On Inverse Problems from the Volatility Calibration with American Options

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0) Introduction to Black & Scholes equations

A European (vanilla) option on a given asset is the right to buy or sell this asset at the future date$T$ and at the price K.

- T is called the **maturity or expiration date**.
- K is called the **strike**.

For a given asset, (ex. a given action), there exist on the markets several options with different maturities and stroke.

An option to buy (to sell) is called a **call (put)**.

Main utility of derivatives: enable the financial institutions to construct hedged funds i.e. funds with low short rate risks

Problem : what is the price of an option?

It is easy to compute the price of the option at maturity T as a function of the price S_T of the underlying asset at this date:

for a call :

- if $S_T < K$, the right to buy is useless, and the option's price is 0 .
- if $S_T > K$, by exercizing the option and by selling immediately the asset, one gets a benefit of $S_T - K$: the option's price is $S_T - K$.

Finally

$$C_T = (S_T - K)_+$$

For a put

$$P_T = (S_T - K)_-$$

Other possible contracts $\rightarrow$ different values at maturity $P_T = h(S_T) : h$ **is called payoff function** . Example $h(S) = 1_{S<K}$

The Black & Scholes equation **How to fix C_t for $t < T$?**

for that, several assumptions:

1. No arbitrage assumption: one cannot make a risk-less immediate benefit.

2. Instantaneous and free transactions.

3. Every risk-less asset has the same remuneration, and we note $r(t)$ **the risk-less asset interest rate** .

4. The price (S_t) of the underlying asset is a stochastic process satisfying the SDE

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

   - $\mu S_t dt$: drift term
   - (W_t) : standard Brownian motion
   - (σ_t) : **volatility**

The no-arbitrage argument

Assume that the price C_t is a function $C(t, S_t)$.

How to choose C?

For that, Itô's formula:

$$\begin{aligned} d[C(t, S_t)] &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma_t^2 S_t^2 dt \\ &= \left(\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma_t^2 S_t^2 \right) dt + \sigma S_t \frac{\partial C}{\partial S} dW_t \end{aligned}$$

Let a portfolio contain 1 share of the asset whose price is S_t and Δ options whose price is C_t :

$$\text{the value of the portfolio:} \quad F_t = S_t + \Delta C_t$$

Idea : one can choose Δ for the portfolio to be **hedged** or **with no short term risk**, i.e. dF_t to be deterministic :

$$\text{One must choose} \quad \Delta = -\left(\frac{\partial C}{\partial S}\right)^{-1},$$

$$dF_t = \left(\mu S_t - \frac{\frac{\partial C}{\partial t} + \mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma_t^2 S_t^2}{\frac{\partial C}{\partial S}}\right) dt$$

Risk-less Portfolio,

$$dF_t = r(t) F_t dt = r(t)\left(S_t - \left(\frac{\partial C}{\partial S}\right)^{-1} C\right) dt$$

So the Black & Scholes equation for the European call is:

$$\partial_t C + \frac{\sigma_t^2 S^2}{2} \partial_{SS}^2 C + rS\partial_S C - rC = 0 \quad t \in [0, T[, \; S > 0,$$
$$C(T, S) = (S - K)_+ \quad S > 0,$$

Backward parabolic PDE (the space dimension is one), independent from μ.

If the option concerns a basket of d assets, one obtains a similar backward parabolic PDE (the space dimension is d).

Numerical difficulties

- The Cauchy data is **singular**.
- Financial analysts need to compute accurately partial derivatives of C.
- The dimension d may be large.
- More complicated contracts...

American Options

An American (vanilla) option on a given asset is the right to buy or sell this asset **before** the future date T and at the price K.

With the same assumptions as above and with a similar argument, one obtains a parabolic variational inequality instead of a pde :

$$
\left|
\begin{array}{l}
\partial_t P + \dfrac{\sigma_t^2 S^2}{2} \partial^2_{SS} P + r S \partial_S P - r P \leq 0, \qquad t \in [0, T[,\ S > 0, \\
P \geq (K - S)_+, \qquad t \in [0, T[,\ S > 0, \\
\left(\partial_t P + \dfrac{\sigma_t^2 S^2}{2} \partial^2_{SS} P + r S \partial_S P - r P \right) (P - (K - S)_+) = 0, \quad t \in [0, T[,\ S > 0, \\
P(T, S) = (K - S)_+, \qquad S > 0.
\end{array}
\right.
$$

The option should be exercised for (t, S_t) at the free boundary $S = \gamma(t)$:

$$
P(t, S) = (K - S)_+ \Leftrightarrow S \leq \gamma(t)
$$

The price $\gamma(t)$ is called **the price of exercise** at t.

How to choose the volatility σ ?

• Simplest models: σ_t constant. These models fail to match the prices actually observed on the markets.

• **Local volatility:** take $\sigma_t = \sigma(t, S_t)$, and try to **calibrate the volatility in order to recover the prices of the options available on the market**

• **Stochastic volatility:** let the volatility be also a stochastic process (Orstein-Uhlenbeck), and obtain a more sophisticated degenerate pde for $C(t, S, \sigma)$: see Stein and Stein, Fouque,Papanicolaou and Sircar, Y.A, Franchi and Tchou

Other models The price of the underlying asset is given by a more general Levy process. The Black and Scholes equation becomes an integro-partial differential equation : (see C. Schwab)

Notations

A **Vanilla American Option (put) with Strike** K **and Maturity** T is a right to sell an asset (the underlying asset) at price K before the date T.

- price of the underlying asset: x
- payoff function:

$$u_\circ(x) = (K - x)_+.$$

- interest rate r, (assumed constant for simplicity)
- local volatility : $\sigma(t,x)$. Denote

$$\eta(t,x) = \sigma^2(t,x).$$

- price of the option

$$u(t,x).$$

Calibration: Solve the Black and Scholes variational inequality for pricing the American put with volatility η. Look for η in order to recover the prices of the options available on the market.

Inverting time, we obtain: for $0 < t < T$ and $x > 0$

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\eta(t,x)x^2}{2}\frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru &\geq 0 \\
u &\geq u_\circ \\
\left(\frac{\partial u}{\partial t} - \frac{\eta(t,x)x^2}{2}\frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru\right)(u - u_\circ) &= 0,
\end{aligned}
$$

with Cauchy data

$$u(t = 0, x) = u_\circ(x)$$

and with

$$u_\circ(x) = (K - x)_+.$$

Content

1. Results on the Variational Inequality and the Free Boundary (Price for which the option should be exercised)
2. Sensitivity w.r.t. volatility variations.
3. The Calibration Problem: Optimality Conditions.
4. Differentiability : partial results.
5. First Numerical Results.

