}}{\Delta}) \cdot \Delta \right] \cdot \Delta - \frac{\partial \boldsymbol{\tau}_o}{\partial \zeta} + \frac{\partial^2 \boldsymbol{\tau}_f}{\partial \zeta^2}$$

Generalized Wave Formulation

 Alternative Continuity Equation



$$\begin{aligned} \mathbf{n}^0 &= (\mathbf{x}, 0) \mathbf{n} \text{ on } \mathcal{U}. \\ \mathbf{n}_\downarrow &= \mathbf{n} \text{ on } \partial\mathcal{U}, \\ \zeta^0 &= (\mathbf{x}, 0) \zeta \text{ on } \mathcal{U}, \\ \zeta_\downarrow &= \zeta \text{ on } \partial\mathcal{U}, \end{aligned}$$

$$\begin{aligned} \{0 \leq \mathbf{u} \cdot \mathbf{n} : \mathcal{U} \ni \mathbf{x}\} &= \mathcal{U}^o \text{ outflow} \\ \{0 > \mathbf{u} \cdot \mathbf{n} : \mathcal{U} \ni \mathbf{x}\} &= \mathcal{U}^i \text{ inflow} \end{aligned}$$

- Let $\mathcal{U} \in IR^2$ have Lipschitz boundary $\partial\mathcal{U}$ decomposed into
- Assume $h_b = \text{constant}$, neglect advective terms in momentum equation

$$\boxed{\begin{aligned} 0 < t & \quad \mathcal{F} = \mathbf{n} \nabla h - \zeta \Delta b + \mathbf{n}^t \mathcal{Q} \\ 0 < t & \quad 0 = (\zeta \mathbf{n}) \cdot \Delta + \zeta^t \mathcal{Q} \end{aligned}}$$

Find ζ and \mathbf{n} such that

Simplified Model

Momentum equation: solve for momentum $\mathbf{u} \Rightarrow$ NIPG
Continuity equation: solve for elevation $\zeta \Rightarrow$ discontinuous Galerkin

Momentum equation: solve for momentum $\mathbf{u} \Rightarrow$ continuous Galerkin
Continuity equation: solve for elevation $\zeta \Rightarrow$ discontinuous Galerkin

— “Multi-algorithmic” Coupling by Equations Strategy —

$$v(\boldsymbol{m}, \boldsymbol{\zeta}) = v(\boldsymbol{m}_\Delta, \boldsymbol{n}_\Delta) + \langle \boldsymbol{u} \cdot \boldsymbol{n}, [\boldsymbol{\zeta}] b \rangle \sum_{\forall i \in \text{Int}} - {}^o v(\boldsymbol{m}, \boldsymbol{\zeta}_\Delta b) \sum_{\forall e \in L_h} + v(\boldsymbol{m}, \boldsymbol{n}_\partial)$$

Apply CEM method to momentum equation: for $\boldsymbol{u} \in H_1(\mathcal{U})$

$$\begin{aligned} \langle \boldsymbol{u} \cdot \boldsymbol{n}, [\boldsymbol{\zeta}] b \rangle &= \langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{\zeta} \rangle - \sum_{\forall i \in \text{Int}} \langle [\boldsymbol{u}], \boldsymbol{u} \cdot \boldsymbol{n} \cdot \boldsymbol{\zeta} \rangle + \sum_{\forall e \in L_h} \langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{\zeta} \rangle - {}^o v(\boldsymbol{u}_\Delta, \boldsymbol{\zeta}_\Delta b) \sum_{\forall e \in L_h} \\ &\quad + {}^o v(\boldsymbol{u}_\Delta, \boldsymbol{n}_\partial) \sum_{\forall e \in L_h} - {}^o v(\boldsymbol{u}, \boldsymbol{\zeta}_\partial) \sum_{\forall e \in L_h} \end{aligned}$$

Apply DG method to continuity equation: for $v \in H_1(\mathcal{U})$

$$\begin{aligned} \cdot^{h,0}M \ni \boldsymbol{m} \wedge v(\boldsymbol{m}, \boldsymbol{\varphi}) = v(\boldsymbol{m}\Delta, \boldsymbol{\Omega}\Delta\eta) + \\ \quad \sum_{\forall i \in I^{int}} -^e v(\boldsymbol{m}, Z\Delta\eta) \sum_{\forall e \in T_h} + v(\boldsymbol{m}, \boldsymbol{\Omega}\varphi) \end{aligned}$$

$$\begin{aligned} \cdot^h V \ni u \wedge \langle \boldsymbol{u} \cdot \boldsymbol{n} \rangle \zeta = ^e v \varphi \langle u, \boldsymbol{u} \cdot \boldsymbol{n} Z \rangle + \\ \quad \sum_{\forall i \in I^{int}} + ^e v(u\Delta, Z\Omega) \sum_{\forall e \in T_h} - ^e v(u, Z\varphi) \sum_{\forall h \in \mathcal{V}_h} \end{aligned}$$

Define $V_h \subset (H_1(U))^2$ consisting of continuous, piecewise polynomials of degree at most k . Let W_h^0 be the corresponding subspace of $(H_1^0(U))^2$. Define $W_h \subset (H_1(U))^2$ by $U \in W_h$. Approximate \boldsymbol{u} by $\boldsymbol{U} \in W_h$.

and K is a constant independent of h and k .

$$\begin{aligned} & \tau p^{\frac{1}{2}} \|(\boldsymbol{\Omega} - \mathbf{n}) \Delta_{\mathcal{L}/\Gamma} u\| \int_L^0 + \tau p^{\frac{1}{2}} \partial \langle (\boldsymbol{\Omega} - \zeta), |\mathbf{u} \cdot \boldsymbol{\Omega}| \rangle \int_L^0 + \\ & \tau p^{\frac{1}{2}} \partial \langle (\boldsymbol{\Omega} - \zeta), |\mathbf{u} \cdot \mathbf{n}| \rangle \int_L^0 + \tau p^{\frac{1}{2}} \partial \langle (\boldsymbol{\Omega} - \zeta), |\mathbf{u} \cdot \boldsymbol{\Omega}| \rangle \sum_{\forall i \in \text{Int}} \int_L^0 + \\ & 2 \|(\boldsymbol{\Omega} - \mathbf{n} - \boldsymbol{\Omega})(L)\|_2^2 + \|(\boldsymbol{\Omega} - \mathbf{n})(Z - \zeta)\|_2^2 = \|(\boldsymbol{\Omega} - \mathbf{n} - \boldsymbol{\Omega}, Z - \zeta)\|_2^2 \end{aligned}$$

where

$$|\mathbf{u} \cdot \boldsymbol{\Omega}| \leq K h_k$$

estimate

Theorem. For ζ, \mathbf{u} sufficiently smooth, the scheme satisfies the error

$k = 1$	N	Z	L^2 error	Z	L^2 error	U	L^2 error	U	L^2 error	Z	N	$k = 1$
	32	32	64	2.4	64	1.99	2.01	1.99	1.98	1.97	256	1.97
	64	64	128	2.5	128	1.98	1.99	1.99	1.97	1.97	512	1.97
	128	128	256	2.3	256	1.97	1.99	1.99	1.97	1.97	512	2.03
	256	256	512	2.1	512	2.03	2.01	2.01	2.01	2.01	512	2.3

$$\cdot \\ z(x, t) = \cos(x - t) \quad u = \sin(x + t).$$

where $g = 9.81$, $\mu = 0.01$ and the exact solution is

$$u_0 = (0, x, 0) \quad z_0 = (0, x, z)$$

$$u \cdot n = \hat{u} \cdot n \quad \text{at } x = 0, \pi/4,$$

$$z = z \quad \text{at } x = 0,$$

$$\partial_t u + g \partial_x z - \mu \partial_x u = f \quad \text{on } [0, \pi/4]$$

$$\partial_t z + z \cdot \partial_x (uz) = 0 \quad \text{on } [0, \pi/4]$$

$$\begin{aligned}
 {}^e\mathcal{V}(\boldsymbol{n}, \mathcal{L}) \sum_{\forall i \in \mathcal{T}_{int}} &= \mathcal{V}\varrho \langle \boldsymbol{n}, (\boldsymbol{n} - \boldsymbol{n}) \omega \rangle + \\
 {}^i\mathcal{U} \langle [\boldsymbol{n}], [\boldsymbol{n}] \omega \rangle \sum_{\forall i \in \mathcal{T}_{int}} &+ \mathcal{V}\varrho \langle \boldsymbol{n} - \boldsymbol{n}, \boldsymbol{u} \cdot \boldsymbol{n}_\Delta \eta \rangle + \mathcal{V}\varrho \langle \boldsymbol{n}, \boldsymbol{u} \cdot \boldsymbol{n}_\Delta \rangle + \\
 {}^i\mathcal{U} \langle [{}^i\boldsymbol{u} \cdot \boldsymbol{n}], [{}^i\boldsymbol{u} \cdot \boldsymbol{n}_\Delta] \eta \rangle \sum_{\forall i \in \mathcal{T}_{int}} &+ {}^i\mathcal{U} \langle [{}^i\boldsymbol{u} \cdot \boldsymbol{n}], [{}^i\boldsymbol{u} \cdot \boldsymbol{n}_\Delta] \eta \rangle \sum_{\forall i \in \mathcal{T}_{int}} - \\
 {}^e\mathcal{V}(\boldsymbol{n}_\Delta, \boldsymbol{n}_\Delta \eta) \sum_{\forall i \in \mathcal{T}_{int}} &+ {}^i\mathcal{U} \langle {}^i\boldsymbol{u} \cdot \underline{\boldsymbol{n}}, [\zeta] \delta \rangle \sum_{\forall i \in \mathcal{T}_{int}} - {}^e\mathcal{V}(\boldsymbol{n}, \zeta_\Delta \delta) \sum_{\forall i \in \mathcal{T}_{int}} + {}^e\mathcal{V}(\boldsymbol{n}, \boldsymbol{n} \cdot \boldsymbol{Q}) \sum_{\forall i \in \mathcal{T}_{int}} \\
 \text{Apply } \textcolor{red}{NIPG} \text{ method to momentum equation: for } u \in H_1(\mathcal{U}) \quad
 \end{aligned}$$

