On the control of some quantum systems

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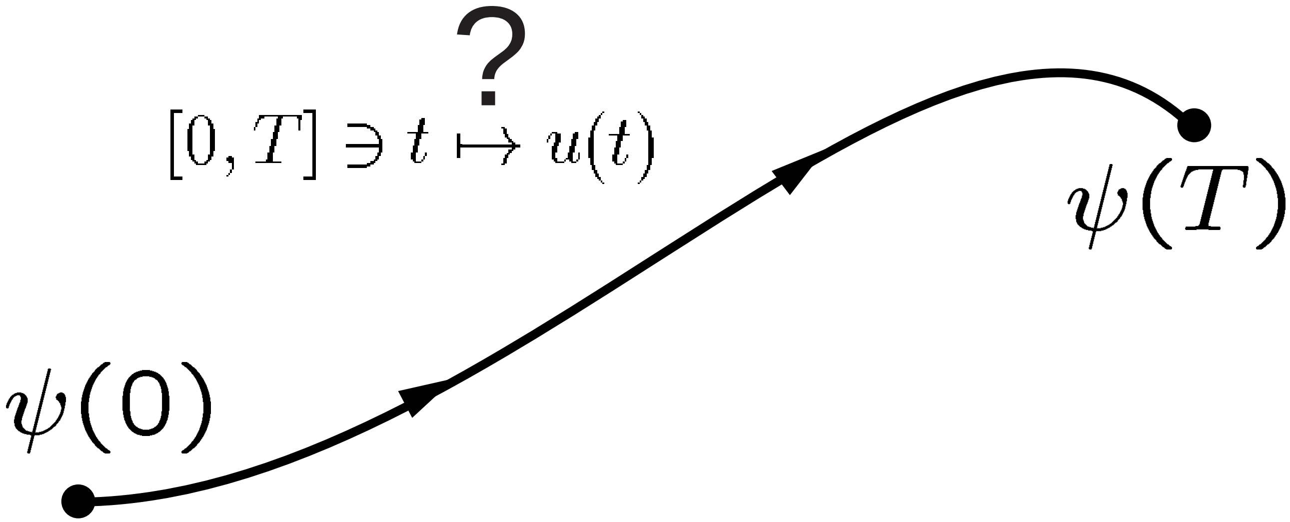
Motion planning for some quantum systems ψ



Difficult problem because it requires the integration of

$$\imath\hbar\frac{d}{dt}\psi = (H_0 + u(t)H_1)\psi.$$

Outline $\imath\hbar\frac{d}{dt}\psi = (H_0 + uH_1)\psi$

- 1D harmonic oscillator $H_0 = (p^2 + q^2)/2$, $H_1 = q$, $p \equiv -\imath\frac{\partial}{\partial q}$.
  Controllable part of dimension 2: $\frac{d^2\langle q \rangle}{dt^2} = -\langle q \rangle + u$.

- The periodic potential (impulse control).

- A particle in a moving box (steady-state controllability).

- A 1D particle with $H_0 = p^2/2 + V(q)$ and $H_1 = -q$ in the quasi-classic approximation ($\hbar \approx 0$).

- Two and three states (tunnelling control).

The harmonic oscillator ($\hbar = 1$ and $p = \frac{1}{\imath}\frac{\partial}{\partial q}$ here)

$$\imath \frac{\partial \psi}{\partial t} = ((p^2 + q^2)/2 - uq)\psi = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} + \frac{1}{2} q^2 \psi - uq\psi$$

with $\psi(\cdot, t)$ in $L^2(\mathbb{R})$. Notice that $\frac{d}{dt}\left(\int |\psi(q,t)|^2 dq\right) = 0$.

Controllability and commutators obtained with $H_0/\imath$ and $H_1/\imath$ using Heisenberg relation $[q, p] = \imath$:

$$[H_0, H_1] = \imath p, \quad [H_0, p] = -\imath q, \quad [H_1, p] = -\imath.$$

Controllable part of dimension 2 (I is not interesting since it corresponds to a global phase change: $\psi = \exp(\imath\theta(t, q))\ \phi$) given by Ehrenfest theorem

$$\frac{d}{dt}\langle q \rangle = \langle p \rangle, \quad \frac{d}{dt}\langle p \rangle = -\langle q \rangle + u$$

with $\langle q \rangle = \int q|\psi(q,t)|^2 dq$, ...