I) Analysis of the Variational Inequality

Hilbert space: $$V = \left\{ v \in L^2(\mathbb{R}_+), x\frac{\partial v}{\partial x} \in L^2(\mathbb{R}_+) \right\},$$

norm: $$\|v\|_V^2 = \|v\|_{L^2(\mathbb{R}_+)}^2 + \|x\frac{\partial v}{\partial x}\|_{L^2(\mathbb{R}_+)}^2.$$

V is separable and reflexive, and $\mathcal{D}(\mathbb{R}_+)$ is dense in V.

Closed Convex of $L^2(0,T;V)$: $\mathcal{K} = \{v \in L^2(0,T;V), v \geq u_\circ \text{ a.e. in } (0,T)\times\mathbb{R}_+\}$.

Variational Setting: find $u \in \mathcal{K} \cap C^0([0,T];L^2(\mathbb{R}_+))$, $\frac{\partial u}{\partial t} \in L^2(0,T;V')$
s.t.

$$< \frac{\partial u}{\partial t} + A(t)u, v - u > \quad \geq 0, \qquad \forall v \in \mathcal{K}, \qquad u(t=0) = u_\circ$$

with

$$< A(t)v, w >= \int_{\mathbb{R}_+} \left(\frac{\eta}{2} x^2 \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + (\eta + x\frac{\partial \eta}{\partial x} - r)x\frac{\partial v}{\partial x} w + rvw \right) dx.$$

Assumption I

For two positive constants $\underline{\eta} \leq \overline{\eta}$,

$$0 < \underline{\eta} \leq \eta \leq \overline{\eta} \quad \text{a.e.}$$

and for $M > 0$,

$$|x \frac{\partial \eta}{\partial x}| \leq M \quad \text{a.e.}$$

Gärding Inequality : there exists $\alpha \geq 0$ s.t. $\forall v \in V$,

$$< A(t)v, v > \quad \geq \quad \frac{1}{4}\underline{\eta}\|v\|_V^2 - \alpha\|v\|_{L^2(\mathbb{R}_+)}^2.$$

Theorem Under Assumptions I, the V.I. has a unique solution u and

- $u \in C^0([0,T] \times [0,+\infty))$, $u(t,0) = K$,
- $x\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \in L^2(0,T;V)$,
- $\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\mathbb{R}_+))$.
- $u \geq u_e$ (u_e price of the European put)
- The quantities $\|u\|_{L^2(0,T;V)}$, $\|u\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$, $\|x\frac{\partial u}{\partial x}\|_{L^2(0,T;V)}$, $\|\frac{\partial u}{\partial x}\|_{L^2(0,T;V)}$, $\|x\frac{\partial u}{\partial x}\|_{L^\infty(0,T;L^2(\mathbb{R}_+))}$, $\|\frac{\partial u}{\partial t}\|_{L^2(0,T;L^2(\mathbb{R}_+))}$ are bounded independently of η.
- $$-1 \leq \frac{\partial u}{\partial x} \leq 0, \quad \forall t \in (0,T], \text{ a.e. } x > 0.$$
- $$\frac{\partial u}{\partial t} + A(t)u = \mu, \quad \mu = rK1_{\{u=u_\circ\}}.$$

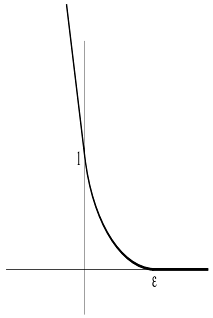
Principle of the proof : $\frac{\partial u_\circ}{\partial t} - \frac{\eta x^2}{2}\frac{\partial^2 u_\circ}{\partial x^2} - rx\frac{\partial u_\circ}{\partial x} + ru_\circ = rK1_{\{x<K\}}$

Penalized Problem : find u_ϵ s.t.

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} - \frac{\eta x^2}{2}\frac{\partial^2 u_\epsilon}{\partial x^2} - rx\frac{\partial u_\epsilon}{\partial x} + ru_\epsilon - rK1_{\{x<K\}}\mathcal{V}_\epsilon(u_\epsilon - u_\circ) &= 0 \\ u_\epsilon(t=0,x) &= u_\circ(x) \end{aligned}$$

where $\mathcal{V}_\epsilon$ **is a smooth non increasing convex function** s.t.

$$\begin{aligned} \mathcal{V}_\epsilon(x) &\geq 1, \quad x \leq 0; \\ \mathcal{V}_\epsilon(x) &= 0, \quad x \geq \epsilon \\ 0 \geq \mathcal{V}'_\epsilon(x) &\geq -\tfrac{2}{\epsilon} \quad 0 \leq x \leq \epsilon. \end{aligned}$$



Minty's trick for passing to the limit and maximum principles for all the estimates.

Other results

- $$u_\epsilon(t,x) - \epsilon \leq u(t,x) \leq u_\epsilon(t,x),$$

therefore $u_\epsilon \to u \quad \text{in} \quad C^0.$

- With $\mu = rK1_{\{u=u_\circ\}}$, the sequence $\mu_\epsilon = rK1_{x<K}\mathcal{V}_\epsilon(u_\epsilon - u_\circ)$ satisfies

$$\mu_\epsilon \to \mu \quad \text{in} \quad L^p((0,T)\times\mathbb{R}_+) \quad 1 \leq p < +\infty.$$

- $x\frac{\partial u_\epsilon}{\partial x} \to x\frac{\partial u}{\partial x} \quad \text{in} \quad C^0([0,T], L^2(\mathbb{R}_+)).$

Price of Exercise (Free Boundary)

Lemma Under Assumption I, there exists a function γ , $[0,T] \to [0,K]$, (called **price of exercise**), s.t.

$$\forall t \in (0,T), \qquad u(t,x) = u_\circ(x) \Leftrightarrow x \leq \gamma(t).$$