$$\begin{aligned}
 {}^i\mathcal{U} \langle u, \boldsymbol{u} \cdot \boldsymbol{n} \zeta \rangle - = {}^o\mathcal{V}\varrho \langle u, \boldsymbol{u} \cdot \boldsymbol{n} \zeta \rangle + \\
 {}^i\mathcal{U} \langle \boldsymbol{n} \zeta, \Delta u \rangle + \sum_{\forall i \in \mathcal{T}_{int}} \langle \zeta \boldsymbol{n} \cdot \boldsymbol{u}, [u] \rangle - \sum_{\forall i \in \mathcal{T}_{int}} (Q \zeta, u) \mathcal{U} - \sum_{\forall i \in \mathcal{T}_{int}} (\boldsymbol{n} \zeta, \Delta u) \mathcal{U} \\
 \text{Apply } \textcolor{green}{DG} \text{ method to continuity equation: for } u \in H_1(\mathcal{U}) \quad
 \end{aligned}$$

$$\begin{aligned}
 {}^o\mathcal{U}(\boldsymbol{m}, \boldsymbol{\Omega}) \sum_{\substack{h \\ \forall i \in \mathbb{I}^{int}}} &= {}^o\mathcal{U}\langle \boldsymbol{m}, (\boldsymbol{n} - \boldsymbol{\Omega})\boldsymbol{o} \rangle + \\
 &\quad {}^i\mathcal{U}\langle [\boldsymbol{m}]^i [\boldsymbol{\Omega}]^i \boldsymbol{o} \rangle \sum_{\substack{h \\ \forall i \in \mathbb{I}^{int}}} + {}^o\mathcal{U}\langle \boldsymbol{n} - \boldsymbol{\Omega}, \boldsymbol{u} \cdot \boldsymbol{m}_\Delta \boldsymbol{o} \rangle + {}^o\mathcal{U}\langle \boldsymbol{m}, \boldsymbol{u} \cdot \boldsymbol{\Omega}_\Delta \rangle + \\
 &\quad {}^i\mathcal{U}\langle [\boldsymbol{u} \cdot \boldsymbol{\Omega}]^i \boldsymbol{u} \cdot \underline{\boldsymbol{m}_\Delta \boldsymbol{o}} \rangle \sum_{\substack{h \\ \forall i \in \mathbb{I}^{int}}} + {}^i\mathcal{U}\langle [\boldsymbol{u} \cdot \boldsymbol{m}]^i \boldsymbol{u} \cdot \underline{\boldsymbol{\Omega}_\Delta \boldsymbol{o}} \rangle \sum_{\substack{h \\ \forall i \in \mathbb{I}^{int}}} - \\
 {}^o\mathcal{U}(\boldsymbol{m}_\Delta, \boldsymbol{\Omega}_\Delta \boldsymbol{o}) \sum_{\substack{h \\ \forall i \in \mathbb{I}^{ext}}} &+ {}^i\mathcal{U}\langle \boldsymbol{u} \cdot \underline{\boldsymbol{m}}^i [Z] \boldsymbol{o} \rangle \sum_{\substack{h \\ \forall i \in \mathbb{I}^{ext}}} - {}^o\mathcal{U}(\boldsymbol{m}, Z \Delta \boldsymbol{o}) \sum_{\substack{h \\ \forall i \in \mathbb{I}^{ext}}} + {}^o\mathcal{U}(\boldsymbol{m}, \boldsymbol{\Omega} \boldsymbol{o}) \sum_{\substack{h \\ \forall i \in \mathbb{I}^{ext}}}
 \end{aligned}$$

$$\begin{aligned}
 {}^i\mathcal{U}\langle \boldsymbol{u}, \boldsymbol{n}_Z \rangle - = {}^o\mathcal{U}\langle \boldsymbol{u}, \boldsymbol{\Omega}_Z \rangle + \\
 &\quad {}^i\mathcal{U}\langle [\boldsymbol{u}]^i \boldsymbol{u} \cdot \underline{\boldsymbol{\Omega}}_Z \rangle \sum_{\substack{h \\ \forall i \in \mathbb{I}^{int}}} + {}^o\mathcal{U}(\boldsymbol{u}_\Delta, Z \boldsymbol{\Omega}) \sum_{\substack{h \\ \forall i \in \mathbb{I}^{ext}}} - {}^o\mathcal{U}(\boldsymbol{u}, Z \boldsymbol{\Omega}) \sum_{\substack{h \\ \forall i \in \mathbb{I}^{ext}}}
 \end{aligned}$$

On each \mathcal{Q}_e , define $V_h = \{v : v|_{\mathcal{Q}_e} \in P_{k_e}(\mathcal{Q}_e)\}$ and $W_h = \{\boldsymbol{m} : \boldsymbol{m}|_{\mathcal{Q}_e} \in P_{k_e}(\mathcal{Q}_e)\}$ where k_e and k_u may be different orders of approximation. Approximate $\boldsymbol{Z}(\cdot, t) \in V_h$, and $\boldsymbol{n}(\cdot, t)$ by $\boldsymbol{Z}(\cdot, t) \in V_h$, and $\boldsymbol{n}(\cdot, t)$ by

$$\begin{aligned}
& + \int_L^0 \| \omega_{1/2}(\boldsymbol{\Omega} - \boldsymbol{n}) \|^2 dt, \\
& t p^{\nu_2} \| [\boldsymbol{\Omega} - \boldsymbol{n}]_{\nu_2} \omega \| \sum_{\forall i \in \text{Int}} \int_L^0 + t p^{\nu_2} \langle Z - \zeta, |\boldsymbol{u} \cdot \boldsymbol{\Omega}| \rangle \int_L^0 + \\
& t p^{\nu_2} \langle [Z - \zeta], |\boldsymbol{u} \cdot \boldsymbol{\Omega}| \rangle \sum_{\forall i \in \text{Int}} \int_L^0 + t p^{\nu_2} \| (\boldsymbol{\Omega} - \boldsymbol{n}) \Delta_{\nu_2} u \| \int_L^0 + \\
& 2 \| (\boldsymbol{\Omega} - \boldsymbol{n}) \|_2 + \| (L)(Z - \zeta) \| = \| (\boldsymbol{\Omega} - \boldsymbol{n}, Z - \zeta) \|_2
\end{aligned}$$

where

$$\left\{ t p \left[\left(\omega_{1+e} H_{k_n^e} \right) \left(\omega_{1+e} H_{k_n^e} \right)^T \right] \sum_{\forall L \in \partial \Omega} \int_L^0 \right\} \leq K^2 \| (\boldsymbol{\Omega} - \boldsymbol{n}, Z - \zeta) \|$$

Theorem. For \boldsymbol{n}, ζ sufficiently smooth and penalty parameter $\omega = O(h_{-1}^e)$, the scheme satisfies the error estimate

$k = 1$	N	Z	U	L^2 error	M_{mesh}
	32	32	32	1.00	Nonuniform Mesh
	64	64	64	1.00	Uniform Mesh
	128	128	128	1.00	
	256	256	256	1.00	
	512	512	512	1.00	
	1.00	1.00	1.00	1.00	
	1.00	1.00	1.00	1.00	
	1.00	1.00	1.00	1.00	
	1.92	1.92	1.92	1.92	
	1.98	1.98	1.98	1.98	
	2.00	2.00	2.00	2.00	
	64	64	64	64	
	128	128	128	128	
	256	256	256	256	
	512	512	512	512	
	1.00	1.00	1.00	1.00	
	1.00	1.00	1.00	1.00	
	1.00	1.00	1.00	1.00	

$$\cdot \cdot .(t + x, t) = \cos(x - t) \quad u = \sin(x + t)$$

where $g = 9.81$, $\mu = 0.01$ and the exact solution is

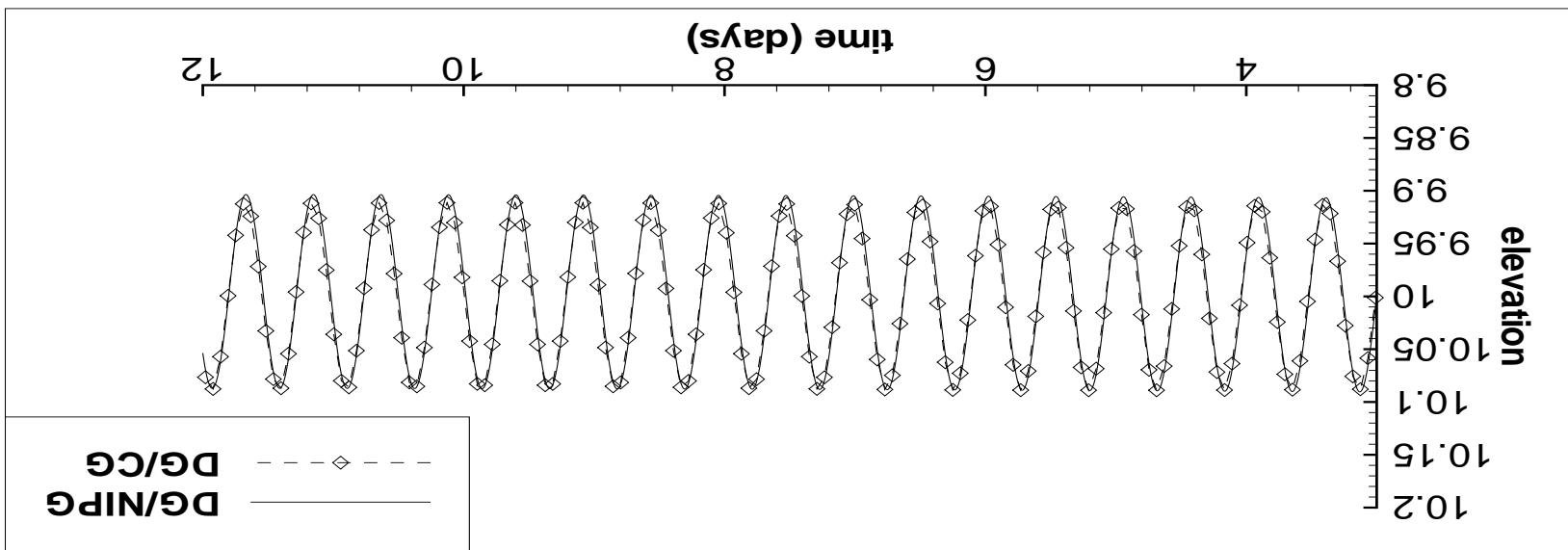
$$u_0 = (0, x, 0) \quad z_0 = (0, x, 0)$$