The harmonic oscillator (continued)

Controllable part: $\frac{d^2\langle q\rangle}{dt^2} = -\langle q\rangle + u$.

The uncontrollable part corresponds to an autonomous Schrödinger equation without control

$$\imath\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial z^2} + \frac{1}{2}z^2\phi$$

with $z = q - \langle q\rangle$ and

$$\psi(t,q) = \exp\left(\imath\underbrace{\left[\langle\dot{q}\rangle z - \int_0^t (\langle q\rangle^2/2 - (\langle\dot{q}\rangle)^2/2 - u\langle q\rangle)dt\right]}_{\theta(t,z)}\right)\ \phi(t,z).$$

The harmonic oscillator (continued)

$\psi_0(q)$ the state of the particle at $t = 0$ with definite energy E:

$$-\frac{1}{2}\frac{\partial^2\psi_0}{\partial q^2} + \frac{q^2}{2}\psi_0 = E\psi_0.$$

Then $\langle q\rangle(0) = \langle p\rangle(0) = 0$. Thus, $\phi(0, z) = \psi_0(0, z)$. Take any C^2 function $[0, T] \ni t \mapsto y(t) \in \mathbb{R}$ such that

$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0, \quad y(T) = a, \quad \dot{y}(T) = \ddot{y}(T) = 0.$$

Then the control $u(t) = \ddot{y}(t) + y(t)$ steers to a new bounded state of definite energy and centered around $q = a$:

$$\phi(T, q) = \exp(\imath\theta)\psi_0(q - a).$$

The harmonic oscillator (end)

Classical decomposition of $\frac{d}{dt}\xi = f(\xi, u)$ via a nonlinear change of coordinates $\xi \mapsto \chi$:

$$\chi = (\chi_1, \chi_2), \quad \frac{d}{dt}\chi_1 = g_1(\chi_1), \quad \frac{d}{dt}\chi_2 = g_2(\chi_1, \chi_2, u).$$

Take the controlled Schrödinger equation (infinite dimension case)

$$\imath \frac{d}{dt}\psi = (H_0 + \sum_1^n u_k H_k)\psi$$

and assume that the Lie algebra spanned by the shew-Hermitean operators $H_0/\imath$ and $H_k/\imath$ is of finite dimension. Does there exists a decomposition into a finite dimensional controllable part and an infinite dimensional uncontrollable part.

The 3D harmonic oscillator

$$\imath \frac{\partial \psi}{\partial t} = - \sum_{k=1}^{3} \left(\frac{1}{2} \frac{\partial^2 \psi}{\partial {q_k}^2} + \frac{q_k^2}{2} \psi - u_k q_k \psi \right).$$

The controllable parts:

$$\frac{d}{dt} \langle q_k \rangle = \langle p_k \rangle \,, \qquad \frac{d}{dt} \langle p_k \rangle = - \langle q_k \rangle + u_k.$$

The transformation

$$\psi(t, q_1, q_2, q_3) = \prod_{k=1}^{3} \exp \left(\imath \left(\langle p_k \rangle (q_k - \langle q_k \rangle) - \int_0^t (\langle q_k \rangle^2 /2 - \langle p_k \rangle^2 /2 - u_k \langle q_k \rangle) \right) \right) \ldots$$
$$\ldots \chi(t, q_1 - \langle q_1 \rangle, q_2 - \langle q_2 \rangle, q_2 - \langle q_2 \rangle)$$

leads to an autonomous oscillator with $z_k = q_k - \langle q_k \rangle$:

$$\imath \frac{\partial \chi}{\partial t} = - \sum_{k=1}^{3} \left(\frac{1}{2} \frac{\partial^2 \chi}{\partial {z_k}^2} + \frac{z_k^2}{2} \chi \right).$$

The periodic potential

$$\imath\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} + (V(q) - uq)\psi$$

With $\ddot{v} = -u$ this equation represents also the dynamics of a particle in a non Galilean frame of absolute position v. Change of independent variables $(t, q) \mapsto (t, z)$ and dependent variable $\psi \mapsto \phi$,

$$q = z - v, \quad \psi(t, z - v) = \exp\left(\imath\left(-z\dot{v} - v\dot{v} + \frac{1}{2}\int_0^t \dot{v}^2\right)\right)\phi(t, z)$$

yields

$$\imath\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial z^2} + V(z - v)\phi.$$

The periodic potential (end)

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} + (V(q) - uq)\psi, \quad \imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2} + V(z-v)\phi.$$