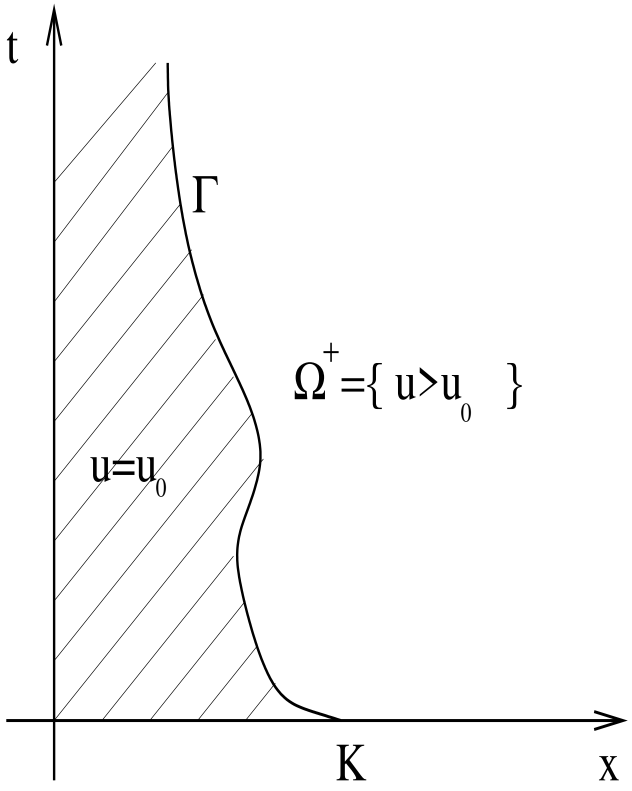
Proof comes from $-1 \leq \frac{\partial u}{\partial x} \leq 0$.

Assumptions II There exists $M > 0$ s.t.

$$|x^2 \frac{\partial^2 \eta}{\partial x^2}| + |\frac{\partial \eta}{\partial t}| + |x \frac{\partial^2 \eta}{\partial x \partial t}| \leq M, \quad \text{p.p.}$$

Theorem Under Assumptions I and II,

- **the price of exercise γ is continuous w.r.t. t.**
- The function $\mu = rK 1_{\{u=u_\circ\}}$ belongs to $C^0([0,T];L^p)$, $1 \leq p < +\infty$, and $\frac{\partial u}{\partial t} \in C^0([0,T];V')$.



In order to prove the previous result, we need the

Lemma : Under Assumptions I and II,

$$\frac{\partial^2 u}{\partial x^2} \geq 0 \quad \text{a.e.}$$

Theorem There exists $\gamma_0 > 0$ s.t.

$$\gamma(t) \geq \gamma_0, \quad \forall t \in [0, T].$$

Idea of the proof of $\gamma(t) \geq \gamma_0$

- For the solution $\tilde{u}$ corresponding to $\eta = \bar{\eta}$, we have $u \leq \tilde{u}$.
- The exercise price $\tilde{\gamma}$ for $\tilde{u}$ is such that $\tilde{\gamma} \leq \gamma$, $\forall t$.
- $\tilde{\gamma}$ is continuous and non increasing (because $\tilde{u}$ in non increasing w.r.t. t).

But $\tilde{\gamma}$ cannot vanish for t_0, because if it was so, $\tilde{\gamma}(t) = 0$ for $t \geq t_0$ and $\tilde{u}$ would satisfy the Black and Scholes eq. for $t > t_0$, which would imply $\tilde{u}(t, 0) < K$. This contradicts $\tilde{u} \geq u$.

Continuity of γ : Idea of the proof

If γ was monotone w.r.t t, then the continuity would be easy to obtain (see Kinderlehrer-Stampacchia).

But γ is not always monotone w.r.t t because η is not monotone w.r.t t.

We are going to introduce auxiliary problems for which the free boundary is the graph of a monotone and continuous function and make comparisons.

Three Steps

1.
$$\limsup_{\tau \to t} \gamma(\tau) \leq \gamma(t)$$

   by contradiction, using the fact that u is continuous.

2.

$$\liminf_{\tau\to t^+} \gamma(\tau) = \gamma(t), \quad \text{and} \quad \liminf_{\tau\to t^-} \gamma(\tau) = \limsup_{\tau\to t^-} \gamma(\tau) \leq \gamma(t) :$$

indeed for $\tau > t$,

$$u(\tau) \leq K - \gamma(t) + \tilde{u}(\tau),$$

where

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial \tau} - \bar{\eta}\frac{x^2}{2}\frac{\partial^2 \tilde{u}}{\partial x^2} - rx\frac{\partial \tilde{u}}{\partial x} + r\tilde{u} &\geq 0 \\
\tilde{u} &\geq (\gamma(t) - x)_+ \\
(\frac{\partial \tilde{u}}{\partial \tau} - \frac{x^2}{2}\bar{\eta}\frac{\partial^2 \tilde{u}}{\partial x^2} - rx\frac{\partial \tilde{u}}{\partial x} + r\tilde{u})(\tilde{u} - (\gamma(t) - x)_+) &= 0, \\
\tilde{u}(\tau = t, x) &= (\gamma(t) - x)_+.
\end{aligned}$$

Thus $\tilde{\gamma}(\tau) \leq \gamma(\tau)$, and $\tilde{\gamma}$ est continuous $\Rightarrow \liminf_{\tau\to t^+} \gamma(\tau) = \gamma(t)$

3.

$$\lim_{\tau \to t^-} \gamma(\tau) = \gamma(t)$$

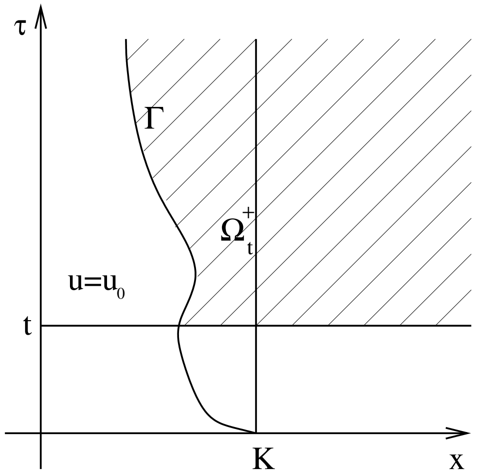
by contradiction, using the strong maximum principle on $\frac{\partial^2 u}{\partial x^2}$ along with the estimate $\frac{\partial^2 u}{\partial x^2} \geq 0$.

Open Question Is γ Lipschitz for $t > 0$?

- If η is monotone w.r.t. t, yes

(and stronger results : L. Caffarelli)

- Else, ? (cf R. Monneau).