$$u \cdot n = \hat{u} \cdot n \quad \text{at } x = 0, \pi/4,$$

$$z \cdot \hat{z} = z \quad \text{at } x = 0,$$

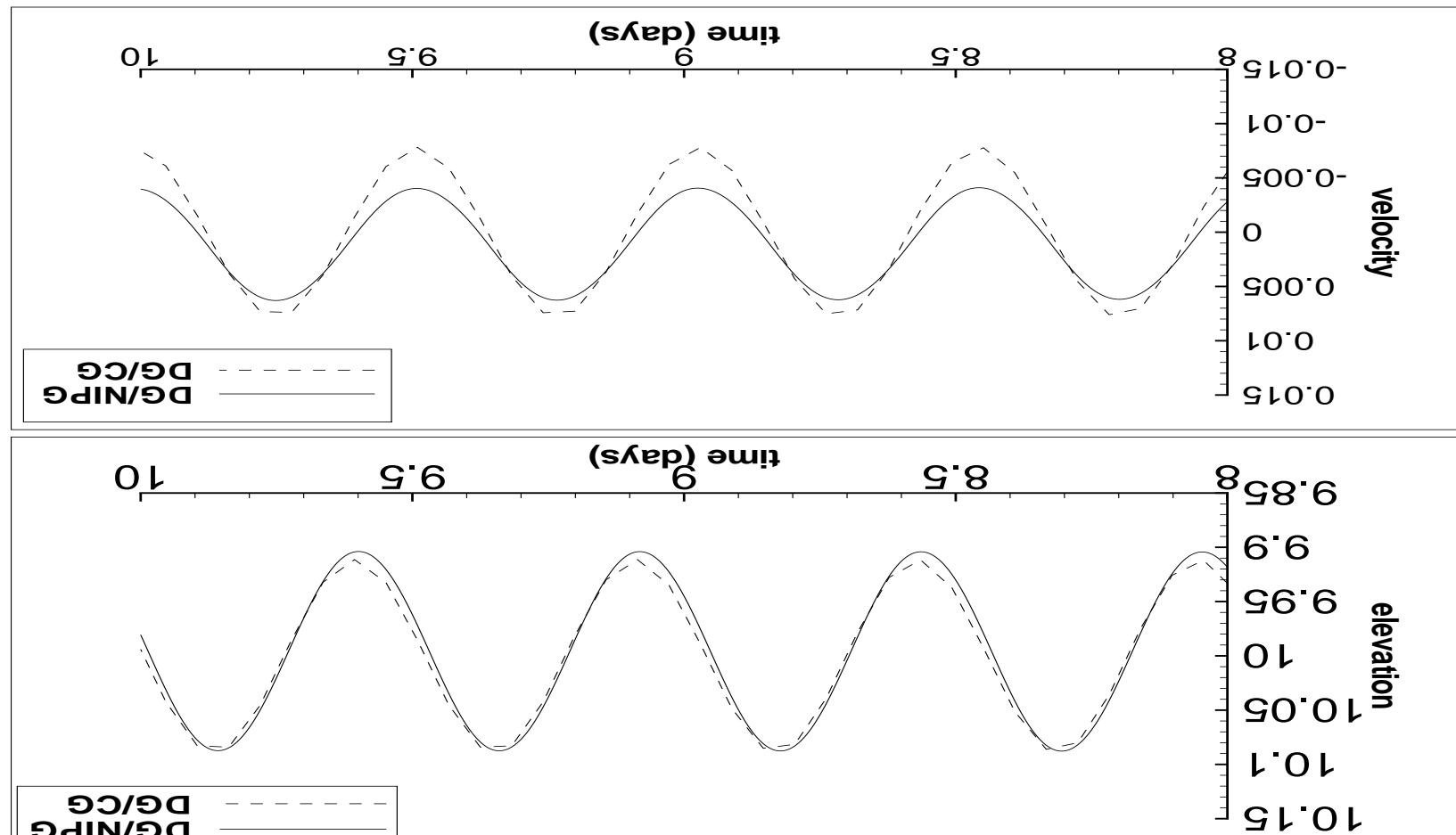
$$f = u^{xx} \varrho - z^x \varrho^x + g^x \varrho + u^t \varrho^t \quad \text{on } [0, \pi/4]$$

$$(z u)^x + z^x \varrho^x = 0 \quad \text{on } [0, \pi/4]$$

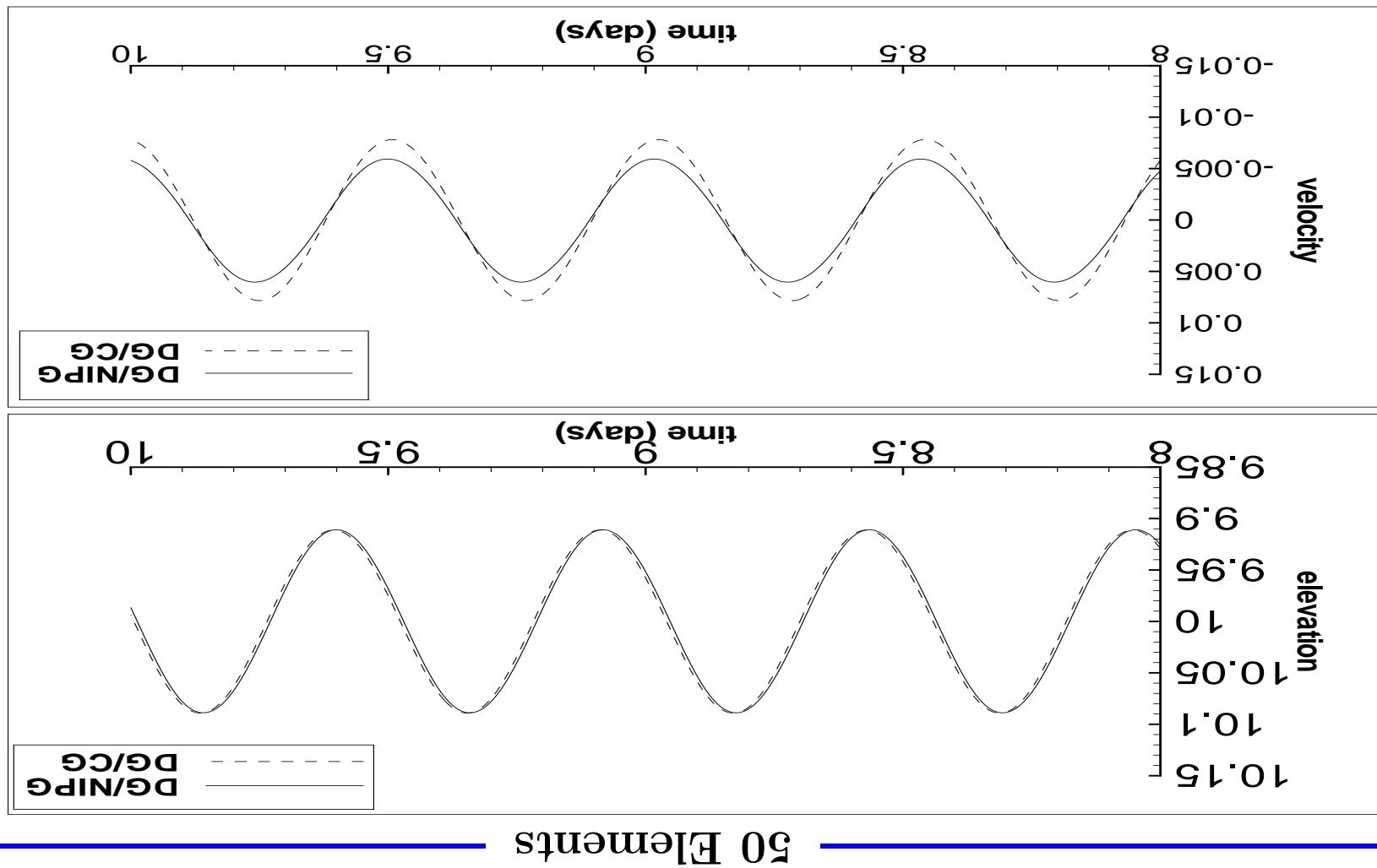


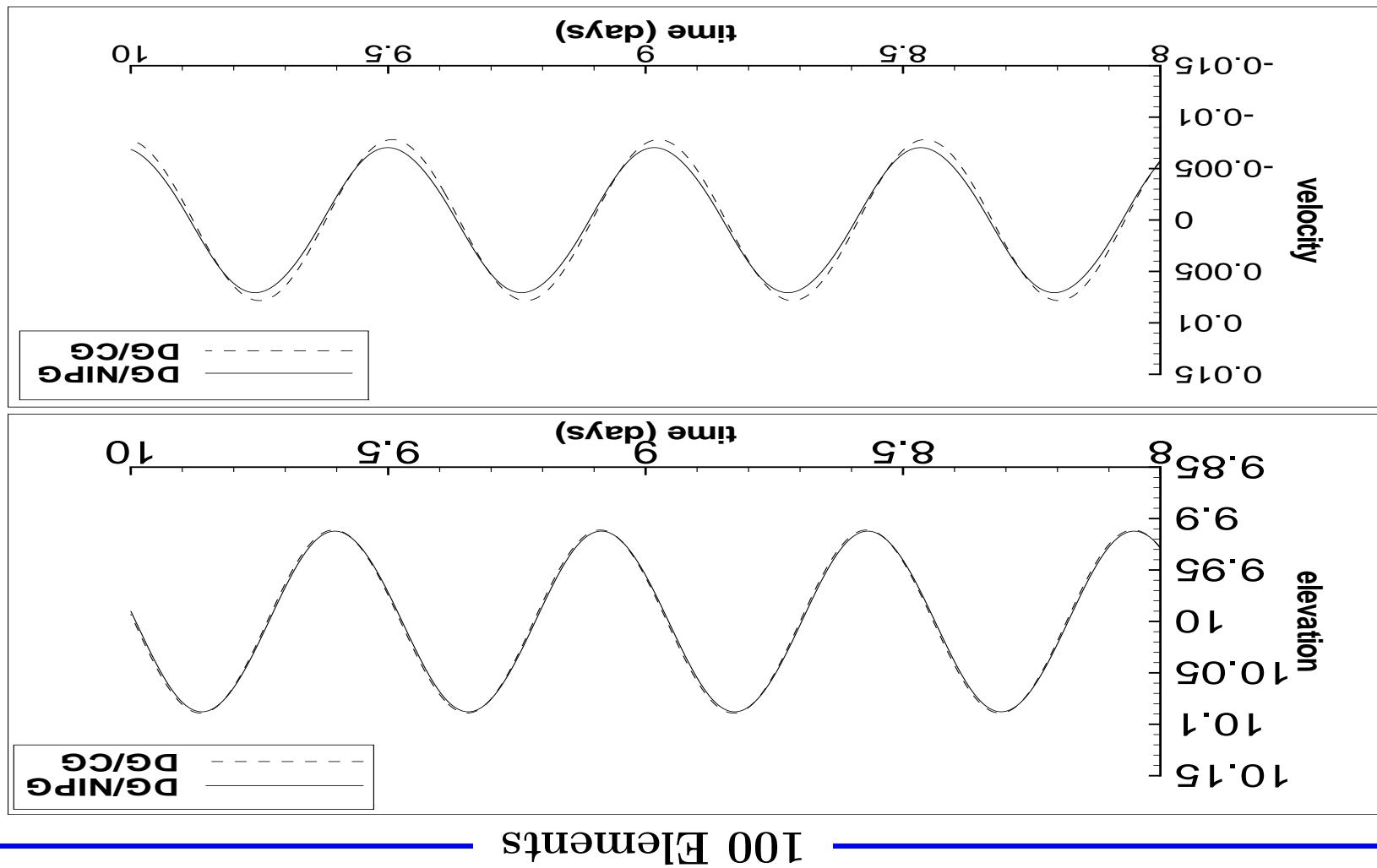
- graph solution at point $x=3600$
- mixed boundary conditions for U
- left boundary condition for $Z = 1 * \cos(0.0001459 * t)$
- computational domain $[0, 10000]$
- $g=9.81$, $\mu=25$