Assume that $V(q+a) = V(q), \forall q$. Take $[0,1] \ni \alpha \mapsto y(\alpha) \in \mathbb{R}$ such that

$$y(0) = \dot{y}(0) = \ddot{y}(0) = 0, \quad y(1) = ka, \quad \dot{y}(1) = \ddot{y}(1) = 0.$$

For $\varepsilon > 0$, the control $u(t) = -\ddot{v}(t) = -y''(t/\varepsilon)/\varepsilon^2$ steers approximatively the system from ψ_1 and ψ_2 such that

$$\psi_2(q) = \psi_1(q - ka).$$

Particle in a moving box of position v

In a Galilean frame

$$\imath\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial z^2}, \qquad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

where v is the position of the box and z is an absolute position .

In the box frame:

$$\imath\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} + \ddot{v}q\psi, \qquad q \in [-\frac{1}{2}, \frac{1}{2}],$$
$$\psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0$$

Tangent linearization around eigen-state $\bar{\psi}$ of energy $\bar{\omega}$

$$\psi(t,q) = \exp(-\imath\bar{\omega}t)(\bar{\psi}(q) + \Psi(q,t))$$

and Ψ satisfies

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2} + \ddot{v}q(\bar{\psi} + \Psi)$$
$$0 = \Psi(-\frac{1}{2},t) = \Psi(\frac{1}{2},t).$$

Assume Ψ and $\ddot{v}$ small and neglecte the second order term $\ddot{v}q\Psi$:

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2} + \ddot{v}q\bar{\psi}, \quad \Psi(-\frac{1}{2},t) = \Psi(\frac{1}{2},t) = 0.$$

Operational computations $s = d/dt$

The general solution of

$$(\imath s + \bar{\omega})\Psi = -\frac{1}{2}\Psi'' + s^2 vq\bar{\psi}$$

is

$$\Psi = A(s,q)a(s) + B(s,q)b(s) + C(s,q)v(s)$$

where

$$A(s,q) = \cos\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)$$
$$B(s,q) = \frac{\sin\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}$$
$$C(s,q) = (-\imath sq\bar{\psi}(q) + \bar{\psi}'(q)).$$

Case $q \mapsto \bar{\phi}(q)$ even

The boundary conditions imply

$$A(s,1/2)a(s) = 0, \quad B(s,1/2)b(s) = -\psi'(1/2)v(s).$$

$a(s)$ is a torsion element: the system is not controllable.

Nevertheless, for steady-state controllability, we have

$$b(s) = -\bar{\psi}'(1/2)\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}\ y(s)$$

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}y(s)$$

$$\Psi(s,q) = B(s,q)b(s) + C(s,q)v(s)$$

Series and convergence

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) = F(s)y(s)$$

where the entire function $s \mapsto F(s)$ is of order 1/2,

$$\exists K, M > 0, \forall s \in \mathbb{C}, \quad |F(s)| \leq K \exp(M|s|^{1/2}).$$

Set $F(s) = \sum_{n\geq 0} a_n s^n$ where $|a_n| \leq K^n/\Gamma(1+2n)$ with $K > 0$ independent of n. Then $F(s)y(s)$ corresponds in the time domain to

$$\sum_{n\geq 0} a_n y^{(n)}(t)$$

that is convergent when $t \mapsto y(t)$ is a C^∞ time function of Gevrey order < 1: i.e. $\exists M > 0$ and $\exists \sigma \in [0, 1[$ such that, $\forall t$, $\forall n$, $|y^{(n)}(t)| \leq M^n \Gamma(1 + (\sigma + 1)n)$

Steady state controllability

Steering from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following Gevrey function of order σ:

$$[0, T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \dfrac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega}D}{\sin^2(\sqrt{\bar{\omega}/2})}$. The fact that this function is of Gevrey order σ results from its exponential decay of order σ around 0 and 1.

Practical computations via Cauchy formula

$$y^{(n)}(t) = \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi$$

where γ is a closed path around zero, $\sum_{n\geq 0} a_n y^{(n)}(t)$ becomes

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi = \frac{1}{2\imath\pi} \oint_\gamma \left(\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t+\xi)\, d\xi.$$

But

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) ds = B_1(F)(\xi)$$

is the Borel transform of F.