If γ is Lipschitz, we have the following result which is important to study sensitivity of γ w.r.t. η :

Proposition : strong convexity of u

If η is s.t., for all t, $0 < t \leq T$,

- $\gamma|_{[t,T]}$ is Lipschitz
- $\frac{\partial u}{\partial t} \in H^1(\Omega_t^+)$ where $\Omega_t^+ = \{u(\tau, x) > u_0, \tau \geq t\}$,

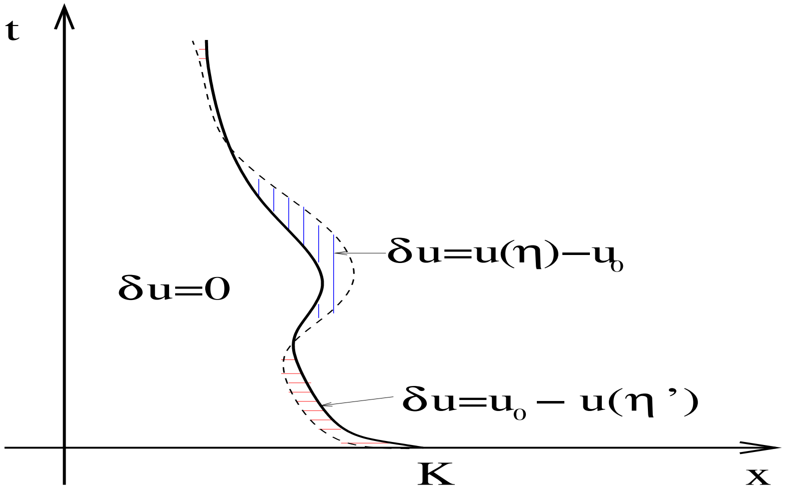
then for t, $0 < t < T$, $\exists m_t > 0$ s.t.

$$\frac{\partial^2 u}{\partial x^2} \geq m_t \quad \text{in } \Omega_t^+ \cap \{x < K\}$$

II) Sensitivity w.r.t. variations of η : Call $u(\eta)$ the price corresp. to η:

Proposition There exists C s.t. for η and η' in the class defined by Assumption I,

- $\|u(\eta)-u(\eta')\|_{L^2(0,T;V)}+\|u(\eta)-u(\eta')\|_{L^\infty(0,T;L^2(\mathbb{R}_+))} \leq C\|\eta-\eta'\|_{L^\infty((0,T)\times\mathbb{R}_+)}$
- $\displaystyle\int_0^T\int_0^K 1_{\{u(\eta)=u_\circ\}}(u(\eta')-u_\circ)+1_{\{u(\eta')=u_\circ\}}(u(\eta)-u_\circ)dtdx \leq C\|\eta-\eta'\|^2_{L^\infty((0,T)\times\mathbb{R}_+)}.$
- If $\|\eta_k-\eta\|_{L^\infty} \to 0$ then $\|1_{\{u(\eta_k)=u_\circ\}}-1_{\{u(\eta)=u_\circ\}}\|_{L^p} \to 0$, for $p<\infty$



Proposition : Sensitivity of the price of exercise

For η and η' in the class defined by Assumptions I and II, and if for all t, $0 < t \leq T$,

- $\gamma(\eta)|_{[t,T]}$ is Lipschitz
- $\frac{\partial u(\eta)}{\partial t} \in H^1(\Omega_t^+)$ with $\Omega_t^+ = \{u(\tau, x) > u_0, \tau \geq t\}$.

Then for t, $0 < t \leq T$, there exists $c_t > 0$ s.t.

$$\|(\gamma(\eta') - \gamma(\eta))_+\|^3_{L^3(t,T)} \leq c_t \|\eta - \eta'\|^2_{L^\infty((t,T)\times\mathbb{R}_+)}$$

Proof: uses the previous result $\frac{\partial^2 u}{\partial x^2} > m_t > 0$.

Remark This result will be important for differentiability of u w.r.t. η.

III) The Calibration Problem

Observations:

- the price x_o of the underlying asset today,
- the prices $(\bar{u}_i)_{i\in I}$ of a family of American puts with different maturities and different strikes $(T_i, K_i)_{i\in I}$. We call $T = \max_{i\in I} T_i$.

Find $\eta \in \mathcal{H}$ minimizing

$$J(\eta) + J_R(\eta), \quad \text{with} \quad J(\eta) = \sum_{i\in I} |u_i(T_i, x_\circ) - \bar{u}_i|^2$$

where

$$\frac{\partial u_i}{\partial t} - \frac{\eta(T_i - t, x)x^2}{2}\frac{\partial^2 u_i}{\partial x^2} - rx\frac{\partial u_i}{\partial x} + ru_i \geq 0 \quad t < T_i, x > 0$$

$$u_i \geq (K_i - x)_+$$

$$\left(\frac{\partial u_i}{\partial t} - \frac{\eta(T_i - t, x)x^2}{2}\frac{\partial^2 u_i}{\partial x^2} - rx\frac{\partial u_i}{\partial x} + ru_i\right)(u_i - (K_i - x)_+) = 0,$$

$$u_i(t = 0, x) = (K_i - x)_+, \quad x > 0$$

Space for η?

For Tychonoff regularization, we use the spaces

$$X = \{\eta \in L^2(\mathbb{R}_+); (x+1)\frac{\partial \eta}{\partial x} \in L^2(\mathbb{R}_+)\}.$$

We have $X \subset L^\infty(\mathbb{R}_+) \cap C^0(\mathbb{R}_+)$ with continuous imbedding and $X \subset L^\infty(\mathbb{R}_+)$ is compact.

$$Y = \{\eta : \eta, x\frac{\partial \eta}{\partial x} \in H^1(0,T;X)\} \subset \{\eta : \eta, x\frac{\partial \eta}{\partial x} \in L^\infty((0,T) \times \mathbb{R}_+)\}.$$

Let $\mathcal{H}$ be the closed convex subset of Y:

$$\mathcal{H} = \left\{\eta \in Y, \underline{\eta} \leq \eta \leq \overline{\eta}, \quad |x\frac{\partial \eta}{\partial x}| \leq M\right\}$$

First Order Optimality Condition

For simplicity only, we suppose that there is only one observation.

$$J(u) = J_R(\eta) + (u(T, x_{ob}) - \bar{u})^2,$$

It is possible to reformulate the V.I. as the identity $\frac{\partial u}{\partial t} + A(t)u = \mu$, with the constraints on u and μ : $u \geq u_\circ$, $\mu \geq 0$, and $< \mu, u - u_\circ >= 0$. Then, one may think to use **Lagrange multipliers** for the pb. in (η, u, μ).