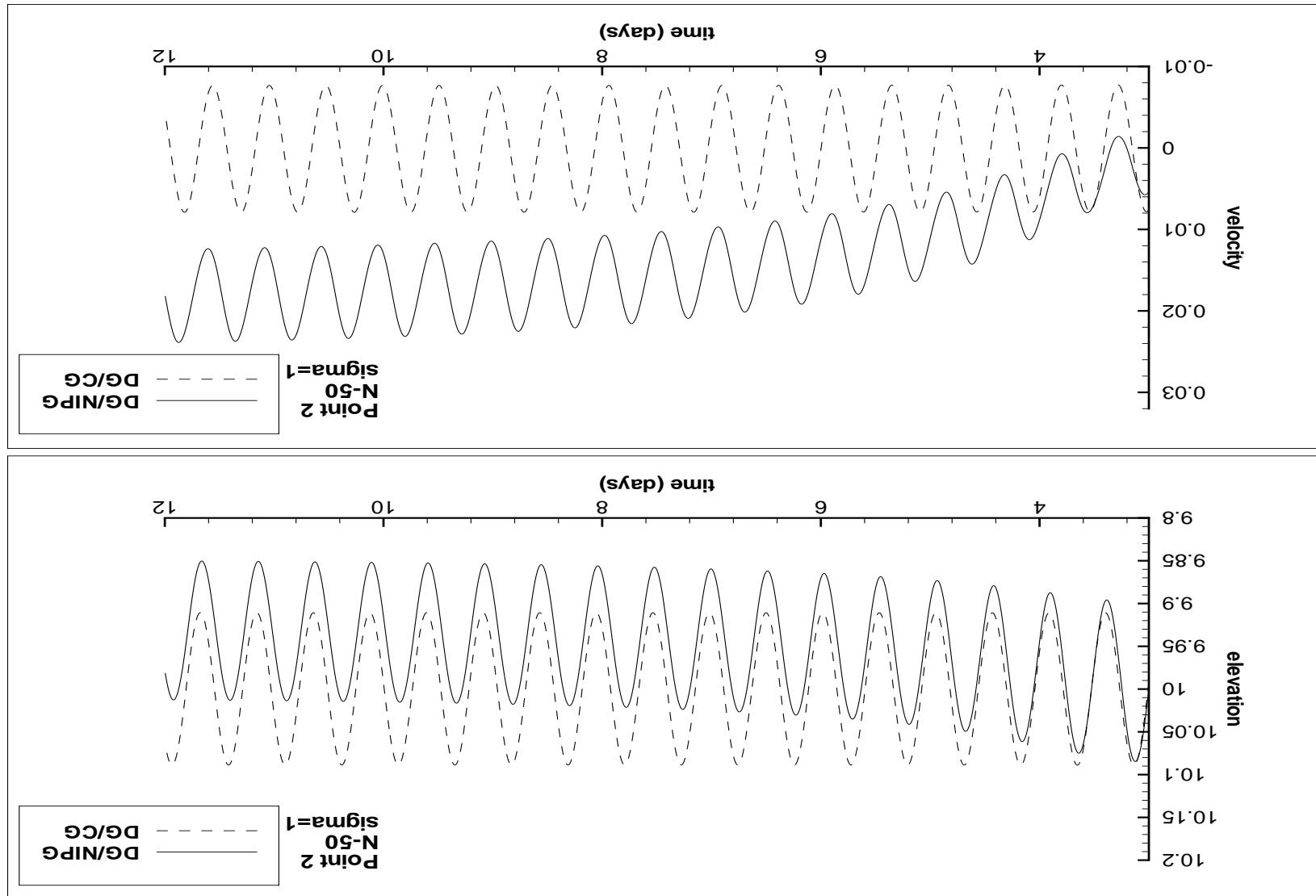
Numerical Results



25 Elements







Penalty Parameter Sensitivity

$$\mathcal{F} = \mathbf{n} \nabla u - \zeta \Delta \boldsymbol{\gamma} + \mathbf{n} \cdot \boldsymbol{\gamma} + \frac{\boldsymbol{\theta}}{\mathbf{n} \cdot \boldsymbol{\theta}}$$

$$0 = \mathbf{b} \cdot \Delta + \frac{\boldsymbol{\theta}}{\zeta \boldsymbol{\theta}}$$

Momentum equation: solve for momentum $\mathbf{u} \Rightarrow$ continuous Galerkin

Continuity equation: solve for elevation $\zeta \Rightarrow$ discontinuous Galerkin

In \mathcal{U}^{II} :

Momentum equation: solve for momentum $\mathbf{u} \Rightarrow$ NIPG

Continuity equation: solve for elevation $\zeta \Rightarrow$ discontinuous Galerkin

In \mathcal{U}^{I} :

Combined Strategies

$$\begin{aligned}
 & {}^{\text{e}}\nabla(\boldsymbol{m}, \mathbf{I}, \underline{\mathbf{f}}) \sum = {}^{\text{I}}\nabla \langle \mathbf{I}\underline{\mathbf{n}} - \mathbf{I}\mathbf{n}, \mathbf{u} \cdot \boldsymbol{m}_{\Delta} \eta \rangle + {}^{\text{I}}\nabla \langle \boldsymbol{m}, \mathbf{u} \cdot \mathbf{I}\mathbf{n}_{\Delta} \eta \rangle - \\
 & \quad \mathbf{I} \langle (\mathbf{I}\mathbf{n} - \mathbf{n}), \mathbf{I}\mathbf{u} \cdot \boldsymbol{m}_{\Delta} \eta \rangle \frac{\zeta}{\mathbf{I}} + \mathbf{I} \langle \boldsymbol{m}, \mathbf{I}\mathbf{u} \cdot \mathbf{I}\mathbf{n}_{\Delta} \eta \rangle - {}^{\text{u}} \langle [\mathbf{n}], \mathbf{u} \cdot \underline{\boldsymbol{m}_{\Delta} \eta} \rangle \sum + \\
 & \quad {}^{\text{u}} \langle [\boldsymbol{m}], \mathbf{u} \cdot \underline{\mathbf{I}\mathbf{n}_{\Delta} \eta} \rangle \sum - {}^{\text{e}}\nabla(\boldsymbol{m}_{\Delta}, \mathbf{I}\mathbf{n}_{\Delta}) \eta \sum + \mathbf{I} \langle \mathbf{I}\mathbf{u} \cdot \boldsymbol{m}, \mathbf{I}\zeta \delta \rangle - \\
 & \quad {}^{\text{u}} \langle \mathbf{u} \cdot \underline{\boldsymbol{m}}, [\mathbf{I}\zeta] \delta \rangle \sum - {}^{\text{e}}\nabla(\boldsymbol{m}, \mathbf{I}\zeta_{\Delta} \delta) \sum + {}^{\text{e}}\nabla(\boldsymbol{m}, \mathbf{I}\mathbf{n}) \mathcal{f} \eta \sum + {}^{\text{e}}\nabla \left(\boldsymbol{m}, \frac{\mathcal{f} \eta}{\mathbf{I}\zeta \rho} \right) \sum \\
 & \text{Apply NIPG method to velocity equation: for } \mathbf{u} \in H_1(\mathcal{U})^2
 \end{aligned}$$

$$\begin{aligned}
 & {}^{\text{e}}\nabla \cup {}^{\text{I}}\nabla \langle \alpha, \mathbf{u} \cdot \mathbf{I}\underline{\mathbf{n}} \rangle - = \mathbf{I} \langle \alpha, \mathbf{u} \cdot \mathbf{I}\underline{\mathbf{n}} \rangle + {}^{\text{o}}\nabla \cup {}^{\text{I}}\nabla \langle \alpha, \mathbf{u} \cdot \mathbf{I}\underline{\mathbf{n}} \rangle + \\
 & \quad {}^{\text{u}} \langle [\alpha], \mathbf{u} \cdot \mathbf{I}\underline{\mathbf{n}} \rangle \sum + {}^{\text{e}}\nabla(\alpha_{\Delta}, \mathbf{I}\underline{\mathbf{n}}) \sum - {}^{\text{e}}\nabla \left(\alpha, \frac{\mathcal{f} \eta}{\mathbf{I}\zeta \rho} \right) \sum \\
 & \text{Apply DG method to continuity equation: for } \alpha \in H_1(\mathcal{U})
 \end{aligned}$$

$$\begin{aligned} \cdot^{\text{II}}\mathcal{U}(\boldsymbol{\omega},^{\text{II}}\boldsymbol{\mathcal{L}}) = & \textcolor{blue}{\langle (\text{II}\boldsymbol{n} - \text{I}\boldsymbol{n}), \text{I}\boldsymbol{u} \cdot \boldsymbol{\omega}_{\Delta} \eta \rangle} \frac{\mathcal{Z}}{\text{I}} + \textcolor{blue}{\langle \boldsymbol{\omega}, \text{I}\boldsymbol{u} \cdot \text{II}\boldsymbol{n}_{\Delta} \eta \rangle} + ^{\text{II}}\mathcal{U}(\boldsymbol{\omega}_{\Delta},^{\text{II}}\boldsymbol{n}_{\Delta} \eta) + \\ & \textcolor{blue}{\langle \text{I}\boldsymbol{u} \cdot \boldsymbol{\omega}, ^{\text{II}}\boldsymbol{\zeta} b \rangle} + \textcolor{blue}{\langle \boldsymbol{u} \cdot \boldsymbol{\omega}, [\text{II}\boldsymbol{\zeta}] b \rangle} \sum_{\forall i \in \text{I}^{\text{int}}} -^{\text{II}}\mathcal{U}(\boldsymbol{\omega}, ^{\text{II}}\boldsymbol{\zeta}_{\Delta} b) + ^{\text{II}}\mathcal{U}(\boldsymbol{\omega}, ^{\text{II}}\boldsymbol{n}) f_L + ^{\text{II}}\mathcal{U}\left(\boldsymbol{\omega}, \frac{\partial}{^{\text{II}}\boldsymbol{n}_{\partial}}\right) \end{aligned}$$