Practical computations via Cauchy formula (end)

In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\imath\pi}\oint_{\gamma} B_1(F)(\xi)y(t+\xi)\ d\xi$$

where γ is a closed path around zero. Such integral representation is very useful when y is defined by convolution with a real signal Y,

$$y(\zeta) = \frac{1}{\varepsilon\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-(\zeta - t)^2/2\varepsilon^2)Y(t)\ dt$$

where $\mathbb{R} \ni t \mapsto Y(t) \in \mathbb{R}$ is any measurable and bounded function:

$$v(t) = \int_{-\infty}^{+\infty} \left[\frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}}\oint_{\gamma} B_1(F)(\xi)\exp(-(\xi-\tau)^2/2\varepsilon^2)\ d\xi\right] Y(t-\tau)\ d\tau.$$

The classical limit $\hbar \approx 0$

1D wave packet $\psi(q,t)$ with Hamiltonian $H = p^2/2 + V(q) - uq$ where $p = \frac{\hbar}{\imath}\frac{\partial}{\partial q}$

$$\imath\hbar \frac{d}{dt}\psi = H\psi.$$

Since $\hbar$ is close to zero, the support of ψ is concentrated around the average position $\langle q \rangle$

$$\dot{\langle q \rangle} = \langle p \rangle\,, \quad \dot{\langle p \rangle} = -\left\langle V'(q) \right\rangle + u.$$

Taylor development (see Messiah book)

$$V'(q) = V'(\langle q \rangle) + (q - \langle q \rangle)V''(\langle q \rangle) + (q - \langle q \rangle)^2 V'''(\langle q \rangle) + ...$$

valid when the potential V is smooth and admits small variations on the support of ψ. Thus

$$\left\langle V' \right\rangle \approx V'(\langle q \rangle) + \frac{\chi}{2}V'''(\langle q \rangle) \quad \text{with} \quad \chi = \left\langle (q - \langle q \rangle)^2 \right\rangle = (\Delta q)^2.$$

The classical limit $\hbar \approx 0$ (continued)

A first approximation is then

$$\dot{\langle q\rangle} = \langle p\rangle\,, \quad \dot{\langle p\rangle} = -V'(\langle q\rangle) - -\frac{\chi}{2}V'''(\langle q\rangle) + u.$$

We have to compute the $\chi = \left\langle (q - \langle q\rangle)^2 \right\rangle$ dynamics: main tool Heisenberg relation $[q,p] = \imath\hbar$, $[q,F(p)] = \imath\hbar F'(p)$, ... and

$$\imath\hbar\frac{d\,\langle A\rangle}{dt} = \langle [A,H]\rangle + \imath\hbar\left\langle \frac{\partial A}{\partial t}\right\rangle.$$

Then

$$\dot{\chi} = \eta, \quad \dot{\eta} = 2(\varpi - \chi V''(\langle q\rangle)), \quad \dot{\varpi} = -\eta V''(\langle q\rangle)$$

where

$$\varpi = \left\langle (p - \langle p\rangle)^2 \right\rangle, \quad \eta = \langle (p - \langle p\rangle)(q - \langle q\rangle) + (q - \langle q\rangle)(p - \langle p\rangle)\rangle\,.$$

The classical limit $\hbar \approx 0$ (continued)

We have the following approximate dynamics (close to the classical one where $\chi, \varpi, \eta = 0$):

$$\frac{d}{dt}\langle q\rangle = \langle p\rangle\,, \quad \frac{d}{dt}\langle p\rangle = -V'(\langle q\rangle) - \frac{\chi}{2}V'''(\langle q\rangle) + u$$
$$\frac{d}{dt}\chi = \eta, \quad \frac{d}{dt}\eta = 2(\varpi - \chi V''(\langle q\rangle)), \quad \frac{d}{dt}\varpi = -\eta V''(\langle q\rangle).$$

The quantity $I = \chi\varpi - \eta^2/4$ is an invariant: $\frac{d}{dt}I = 0$.

Strongly related to Heisenberg uncertainty principle:
assume $I > \hbar^2$, then $\chi\varpi = I + \eta^2/4$ and

$$\Delta q \cdot \Delta p = \sqrt{\chi\varpi} \geq \hbar.$$

The above model is valid for $\hbar$, $I \geq \hbar^2$, χ, ϖ and η small.