But as observed by M. Bergougnioux et al for optimal control, **it is not generally not possible to find optimality conditions with as many Lagrange multipliers as constraints,** for the obtained system has no solution.

Therefore, **we look first for optimality conditions for the penalized pb., then we pass to the limit.**

Optimality Conditions for the Penalized Problem

Lemma Assume that there exits a positive number a such that $u^*_\epsilon(t,x) > u_\circ(x) + \epsilon$ for all (t,x) with $|x - x_{ob}| \le a$ and $T - t \le a$.
There exists a unique $p^*_\epsilon \in L^2(0,T,L^2(\mathbb{R}_+))$ s.t., in a very weak sense,

$$\frac{\partial p^*_\epsilon}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\eta^*_\epsilon x^2}{2} p^*_\epsilon\right) - \frac{\partial}{\partial x}(rxp^*_\epsilon) - rp^*_\epsilon + rK1_{\{x<K\}}\mathcal{V}'_\epsilon(u^*_\epsilon - u_\circ)p^*_\epsilon = 0$$
$$p^*_\epsilon(T) = -2(u^*_\epsilon(T,x_{ob}) - \bar{u})\delta_{x_{ob}}$$

The optimality conditions for the penalized problem are

- u^*_ϵ satisfies the penalized problem for $\eta = \eta^*_\epsilon$.
- there exists an **adjoint state** $p^*_\epsilon \in L^2(0,T;L^2(\mathbb{R}_+))$ solution of the problem above .
- For all $\eta \in \mathcal{H}$,

$$< DJ_R(\eta^*_\epsilon), \eta - \eta^*_\epsilon > + \int_0^T \int_{\mathbb{R}_+} \frac{x^2}{2}(\eta - \eta^*_\epsilon)p^*_\epsilon \frac{\partial^2 u^*_\epsilon}{\partial x^2}\, dt\, dx \ge 0.$$

Theorem Let $\epsilon_n \to 0$ and $(\eta^*_{\epsilon_n}, u^*_{\epsilon_n})$ be a sequence of solutions to the penalized problem converging to (η^*, u^*). Assume that $\exists a > 0$ s.t. $u^*_{\epsilon_n}(t,x) > u_\circ(x) + \epsilon_n$ for all (t,x) with $|x - x_{ob}| \le a$ and $T - t \le a$. There exists a subsequence n_k such that $p^*_{\epsilon_{n_k}} \rightharpoonup p^*$ in $L^2(0,T,L^2(\mathbb{R}_+))$ and a Radon measure α^* s.t., in a very weak sense,

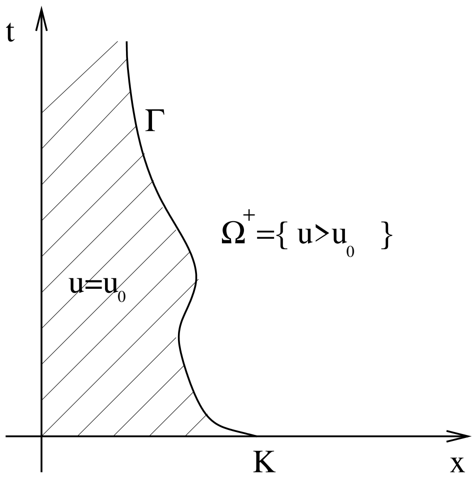
$$\frac{\partial p^*}{\partial t} + \frac{\partial^2}{\partial x^2}(\frac{\eta^* x^2}{2} p^*) - \frac{\partial}{\partial x}(rxp^*) - rp^* - \alpha^* = 0$$
$$p^*(T) = -2(u^*(T, x_{ob}) - \bar{u})\delta_{x_{ob}}$$

Furthermore we have

$$\mu^* |p^*| = 0, \quad |u^* - u_\circ| \alpha^* = 0$$

Finally, for any $\eta \in \mathcal{H}$,

$$< DJ_R(\eta^*), \eta - \eta^* > + \int_0^T \int_{\mathbb{R}_+} \frac{x^2}{2}(\eta - \eta^*) p^* \frac{\partial^2 u^*}{\partial x^2}\, dt\, dx \ge 0.$$



Interpretation If γ^* is Lipschitz, one can show that the adjoint state p^* is the solution of the backward parabolic PDE in the non-cylindrical domain Ω^+ :

$$\frac{\partial p^*}{\partial t} + \frac{\partial^2}{\partial x^2}\left(\frac{\eta x^2}{2} p^*\right) - \frac{\partial}{\partial x}(rxp^*) - rp^* = 0, \quad (t,x) \in \Omega^+$$

$$p^*(T,x) = -2(u^*(T,x_{ob}) - \bar{u})\delta_{x_{ob}} \quad x > \gamma^*(T),$$

$$p^*(t,\gamma^*(t)) = 0$$

and the singular part of the measure α^* is the singular measure

$$< \alpha^*_{\text{sing}}, v >= \quad \frac{1}{2}\int_t (\gamma^*(t))^2 \left(\eta^* \frac{\partial p^*}{\partial x} v\right)(t,\gamma^*(t))dt.$$

IV) A partial result on differentiability For numerical optimization, it is important to compute the differential of J : for that, we assume that η belongs to the smaller class:

$$Y = \left\{ \eta : \begin{array}{l} \eta, x\frac{\partial \eta}{\partial x}, x^2\frac{\partial^2 \eta}{\partial x^2}, \frac{\partial \eta}{\partial t}, x\frac{\partial^2 \eta}{\partial t \partial x} \in H^1(0,T;X), \\ \eta|_{t<t_0} = \eta_0 \end{array} \right\}$$

and $\mathcal{H}$ is the set of functions of Y verifying assumptions and I et II.

Theorem Assume that $\forall t > 0$, $\gamma(\eta)|_{(t,T]}$ is Lipschitz, et $\frac{\partial u(\eta)}{\partial \tau} \in H^1(\Omega_t^+)$. The functional $\eta \mapsto J(u(\eta))$ **is differentiable w.r.t.** η**,** and its differential is

$$\xi \mapsto \int_0^T \int_{\mathbb{R}_+} \xi \frac{x^2}{2} p(\eta) \frac{\partial^2 u(\eta)}{\partial x^2}$$

Proof Uses the previous results on sensitivity of u and η w.r.t. η.