Apply **CEM** method to velocity equation: for $\boldsymbol{u} \in H_1(\mathcal{U})^2$

$$\begin{aligned} \textcolor{brown}{\langle \boldsymbol{\alpha} \cdot \boldsymbol{u}, \text{II}\boldsymbol{u} \cdot \text{II}\boldsymbol{\zeta}^{\text{II}}\boldsymbol{n} \rangle} - = & \textcolor{blue}{\langle \boldsymbol{\alpha}, \text{I}\boldsymbol{u} \cdot \text{II}\boldsymbol{\zeta}^{\text{II}}\boldsymbol{n} \rangle} - ^{\text{o}}\mathcal{U}(\boldsymbol{\alpha}, \boldsymbol{u} \cdot \text{II}\boldsymbol{\zeta}^{\text{II}}\boldsymbol{n}) + \\ & \textcolor{blue}{\langle [\boldsymbol{\alpha}], \boldsymbol{u} \cdot \text{II}\boldsymbol{\zeta}^{\text{II}}\boldsymbol{n} \rangle} \sum_{\forall i \in \text{I}^{\text{int}}} + ^{\text{o}}\mathcal{U}(\boldsymbol{\alpha}_{\Delta}, ^{\text{II}}\boldsymbol{\zeta}^{\text{II}}\boldsymbol{n}) \sum_{\forall e \in \mathcal{L}} - ^{\text{o}}\mathcal{U}\left(\boldsymbol{\alpha}, \frac{\partial}{^{\text{II}}\boldsymbol{\zeta}^{\text{II}}\partial}\right) \sum_{\forall L} \end{aligned}$$

Apply **DG** method to continuity equation: for $v \in H_1(\mathcal{U})$

Formulation in II

$$\begin{aligned}
{}^{\text{e}}\mathcal{V}(\boldsymbol{m}, \mathbf{I}\underline{\boldsymbol{\varphi}}) \sum_{\mathcal{E}^{\text{int}}} = & \textcolor{blue}{\langle \boldsymbol{m}, [\boldsymbol{\Omega}] \boldsymbol{\varphi} \rangle} + {}^{\text{I}}\mathcal{V}\ell \langle \boldsymbol{m}, (\mathbf{In} - \mathbf{I}\boldsymbol{\Omega}) \boldsymbol{\varphi} \rangle + \\
& {}^{\text{I}}\mathcal{U} \langle [\boldsymbol{m}], [\mathbf{I}\boldsymbol{\Omega}] \boldsymbol{\varphi} \rangle \sum_{\mathcal{E}^{\text{int}}} + \textcolor{blue}{\langle [\boldsymbol{\Omega}], \mathbf{I}\boldsymbol{u} \cdot \boldsymbol{m}_{\Delta} \eta \rangle} \frac{2}{1} + \textcolor{blue}{\langle \boldsymbol{m}, \mathbf{I}\boldsymbol{u} \cdot \underline{\boldsymbol{\Omega}_{\Delta} \eta} \rangle} - \\
& {}^{\text{I}}\mathcal{V}\ell \langle \mathbf{In} - \mathbf{I}\boldsymbol{\Omega}, \boldsymbol{u} \cdot \boldsymbol{m}_{\Delta} \eta \rangle + {}^{\text{I}}\mathcal{V}\ell \langle \boldsymbol{m}, \boldsymbol{u} \cdot \mathbf{I}\boldsymbol{\Omega}_{\Delta} \eta \rangle - {}^{\text{I}}\mathcal{U} \langle [\mathbf{I}\boldsymbol{\Omega}], {}^{\text{I}}\boldsymbol{u} \cdot \underline{\boldsymbol{m}_{\Delta} \eta} \rangle \sum_{\mathcal{E}^{\text{int}}} + \\
& {}^{\text{I}}\mathcal{U} \langle [\boldsymbol{m}], {}^{\text{I}}\boldsymbol{u} \cdot \underline{\boldsymbol{\Omega}_{\Delta} \eta} \rangle \sum_{\mathcal{E}^{\text{int}}} - {}^{\text{e}}\mathcal{V}(\boldsymbol{m}_{\Delta}, \mathbf{I}\boldsymbol{\Omega}_{\Delta}) \eta \sum_{\mathcal{E}^{\text{int}}} + \textcolor{blue}{\langle \mathbf{I}\boldsymbol{u} \cdot \boldsymbol{m}, {}^{\text{I}}Z \delta \rangle} - \\
& {}^{\text{I}}\mathcal{U} \langle {}^{\text{I}}\boldsymbol{u} \cdot \underline{\boldsymbol{m}}, [{}^{\text{I}}Z] \delta \rangle \sum_{\mathcal{E}^{\text{int}}} - {}^{\text{e}}\mathcal{V}(\boldsymbol{m}, {}^{\text{I}}Z \Delta \delta) \sum_{\mathcal{E}^{\text{int}}} + {}^{\text{e}}\mathcal{V}(\boldsymbol{m}, \mathbf{I}\boldsymbol{\Omega}) f_{q_L} \sum_{\mathcal{E}^{\text{int}}} + {}^{\text{e}}\mathcal{V} \left(\boldsymbol{m}, \frac{\hbar \ell}{\mathbf{I}\boldsymbol{\Omega} \rho} \right) \sum_{\mathcal{E}^{\text{int}}}
\end{aligned}$$

$$\begin{aligned}
{}^{\text{e}}0 = & {}^{\text{I}}\mathcal{V}\ell \cup {}^{\text{I}}\mathcal{V}\ell \langle \boldsymbol{a}, \boldsymbol{u} \cdot \mathbf{I}\underline{\boldsymbol{\varphi}} \rangle - \textcolor{blue}{\langle \boldsymbol{a}, \mathbf{I}\boldsymbol{u} \cdot \mathbf{I}\underline{\boldsymbol{H}} \underline{\boldsymbol{\Omega}} \rangle} + {}^{\text{o}}\mathcal{V}\ell \cup {}^{\text{I}}\mathcal{V}\ell \langle \boldsymbol{a}, \boldsymbol{u} \cdot \mathbf{I}\boldsymbol{H} \mathbf{I}\underline{\boldsymbol{\Omega}} \rangle + \\
& {}^{\text{I}}\mathcal{U} \langle [\boldsymbol{a}], {}^{\text{I}}\boldsymbol{u} \cdot \mathbf{I}\underline{\boldsymbol{H}} \mathbf{I}\underline{\boldsymbol{\Omega}} \rangle \sum_{\mathcal{E}^{\text{int}}} + {}^{\text{e}}\mathcal{V}(\boldsymbol{a}_{\Delta}, \mathbf{I}\boldsymbol{H} \mathbf{I}\underline{\boldsymbol{\Omega}}) \sum_{\mathcal{E}^{\text{int}}} - {}^{\text{e}}\mathcal{V} \left(\boldsymbol{a}, \frac{\hbar \ell}{{}^{\text{I}}Z \rho} \right) \sum_{\mathcal{E}^{\text{int}}}
\end{aligned}$$

Approximate \boldsymbol{Z} by $Z \in V_{\text{I}, h}$ (DG) and approximate \boldsymbol{u} by \boldsymbol{U} in NIPG .
 $\{(\mathcal{U}_{\text{In}})_{k_{\text{In}}^e}\}_{k_{\text{In}}^e} = {}^h\boldsymbol{W}$
 $\{(\mathcal{U}_{\text{SI}})_{k_{\text{SI}}^e}\}_{k_{\text{SI}}^e} = {}^hV_{\text{I}, h}$

On \mathcal{U} , define finite dimensional approximating spaces for $k_{\text{SI}}^e, k_{\text{In}}^e \geq 1$:

Multi-Coupled Weak Formulation

$$\begin{aligned}
 \cdot^{II}v(\boldsymbol{\omega}, \boldsymbol{\omega}^{II}, \underline{\boldsymbol{\Omega}}) = & \textcolor{blue}{\langle \boldsymbol{\omega}, [\boldsymbol{\Omega}] \boldsymbol{v} \rangle} - \textcolor{blue}{\langle [\boldsymbol{\Omega}] \boldsymbol{u}, \boldsymbol{\omega} \Delta \boldsymbol{u} \rangle} + \frac{2}{1} \\
 & \textcolor{blue}{\langle \boldsymbol{\omega}, \boldsymbol{u} \cdot \underline{\boldsymbol{\Omega} \Delta \boldsymbol{u}} \rangle} + {}^{II}v(\boldsymbol{\omega} \Delta, {}^{II}\boldsymbol{\Omega} \Delta \boldsymbol{u}) + \textcolor{blue}{\langle \boldsymbol{u} \cdot \boldsymbol{\omega}, {}^{II}Z \boldsymbol{v} \rangle} + \\
 & \mathcal{U} \left\langle \boldsymbol{u} \cdot \boldsymbol{\omega}, [{}^{II}Z] \boldsymbol{v} \right\rangle \sum_{\mathcal{U} \in \mathcal{T}_{int}} - {}^{II}v(\boldsymbol{\omega}, {}^{II}Z \Delta \boldsymbol{v}) + {}^{II}v(\boldsymbol{\omega}, {}^{II}\boldsymbol{\Omega}) f_{q_L} + {}^{II}v \left(\boldsymbol{\omega}, \frac{\partial}{{}^{II}\boldsymbol{\Omega} \partial} \right)
 \end{aligned}$$

$$\begin{aligned}
 '0 = & {}^I v \cup {}^I v \langle \boldsymbol{u}, \boldsymbol{u} \cdot {}^{II}h \boldsymbol{\Omega} \rangle - \textcolor{blue}{\langle \boldsymbol{u}, \boldsymbol{u} \cdot {}_L H \underline{\boldsymbol{\Omega}} \rangle} - {}^o v \cup {}^o v \langle \boldsymbol{u}, \boldsymbol{u} \cdot {}^{II}H {}^{II}\boldsymbol{\Omega} \rangle + \\
 & \mathcal{U} \left\langle [\boldsymbol{u}], \boldsymbol{u} \cdot {}^{II}H {}^{II}\boldsymbol{\Omega} \right\rangle \sum_{\mathcal{U} \in \mathcal{T}_{int}} + {}^o v(\boldsymbol{\omega} \Delta, {}^{II}H {}^{II}\boldsymbol{\Omega}) \sum_{\mathcal{U} \in \mathcal{T}_h} - {}^o v \left(\boldsymbol{\omega}, \frac{\partial}{{}^{II}Z \partial} \right) \sum_{\mathcal{U} \in \mathcal{T}_h}
 \end{aligned}$$

(CFEM).