The classical limit $\hbar \approx 0$ (continued)

$$\frac{d}{dt}\langle q\rangle = \langle p\rangle\,, \quad \frac{d}{dt}\langle p\rangle = -V'(\langle q\rangle) - \frac{\chi}{2}V'''(\langle q\rangle) + u$$
$$\frac{d}{dt}\chi = \eta, \quad \frac{d}{dt}\eta = 2(\varpi - \chi V''(\langle q\rangle)), \quad \frac{d}{dt}\varpi = -\eta V''(\langle q\rangle).$$

For any $I > 0$, the restriction of the dynamics on $\chi\varpi - \eta^2/4 = I$ is flat with χ as flat output: assume that instead of knowing the control $t \mapsto u(t)$ (direct problem), we know $t \mapsto \chi(t)$ (inverse problem). Then

$$\eta = \dot{\chi}, \quad \varpi = (I + \eta^2/4)/\chi.$$

Since $V''(\langle q\rangle) = (\varpi - \dot{\eta}/2)/\chi$, $\langle q\rangle$ is an implicit function of $(\chi, \dot{\chi}, \ddot{\chi})$. Thus $\langle p\rangle$ and u are implicit functions of $(\chi, ..., \chi^{(3)})$, and $(\chi, ..., \chi^{(4)})$, respectively. The inverse problem admits no dynamics and thus χ is the flat output.

The classical limit $\hbar \approx 0$ (continued)

Assume that we start with the steady-state

$$(\langle q \rangle, \langle p \rangle, \chi, \eta, \varpi) = (q_1, 0, \chi_1, 0, V''(q_1)\chi_1)$$

and finish with another stable steady-state

$$(\langle q \rangle, \langle p \rangle, \chi, \eta, \varpi) = (q_2, 0, \chi_2, 0, V''(q_2)\chi_2)$$

with $q_1 \neq q_2$, $V'(q_1) = V'(q_2) = 0$, $V''(q_1) > 0$, $V''(q_2) > 0$ and $V''(q_1)(\chi_1)^2 = V''(q_2)(\chi_2)^2$. Then exists a region between q_1 and q_2 where $V'' < 0$: unstable zone avoids a quasi-static strategy. Flatness based strategy to design a steering control $u(t)$ that is not too large with a steering time of the same order of the natural time-constant $\sqrt{1/|V''(q)|}$. However, such design method has to pass through singularities corresponding to potential inflexion $V'''(q) = 0$.

The classical limit $\hbar \approx 0$ (end)

Such quasi-classic approximations is adapted to multiple degree of freedom (not the case of the WKB method). For a single particle, with n degrees of freedom $(q_1, ..., q_n)$ and

$$H_0 = \sum_{i=1}^{n} p_i^2/2 + V(q), \quad H_i(q) = -q_i$$

the approximated dynamics reads for $i = 1, ..., n$:

$$\begin{aligned} \dot{\langle q_i \rangle} &= \langle p_i \rangle \\ \dot{\langle p_i \rangle} &= -V_{,i}(\langle q \rangle) - \chi^{jk} V_{,ijk}(\langle q \rangle) + u_i \end{aligned}$$

where the χ^{jk} dynamics involve second order derivatives of V and the variables

$$\begin{aligned} \eta^{jk} &= \left\langle (q_j - \left\langle q_j \right\rangle)(p_k - \langle p_k \rangle) + (p_j - \left\langle p_j \right\rangle)(q_k - \langle q_k \rangle) \right\rangle \\ \varpi^{j,k} &= \left\langle (p_j - \left\langle p_j \right\rangle)(p_k - \langle p_k \rangle) \right\rangle . \end{aligned}$$

Two states system

$$\imath\hbar\frac{d}{dt}\psi = \left(\begin{pmatrix} -E/2 & 0 \\ 0 & E/2 \end{pmatrix} + u\begin{pmatrix} h_1 & b \\ b^* & h_2 \end{pmatrix}\right)\psi \text{ with } \psi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Take the density matrice

$$\rho = \phi\phi^* = \begin{pmatrix} |a_1|^2 & a_1^* a_2 \\ a_1 a_2^* & |a_2|^2 \end{pmatrix} = 1 + \lambda\sigma_x + \mu\sigma_y + \nu\sigma_z$$

with the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Set $\vec{S} = (\lambda, \mu, \nu) \in \mathbb{S}^2$ (the Bloch sphere, where the meaningless absolute phase is removed) ($\omega_0 = E/\hbar$):

$$\frac{d}{dt}\vec{S} = \vec{S} \wedge (\omega_0 \vec{B}_0 + \frac{u}{\hbar}\vec{B}_1), \quad \vec{B}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{B}_1 = \begin{pmatrix} -2\Re(b) \\ 2\Im(b) \\ h_1 - h_2 \end{pmatrix}.$$