Idea of the proof

$\delta\eta$ admissible variation of $\eta \to (\delta u, \delta\mu)$ variation of (u, μ)

From sensitivity estimates, we have

$$< \frac{\partial \delta u}{\partial t} + A_\eta(t)\delta u, w > + \int \frac{x^2}{2}\delta\eta \frac{\partial u(\eta)}{\partial x}\frac{\partial w}{\partial x} + (x\delta\eta + \frac{x^2}{2}\frac{\partial \delta\eta}{\partial x})\frac{\partial u(\eta)}{\partial x} w - < \delta\mu, w >= o(\delta\eta).$$

$$< \mu, \delta u > + < \delta\mu, u(\eta) - u_\circ > + < \delta\mu, \delta u >= 0 \quad \text{and} \quad < \delta\mu, \delta \mathrm{u} > \leq \mathrm{C}\|\delta\eta\|^2_{\mathrm{L}^\infty((0,\mathrm{T})\times\mathbb{R}_+)}$$

From the definition of the adjoint state,

$$< DJ(u(\eta)), \delta u >=$$
$$\int \frac{x^2}{2}\delta\eta \frac{\partial u(\eta)}{\partial x}\frac{\partial p}{\partial x} + (x\delta\eta + \frac{x^2}{2}\frac{\partial \delta\eta}{\partial x})\frac{\partial u(\eta)}{\partial x} p - < \delta\mu, p > + < \alpha, \delta u > + o(\delta\eta).$$

We need to prove that $< \alpha, \delta u >$ and $< \delta\mu, p >$ are $o(\delta\eta)$

The term $<\delta\mu, p>$

$$\begin{array}{rcl} |<\delta\mu,p>| & = & rK\left|\int_{t=t_0}^{T} 1_{\delta\gamma(t)>0}\int_{\gamma(t)}^{\gamma(t)+\delta\gamma(t)} p\right| \\ & \leq & \int_{t=t_0}^{T} 1_{\delta\gamma(t)>0}\sqrt{\delta\gamma(t)}\sqrt{\int_{\gamma(t)}^{\gamma(t)+\delta\gamma(t)} p^2} \\ & \leq & \int_{t=t_0}^{T} 1_{\delta\gamma(t)>0}(\delta\gamma(t))^{\frac{3}{2}}\sqrt{\int_{\gamma(t)}^{\gamma(t)+\delta\gamma(t)} (\frac{\partial p}{\partial x})^2}. \end{array}$$

and we use

$$\|\delta\gamma_+\|^3_{L^3(t_0,T)} \leq c_{t_0}\|\delta\eta\|^2_{L^\infty((t_0,T)\times\mathbb{R}_+)}$$

and a variant of Lebesgue's theorem.

V) Numerical Aspects

• for $J_R(\eta)$, Tychonoff regularization + quadratic deviation from an a priori volatility.

• Difficulty of the problem : as many V.I. and Adjoint problems as Observations.

Discretization

• We work on a truncated domain $Q = (0, \bar{T}) \times (0, \bar{x}_{\max})$. We impose $u(x_{\max}) = 0$. Not optimal: (**later**, use a transparent boundary condition)

• Singularity around $x = K_i$: $\Rightarrow$ refinement near those points $\Rightarrow$ finite Element with non uniform mesh for u. For the time scheme, backward Euler so far, (**later**, order 2).

• Similarly, Q_1 finite element with non uniform mesh for η or **splines**.

• The mesh for u is a refinement of the one for η (20 to 100 times finer)

Digression : transparent boundary condition Consider the heat equation

$\partial_t u - \partial_{xx} u = 0,\ x > a, 0 < t < T,\ \ u(t,a) = g(t),\ 0 < t < T,\ \ u(0,x) = 0$

Extend by $u(t,x) = g(t)$ in $x < a \rightarrow$ heat equation in $(0,T) \times \mathbb{R}$:

$u(x,t) = \int_0^t (g'(s) \int_{-\infty}^a G(x-y, t-s) dy) ds + \int_0^t q(s) G(t-s, x-a) ds$

where G is the Green kernel of the heat equation and $q(t) = \partial_x u(t,a)$. This yields on $x = a$:

$$u(a) = -\sqrt{\frac{2}{\pi}} \int_0^t \partial_x u(a) \frac{ds}{\sqrt{t-s}}$$

Consider now the following problem

$$\partial_t v - \partial_x \left(\sigma(t,x)\partial_x v\right) = 0,\ x \in \mathbb{R}, 0 < t < T,\ \ v(0,x) = v_0(x)\ x \in \mathbb{R}$$

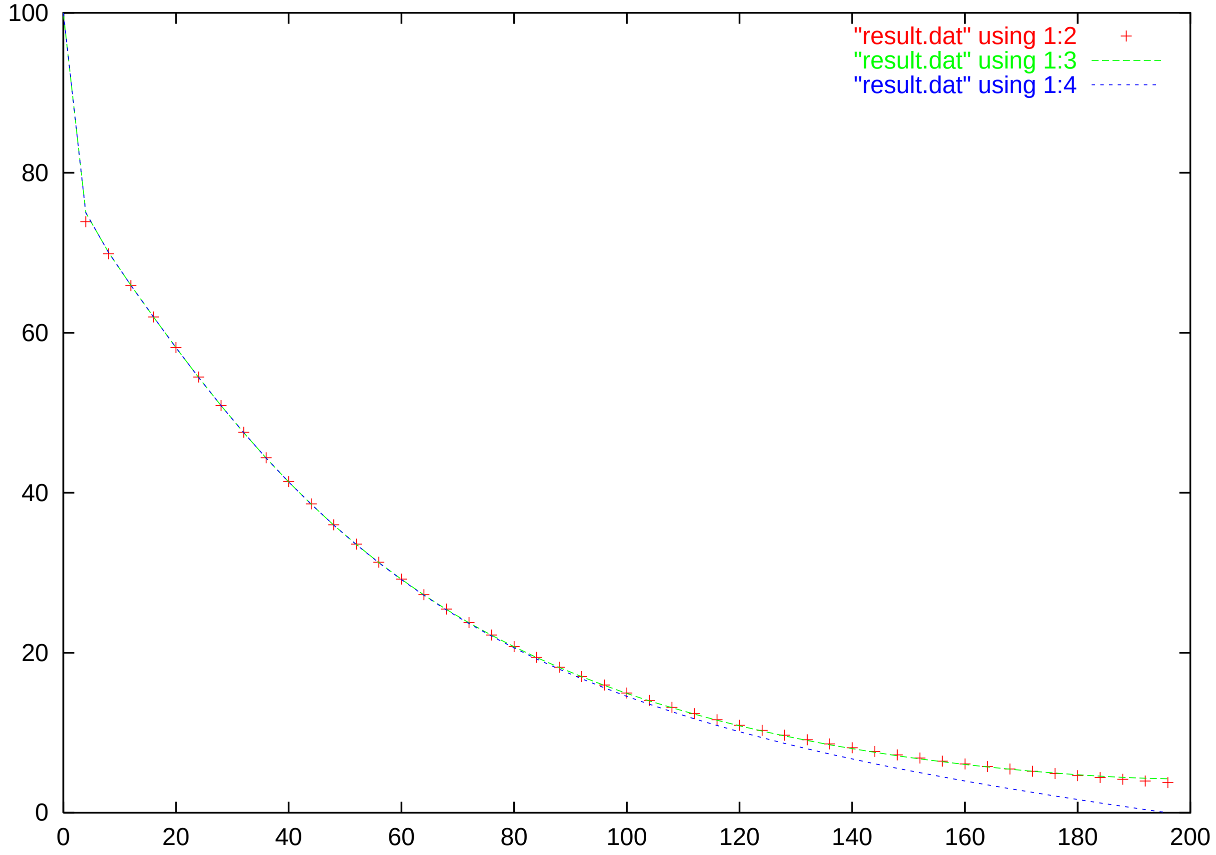
where σ is such that $\sigma(t,x) = \sigma_0$ if $x \geq a$ and where $v_0(x) = 0$ if $x \geq a$; then the problem is equivalent to