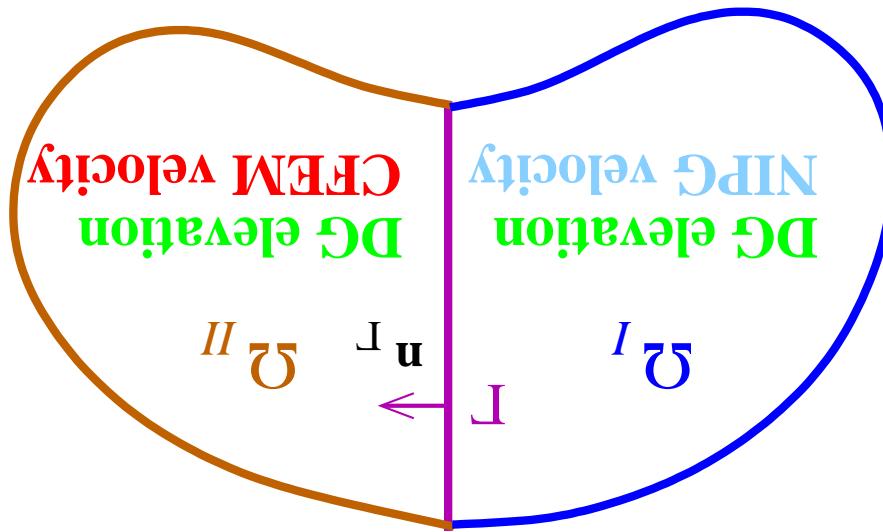
Approximate \boldsymbol{S}^{II} by $Z^{II} \in V^{II,h}$ (DG) and approximate \boldsymbol{u}^{II} by $\boldsymbol{U}^{II} \in R^h$

$$R^h \subset (H_1(\mathcal{U}^{II}))^2 \cup \{ \boldsymbol{n}^{II} : \boldsymbol{n}^{II} = \boldsymbol{u}^{II} \text{ on } \partial \mathcal{U}^{II} \},$$

$$V^{II,h} = \{ u : \mathcal{U}^{II} \rightarrow R : u|_{\mathcal{U}^{II}} \in P_{k_{\mathcal{U}^{II}}}(\mathcal{U}^{II}) \}$$

On \mathcal{U}^{II} , define finite dimensional approximating spaces for $k_{\mathcal{U}^{II}} \geq 1$:

Multi-Coupled Weak Formulation (continued)



$$v \in V^I(U^I), \boldsymbol{w} \in W^h(U^I), \boldsymbol{v} \in V^{II}(U^{II}), \boldsymbol{w} \in W^h(U^{II}), \text{ and } \boldsymbol{\omega} \in \mathbf{R}^{0,h}(U^{II}).$$

such that the previous equations are satisfied for all test functions

$$(Z^I, U^I, Z^{II}, U^{II}) \in V^I(U^I) \times W^h(U^I) \times V^{II}(U^{II}) \times \mathbf{R}^h(U^{II})$$

For each $t > 0$, we seek approximating solutions

Multi-Coupled Weak Formulation (continued)

$$\begin{aligned}
& \eta p^{\frac{1}{2}} \|[\mathbf{I} \mathbf{n}]_{\mathcal{Z}/\mathcal{L}} \varphi\| \int_0^0 + \eta p^{\frac{1}{2}} \|\mathbf{I} \mathbf{n}\|_{\mathcal{Z}/\mathcal{L}} \varphi\| \int_0^0 + \\
& \quad \eta p^{\frac{1}{2}} \|[\mathbf{I} \mathbf{n}]_{\mathcal{Z}/\mathcal{L}} \varphi\| \sum_{\forall i \in \mathcal{I}_{\text{int}}} \int_0^0 + \eta p^{\frac{1}{2}} \|(\mathbf{U} - \mathbf{n}) \Delta_{\mathcal{Z}/\mathcal{L}} \eta\| + \\
& \eta p^{\frac{1}{2}} \|(\mathbf{U} - \mathbf{n})_{\mathcal{Z}/\mathcal{L}}^{f_q}\| + \eta p^{\frac{1}{2}} \langle Z - \zeta, |\mathbf{u} \cdot \mathbf{U}| \rangle \int_0^0 + \eta p^{\frac{1}{2}} \langle [Z - \zeta], |\mathbf{u} \cdot \underline{\mathbf{U}}| \rangle \int_0^0 + \\
& \eta p^{\frac{1}{2}} \langle [\mathbf{u} \cdot \mathbf{U}], |\mathbf{u} \cdot \mathbf{U}| \rangle \int_0^0 + \eta p^{\frac{1}{2}} \langle [\mathbf{u} \cdot \mathbf{U}], |\mathbf{u} \cdot \mathbf{U}| \rangle \int_0^0 + \eta p^{\frac{1}{2}} \langle [\mathbf{u} \cdot \mathbf{U}], |\mathbf{u} \cdot \mathbf{U}| \rangle \sum_{\forall i \in \mathcal{I}_{\text{int}}} \int_0^0 + \\
& \eta p^{\frac{1}{2}} \langle [\mathbf{u} \cdot \underline{\mathbf{U}}], |\mathbf{u} \cdot \underline{\mathbf{U}}| \rangle \sum_{\forall i \in \mathcal{I}_{\text{int}}} \int_0^0 + \eta p^{\frac{1}{2}} \|(\mathbf{U} - \mathbf{n})\| + \eta p^{\frac{1}{2}} \|(Z - \zeta)\| = \|\mathbf{U} - \mathbf{n}, Z - \zeta\|
\end{aligned}$$

$$\left\{ \eta p^{\frac{1}{2}} \left[\left(\eta p^{\frac{1}{2}} \right)_{\mathcal{L}} + h_{\mathcal{L}}^{\frac{1}{2}} H \right] \|\mathbf{n}\|_2 + h_{\mathcal{L}}^{\frac{1}{2}} \|\zeta\|_2 \right\} \sum_{\forall i \in \mathcal{I}_{\text{int}}} \int_0^0 + K^6 h^5 \|\mathbf{U} - \mathbf{n}, Z - \zeta\|$$

Theorem. For \mathbf{n}, ζ sufficiently smooth and positive penalty parameter $\eta = O(h^{-\frac{1}{2}})$, the scheme satisfies the error estimate

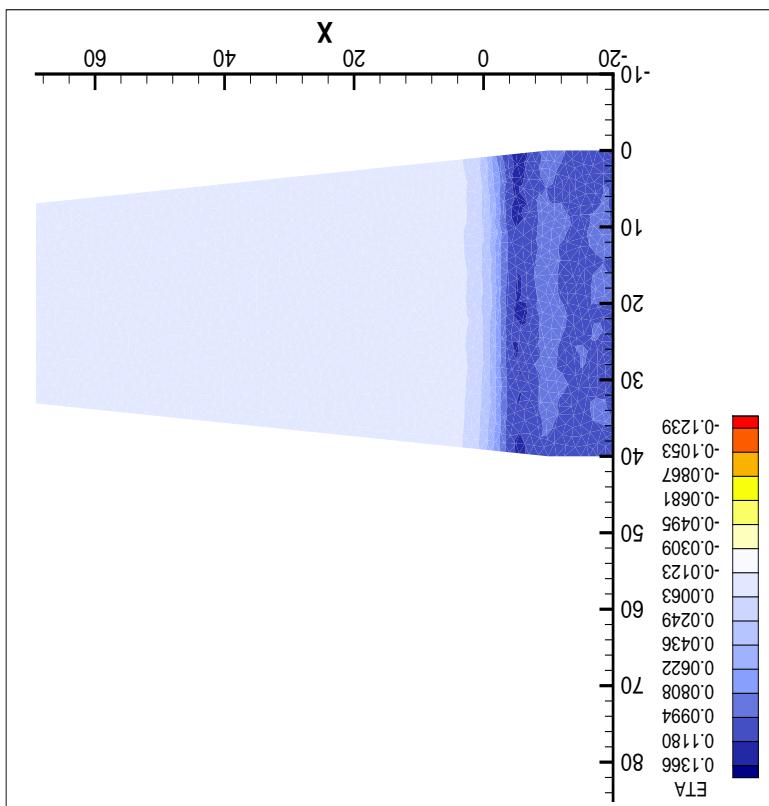
Multi-Coupled A Priori Error Estimate

- Solves momentum by CEM
 - Constant or linear basis functions
 - Revert to PC and solve by **discontinuous Galerkin**
- ADIRC modification

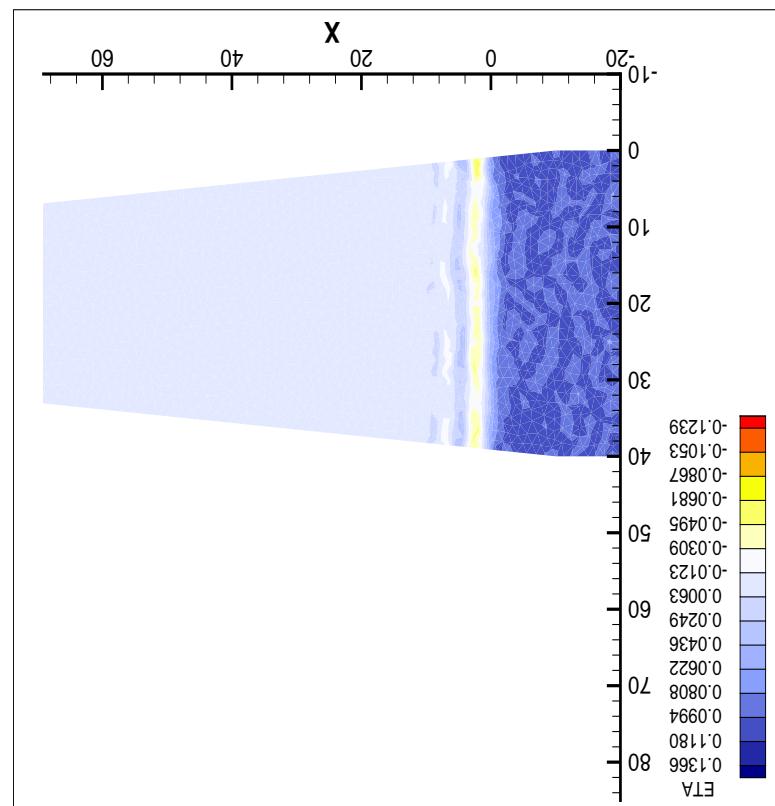
- Luetrich, Westerink, Scheffer (1991)
 - Linear basis functions
 - Solves momentum by CEM
 - Utilizes GWCE formulation solved by **CEM**
- ADIRC