Two states system (continued)

$$\frac{d}{dt}\vec{S} = \vec{S} \wedge (\omega_0 \vec{B}_0 + \frac{u}{\hbar}\vec{B}_1), \quad \vec{S} \in \mathbb{S}^2$$

Set $\tau = \omega_0 t$, $' = d/d\tau$ and $v = \frac{\|\vec{B}_1\|}{\omega_0 \hbar} u$ the new control. The dynamics becomes

$$\vec{S}' = \vec{S} \wedge (\vec{B}_0 + v\vec{J})$$

where $\vec{J}$ is the unitary vector $\frac{1}{\|\vec{B}_1\|}\vec{B}_1$. Denote by $\alpha \in]0, \pi[$ the angle between $\vec{B}_0$ and $\vec{J}$ and consider the ortho-normal frame $(\vec{I}, \vec{J}, \vec{K})$ with $\vec{K} = \vec{B}_0 \wedge \vec{J} / \sin\alpha$ and $\vec{I} = \vec{J} \wedge \vec{K}$. Set $\vec{S} = x\vec{I} + y\vec{J} + z\vec{K}$ ($(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$):

$$x' = -z(\cos\alpha + u), \quad y' = z\sin\alpha, \quad z' = x(\cos\alpha + u) - y\sin\alpha,$$

The dynamics on $\mathbb{S}^2$ is flat with $y \propto \vec{S} \cdot \vec{B}_1$ as flat output.

Two states system (end)

With $\alpha = \pi/2$,

$$x' = -zu, \quad y' = z, \quad z' = xu - y,$$

implies that

$$z = y', \quad x = \pm\sqrt{1 - y^2 - (y')^2}.$$

The smooth trajectory

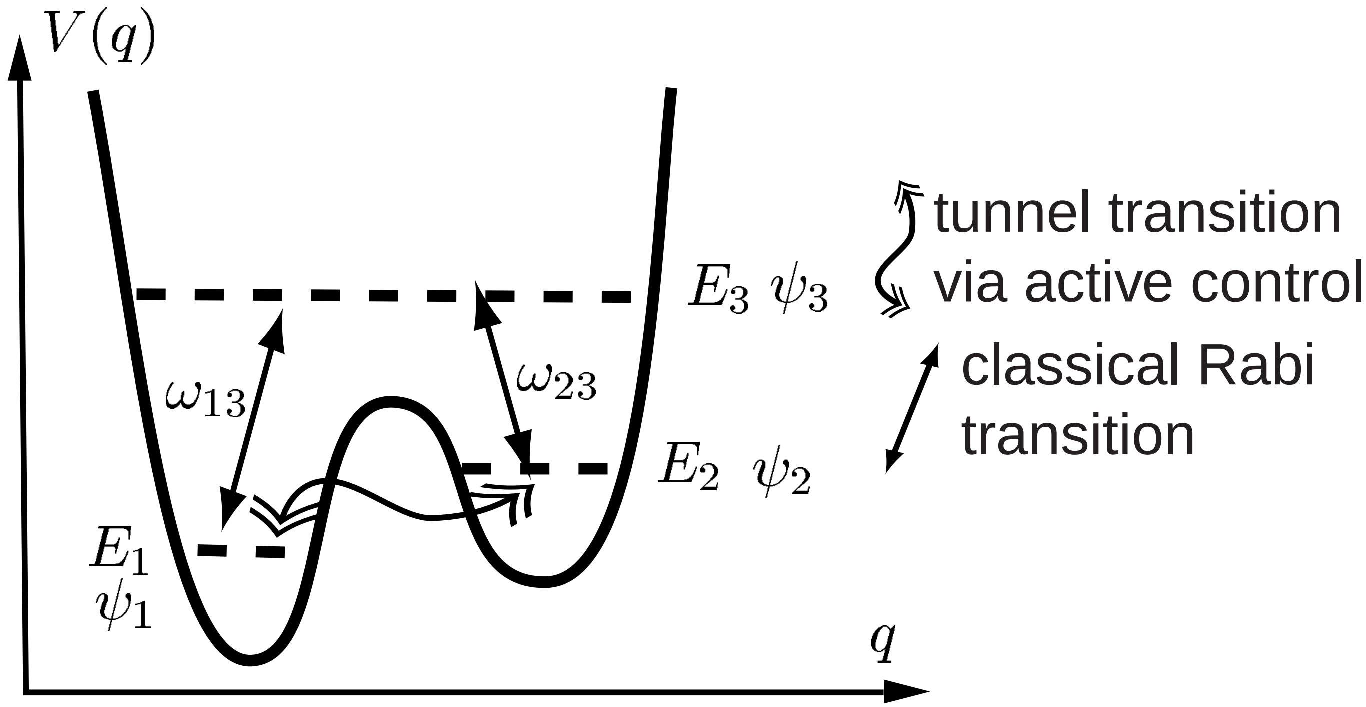
$$x(\tau) = g(\tau), \quad y(\tau) = p(\tau), \quad z(\tau) = p'(\tau), \quad v(\tau) = -g'(\tau)/p'(\tau)$$