$$\begin{aligned}
&\partial_t v - \partial_x \left(\sigma(t,x)\partial_x v\right) = 0,\ x < a, 0 < t < T,\\
&v(a) = -\sqrt{\tfrac{2}{\pi}}\sigma_0 \textstyle\int_0^t \partial_x v(a) \frac{ds}{\sqrt{t-s}}\\
&v(0,x) = v_0(x)\ x < a
\end{aligned}$$

$$v(t_{n+1}, a) = -\sqrt{\frac{2}{\pi}} \int_0^{t_{n+1}} \partial_x v(s,a) \frac{ds}{\sqrt{t-s}}$$

can be discretized by

$$\sqrt{\frac{\pi}{2}} v(t_{n+1}, a) + 2\sqrt{\delta t}\partial_x v(t_{n+1}, a) = -\int_0^{t_{n+1}-\delta t} \partial_x v(s,a) \frac{ds}{\sqrt{t-s}}$$



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"result.dat" using 1:3
"result.dat" using 1:4
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80
100
120
140
160
180
200

At each time step, and for each observation, one needs to solve a discrete V.I.

$$(V - U_i^{n+1})^T \left((M + \Delta t A_{n+1})U_i^{n+1} - MU_i^n\right) \geq 0, \quad \forall V \in \mathcal{K}_i,$$
$$U_i^{n+1} \in \mathcal{K}_i \Leftrightarrow u_i(t^{n+1}, x_j) \geq (K_i - x_j)_+$$

This problem is solved **exactly** by an algorithm for tracking the free boundary, i.e. one looks for $\gamma_i^{n+1} = \gamma_i(t_{n+1})$

Advantages

- **Dramatically faster** (50 to 100) than an iterative method like PSOR.
- **Yields the exact solution**, which is important in view of the calibration problem.

Algorithm for $Au \geq f; \quad u \geq u_0; \quad (u-u_0)^T(Au-f)=0$. Since the free boundary is the graph of a function : at each time step (implicit scheme) move the point left/right according to which constraint is violated.

choose a γ; set found=false;

while(**not** found)

. solve $Au(x)=f(x)$ pour $x>\gamma$ with b.c. $u(\gamma)=u_0(\gamma)$

. **if** $(u(\gamma+h)<u_0(\gamma+h))$

. found=false; $\gamma=\gamma+h$;

. **else**

. compute $r=(Au-f)|_{(\gamma-h)}$;

. **if** $(r<0)$

. found=false; $\gamma=\gamma-h$;

. **else** found=true

If the mesh is fine enough, and if the above algorithm for γ^{k+1} is initialized with γ^k, the algorithm needs in average less than one iteration per time step

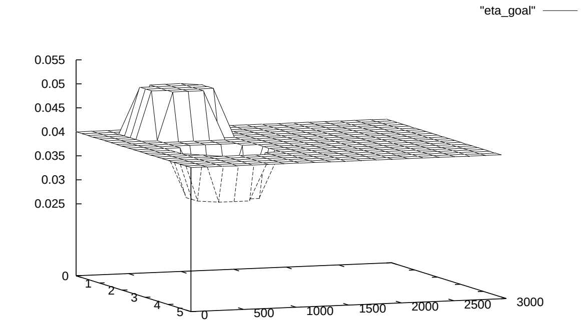
Optimization Algorithm

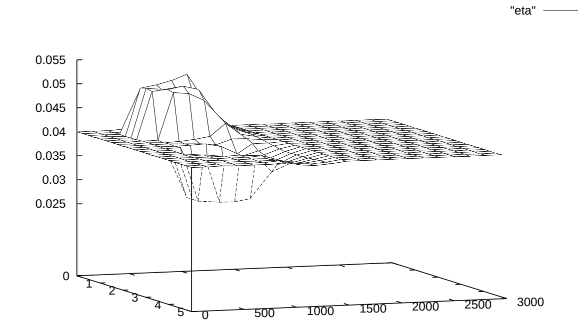
- Quasi-Newton with exact computation of grad$J$.
- For the cost function : N_I discrete V.I. solves + regularization function.
- To compute the gradient:
  - Solve N_I discrete adjoint problem :