Numerical Application

Modified DG implementation

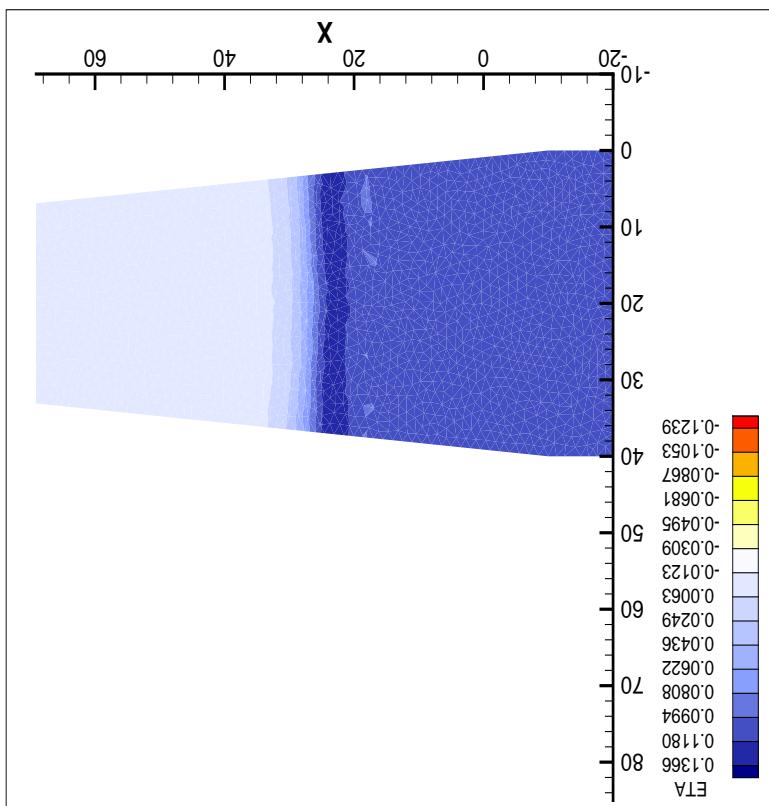


ADClRC implementation

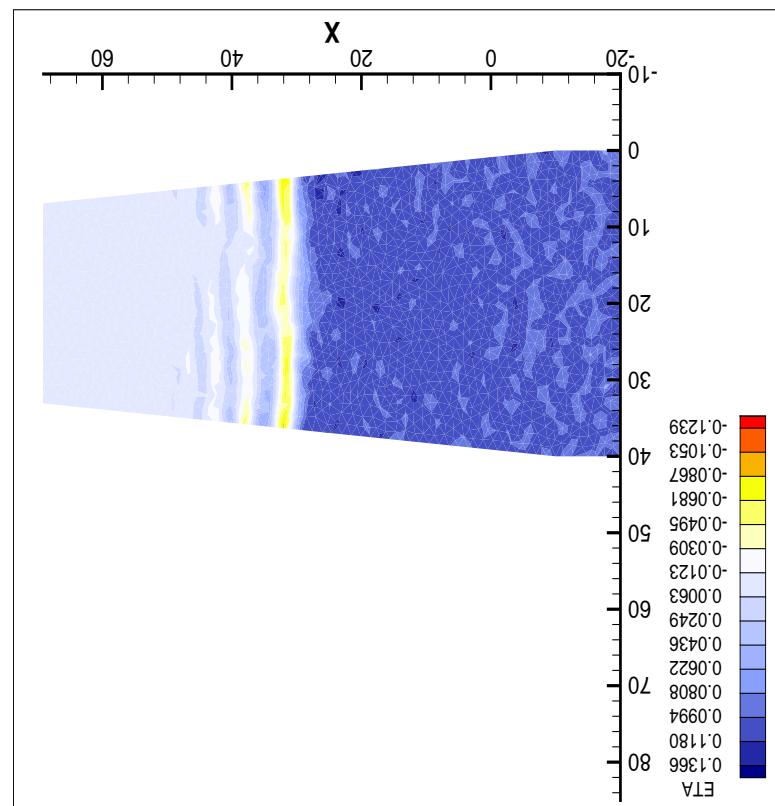


Numerical Results for Ellevation

Modified DG implementation

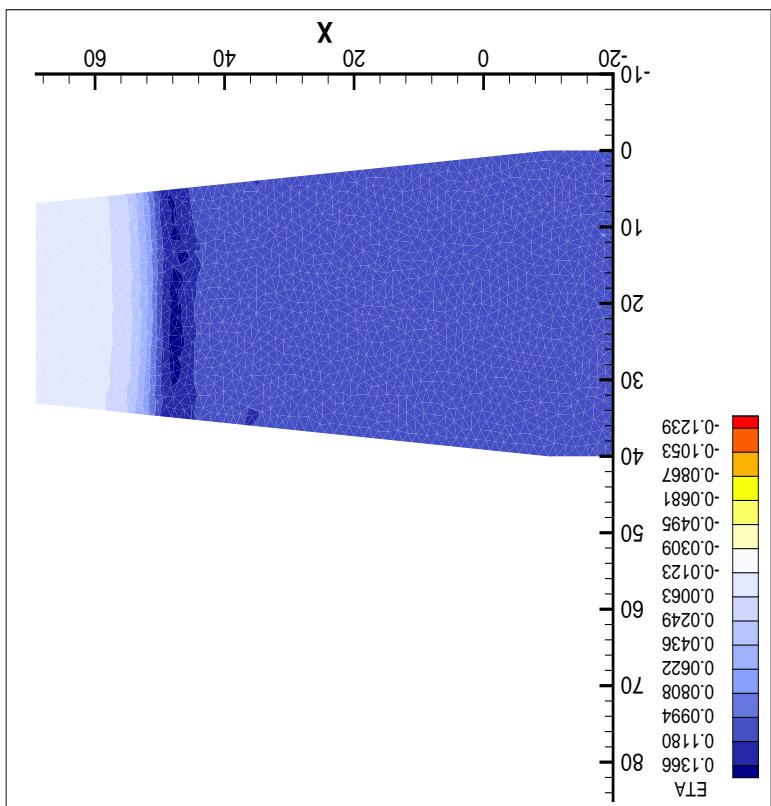


ADClRC implementation

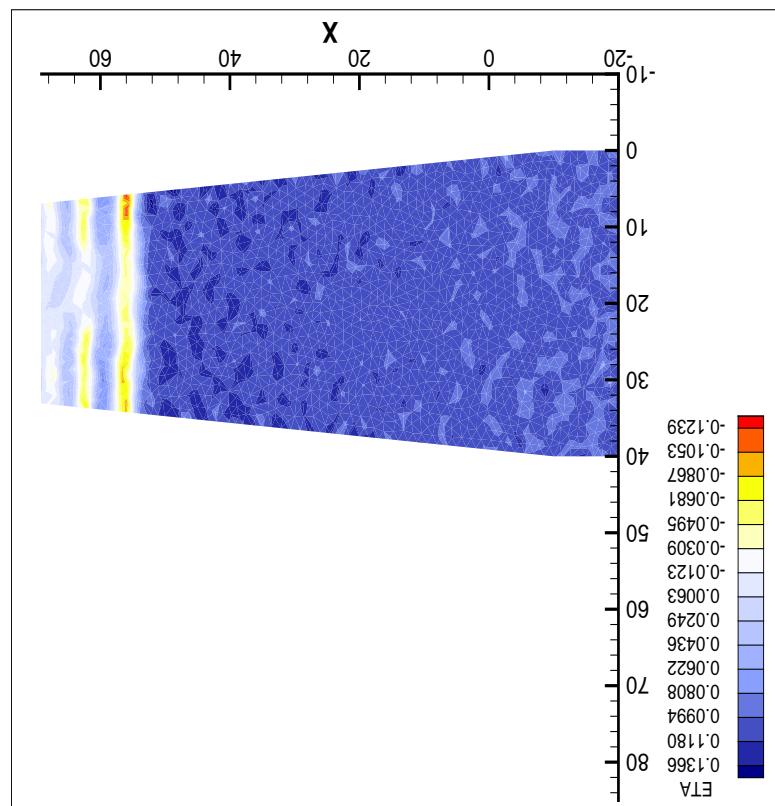


Numerical Results for Ellevation

Modified DG implementation

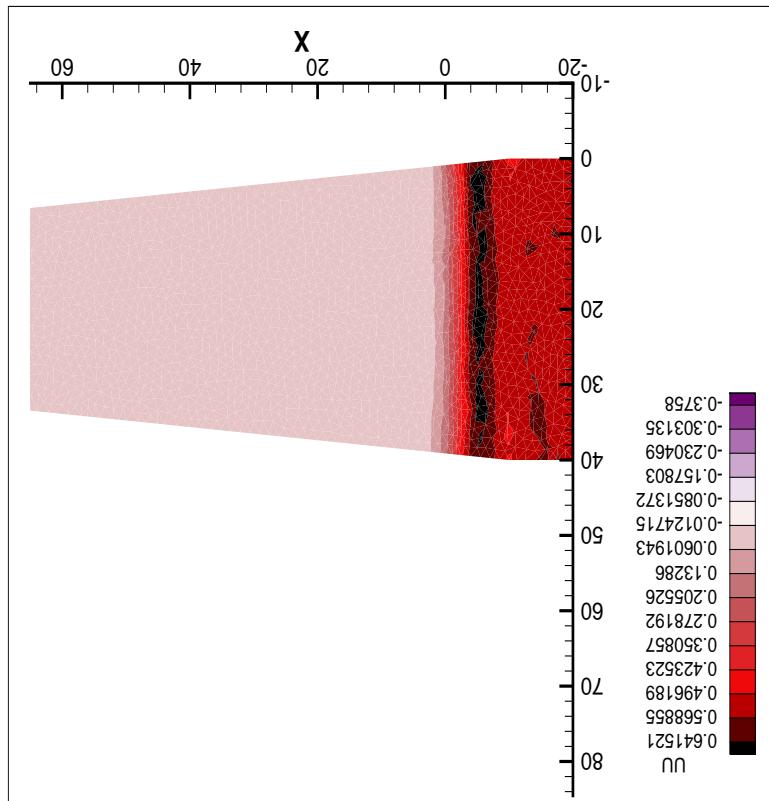


ADClRC implementation

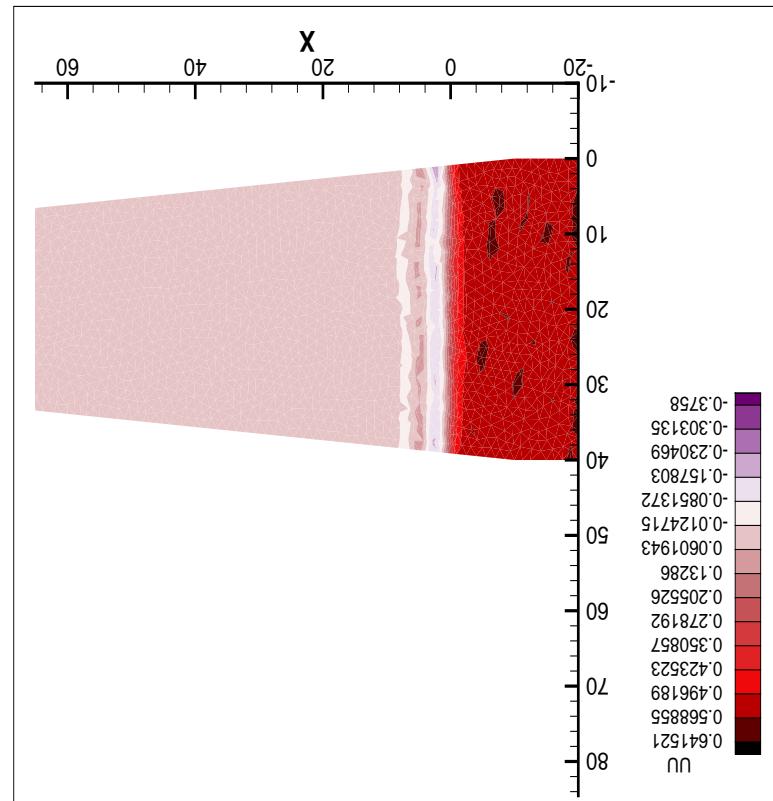


Numerical Results for Ellevation

Modified DG implementation

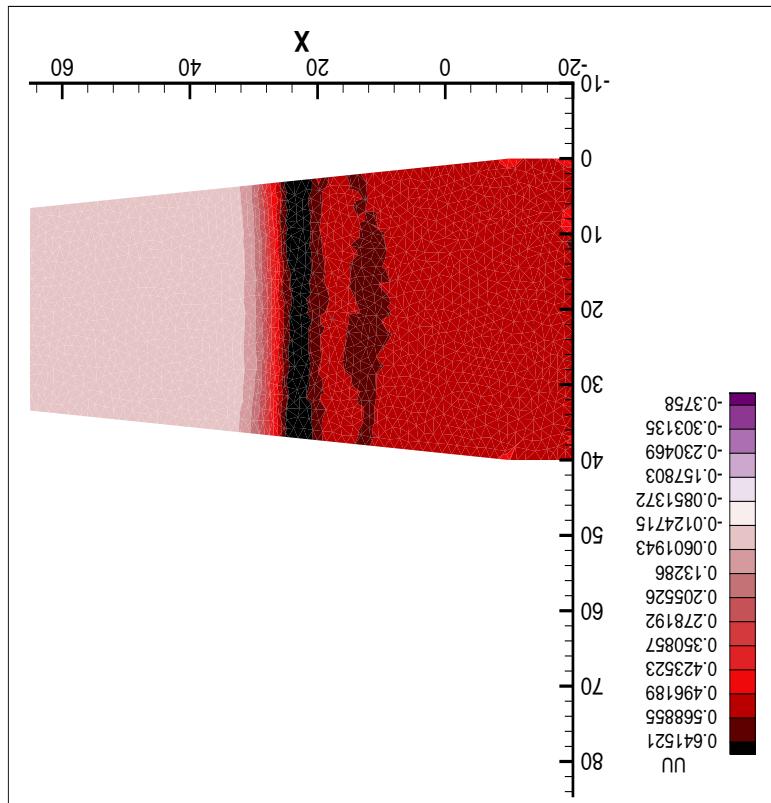