with

$$\begin{array}{l} p(\tau) = (1-\tau)\tau^2(2-\tau) \\ f(\tau) = 1 - p^2(\tau) - (p')^2(\tau) \end{array}, \quad g(\tau) = \begin{cases} -\sqrt{f(\tau)} & \text{if } \tau \in [0,1] \\ \sqrt{f(\tau)} & \text{if } \tau \in [1,2] \end{cases}$$

steers the system from $(-1, 0, 0)$ with $v(0) = 0$ at $\tau = 0$ to $(1, 0, 0)$ with $v(2) = 0$ at $\tau = 2$.

Three states system



Reduced model of a $1D$ particule in the potential $V(q)$ with two minima and bounded states ψ_1 and ψ_2 of low energy E_1 and E_2 separated by a potential barrier and a third bounded state ψ_3 of energy E_3 passing over the barrier.

Three states system (continued)

$$\imath\hbar\frac{d}{dt}\psi = (p^2/2 + V(q) - uq)\psi$$

Projection of $\psi(t,q)$ dynamics on the vector space spanned by (ψ_1, ψ_2, ψ_3) yields,

$$\psi(t,q) \approx \alpha_1(t)\psi_1(q) + \alpha_2(t)\psi_2(q) + \alpha_3(t)\psi_3(q)$$

and

$$\begin{aligned}
\imath\hbar\tfrac{d}{dt}\alpha_1 &= (E_1 + u\langle q\rangle_1)\alpha_1 + u\langle q\rangle_{13}\,\alpha_3 \\
\imath\hbar\tfrac{d}{dt}\alpha_2 &= (E_2 + u\langle q\rangle_2)\alpha_2 + u\langle q\rangle_{23}\,\alpha_3 \\
\imath\hbar\tfrac{d}{dt}\alpha_3 &= (E_3 + u\langle q\rangle_3)\alpha_3 + u\langle q\rangle_{31}^*\,\alpha_1 + u\langle q\rangle_{32}^*\,\alpha_2
\end{aligned}$$

where

$$\langle q\rangle_i = \int |\psi_i|^2 q\ dq, \quad \langle q\rangle_{ij} = \int \psi_i^*\psi_j q\ dq.$$

Three states system (continued)

$$\begin{aligned}
\imath\hbar\frac{d}{dt}\alpha_1 &= (E_1 + u\langle q\rangle_1)\alpha_1 + u\langle q\rangle_{13}\alpha_3 \\
\imath\hbar\frac{d}{dt}\alpha_2 &= (E_2 + u\langle q\rangle_2)\alpha_2 + u\langle q\rangle_{23}\alpha_3 \\
\imath\hbar\frac{d}{dt}\alpha_3 &= (E_3 + u\langle q\rangle_3)\alpha_3 + u\langle q\rangle_{31}^*\alpha_1 + u\langle q\rangle_{32}^*\alpha_2
\end{aligned}$$

Phases changes

$$\alpha_i = \exp\left(-\imath E_3 t - \int_0^t u(s)\langle q\rangle_3/\hbar\; ds\right)\; a_i, \quad i = 1, 2, 3.$$

yields

$$\begin{aligned}
\imath\frac{d}{dt}a_1 &= (\omega_{13} + ue_1)a_1 + ub_1 a_3 \\
\imath\frac{d}{dt}a_2 &= (\omega_{23} + ue_2)a_2 + ub_2 a_3 \\
\imath\frac{d}{dt}a_3 &= ub_1^* a_1 + ub_2^* a_2
\end{aligned}$$

where $\omega_{13} = (E_1 - E_3)/\hbar$ and $\omega_{23} = (E_2 - E_3)/\hbar$ are the Bohr frequencies, $e_i = (\langle q\rangle_i - \langle q\rangle_3)/\hbar$, $b_1 = \langle q\rangle_{13}/\hbar$ and $b_2 = \langle q\rangle_{23}/\hbar$.

Three states system (continued)

$$\begin{aligned} \imath\tfrac{d}{dt}a_1 &= (\omega_{13} + ue_1)a_1 + ub_1 a_3 \\ \imath\tfrac{d}{dt}a_2 &= (\omega_{23} + ue_2)a_2 + ub_2 a_3 \\ \imath\tfrac{d}{dt}a_3 &= ub_1^* a_1 + ub_2^* a_2 \end{aligned}$$

$[0, T] \ni t \mapsto u(t)$ steering from $(1, 0, 0)$, $a = (0, 1, 0)$ without passing via $(0, 1, 0)$ is not obvious: control design mixing standard perturbation techniques and flatness based steering methods.

Take a small control u, $|ub_i|, |ue_j| \ll \omega_{13}, \omega_{23}$ varying slowly (time constant much smaller than $T_{23} = 2\pi/\omega_{23}$ and $T_{13} = 2\pi/\omega_{13}$ the Bohr periods). Set $b_1 = r_1 \exp(\imath\theta_1)$, $b_2 = r_2 \exp(\imath\theta_2)$ with r_i and θ_i real, and

$$u = \frac{2v_1(t)}{r_1}\cos(\omega_{13}t) + \frac{2v_2(t)}{r_2}\cos(\omega_{23}t)$$

with v_1 and v_2 small amplitudes.

Three states system (continued)

With $u = \frac{2v_1(t)}{r_1}\cos(\omega_{13}t) + \frac{2v_2(t)}{r_2}\cos(\omega_{23}t)$ the trajectories of

$$
\begin{aligned}
\imath\frac{d}{dt}a_1 &= (\omega_{13} + ue_1)\ a_1 + ur_1 \exp(\imath\theta_1)\ a_3 \\
\imath\frac{d}{dt}a_2 &= (\omega_{23} + ue_2)\ a_2 + ur_2 \exp(\imath\theta_2)\ a_3 \\
\imath\frac{d}{dt}a_3 &= ur_1 \exp(-\imath\theta_1)\ a_1 + ur_2 \exp(-\imath\theta_2)\ a_2
\end{aligned}
$$

are close to the trajectories of the average system

$$\dot{x}_1 = v_1 x_3, \quad \dot{x}_2 = v_2 x_3, \quad \dot{x}_3 = -v_1 x_1 - v_2 x_2$$

where

$$a_1 = \exp(\imath(\theta_1 - \omega_{13}t))x_1, \quad a_2 = \exp(\imath(\theta_2 - \omega_{23}t))x_2, \quad a_3 = \imath x_3.$$

Three states system (end)

$$\dot{x}_1 = v_1 x_3, \quad \dot{x}_2 = v_2 x_3, \quad \dot{x}_3 = -v_1 x_1 - v_2 x_2$$

Take any increasing smooth bijection $s \mapsto \sigma(s)$ from $[0, 1]$ to $[0, 1]$ with

$$\sigma(0) = 0, \quad \sigma(1) = 1, \quad \frac{d^i\sigma}{ds^i}(0) = \frac{d^i\sigma}{ds^i}(1) = 0, \quad i = 1, 2, 3.$$

Take $T \gg T_{13}, T_{23}$ and set $x_1 = 1 - \sigma(t/T)$ and $x_2 = \sigma(t/T)$. Then the control

$$v_1(t) = \frac{-\sigma'(t/T)}{T\sqrt{2\sigma(t/T)(1 - \sigma(t/T))}}$$

$$v_2(t) = \frac{\sigma'(t/T)}{T\sqrt{2\sigma(t/T)(1 - \sigma(t/T))}}$$

steers $(1, 0, 0)$ at $t = 0$ to $(0, 1, 0)$ at $t = T$.

Conclusion

- Perturbations techniques to get approximate solutions and steering controls.

- Importance of low dimensional models to investigate some basic controllability problems.

- The control corresponds to a classical homogeneous electric field: inhomogeneous and magnetic ?

- Feedback: is there a the quantum analogue of the PID regulator ?