ADCIJC implementation

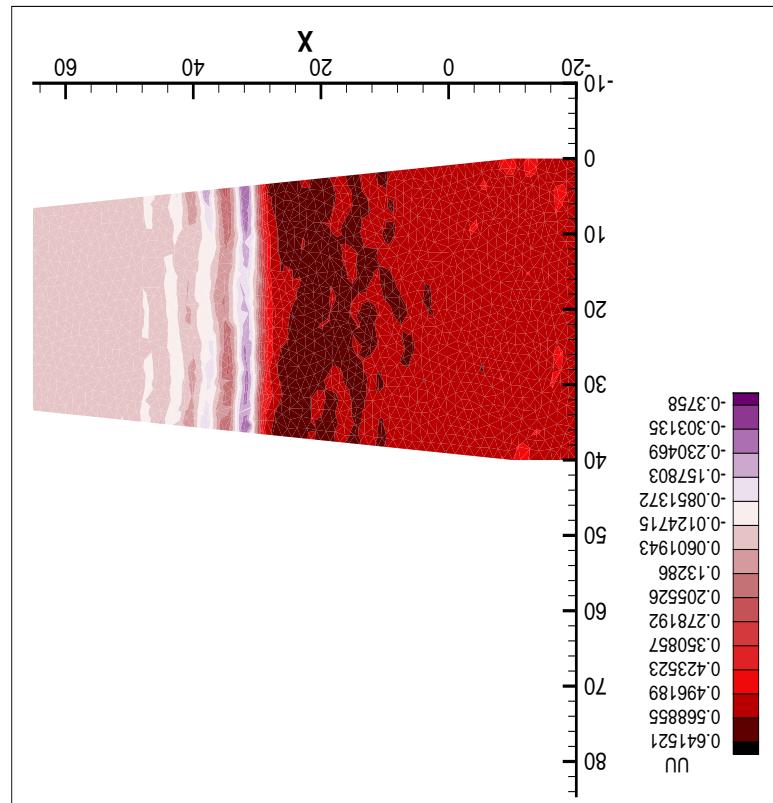


Numerical Results for Velocity

Modified DG implementation

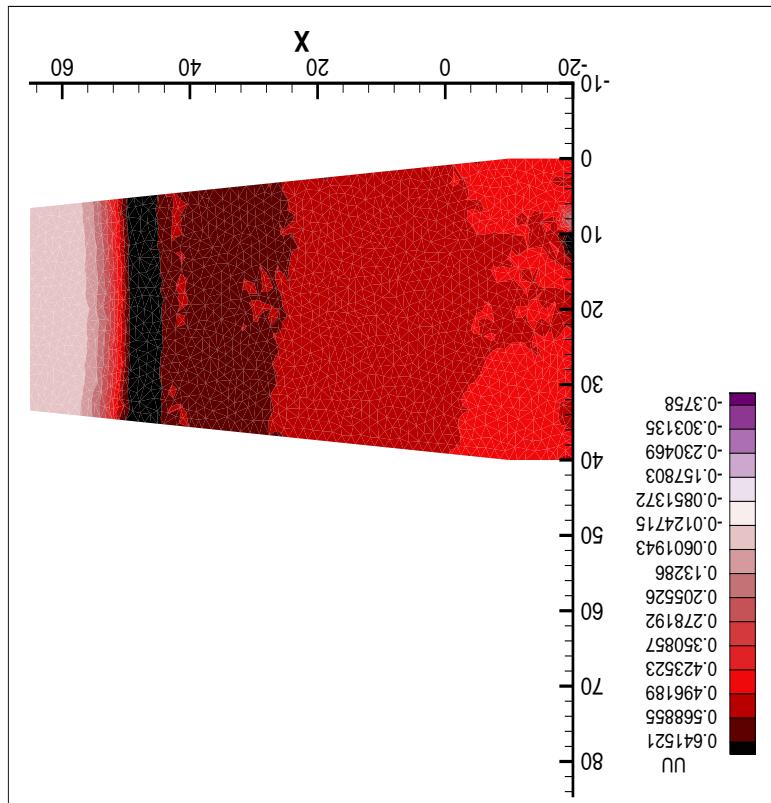


ADCIJC implementation

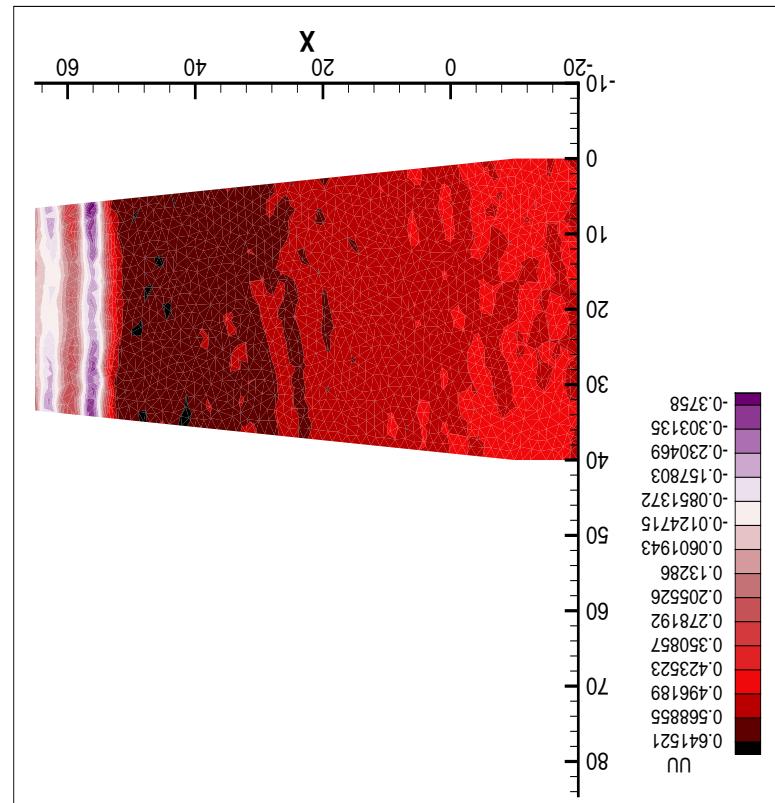


Numerical Results for Velocity

Modified DG implementation



ADIRC implementation



Numerical Results for Velocity

- Adaptive strategies: examine the accuracy and efficiency of adaptivity with respect to the method.
- Adaptive methodologies: develop a simulation methodology combining the simulation methods and strategies.
- A posteriori error estimation.

Current Work