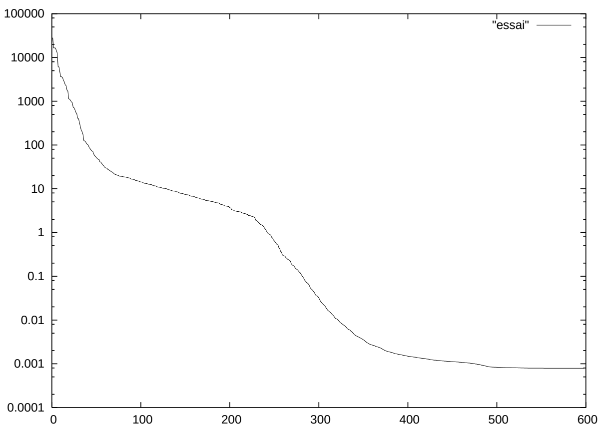
$$\begin{cases} \left[(M+\Delta t A_n^T)P_i^n - MP_i^{n+1}\right]_j = 0 & \text{si} \quad u_i(t^n,x_j) > (K_i - x_j)_+, \\ p_i(t^n,x_j) = 0 & \text{si} \quad u_i(t^n,x_j) = (K_i - x_j)_+, \\ p_i(T_i,x_j) = 2\delta_{x_j=x_0}(u_i(T_i,x_j) - \bar{u}_i) & \end{cases}$$

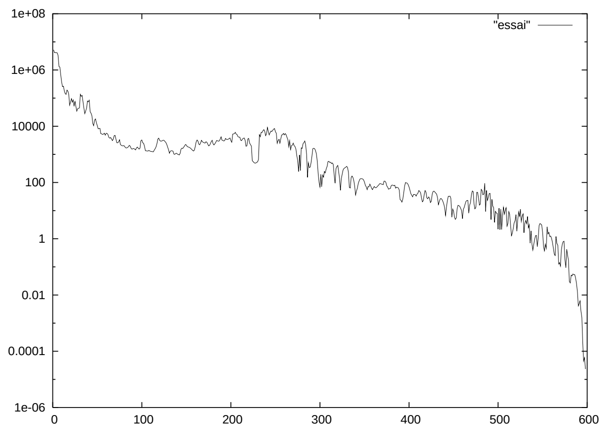
- **The N_I direct and adjoint problems are solved in parallel with N_K procs. Uses the Message Passing Interface (MPI)**

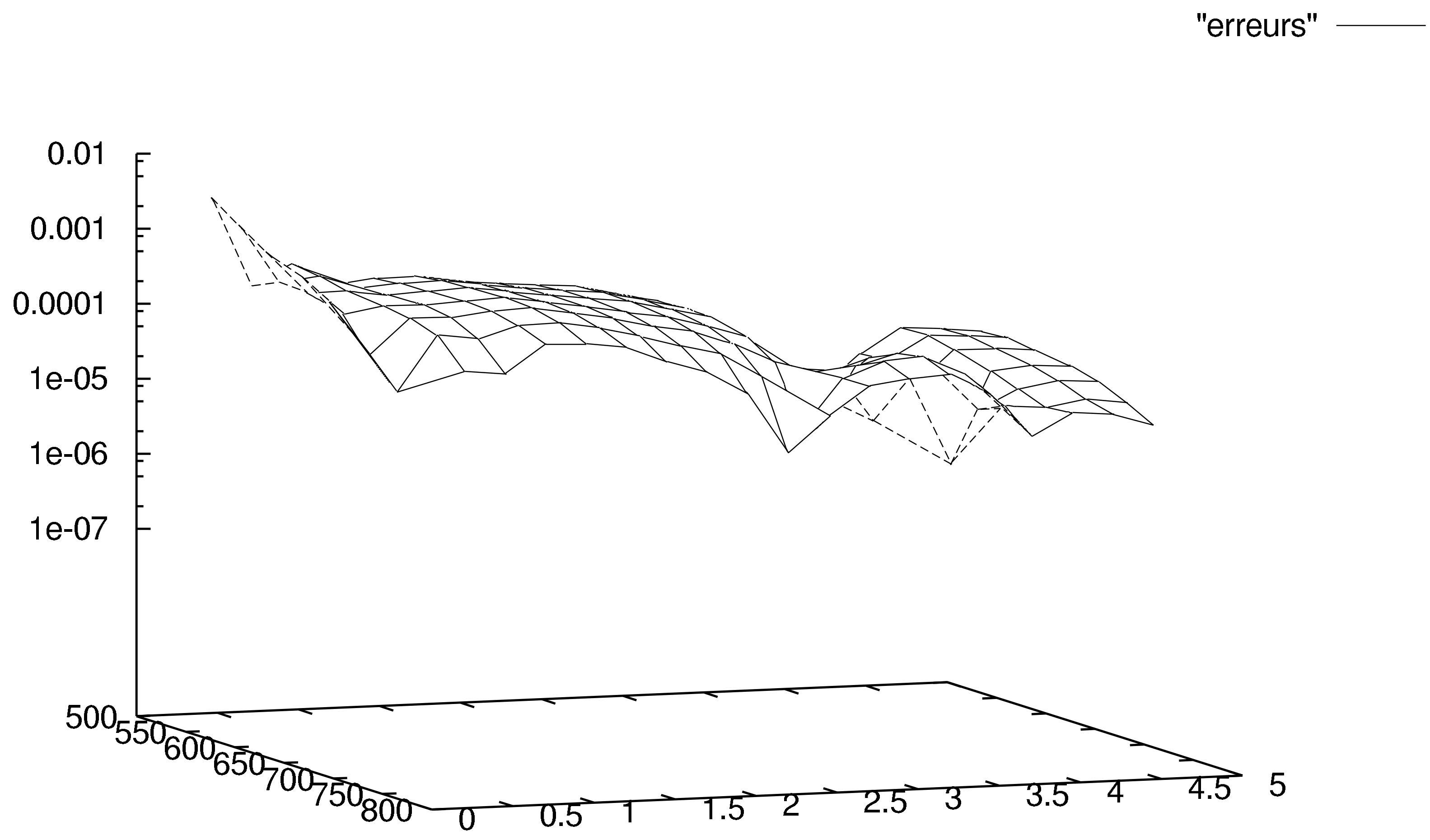
Numerical results for a test problem











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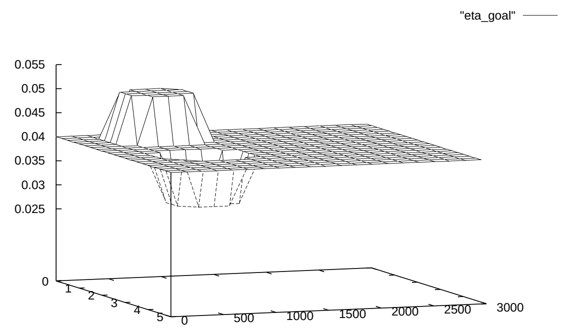
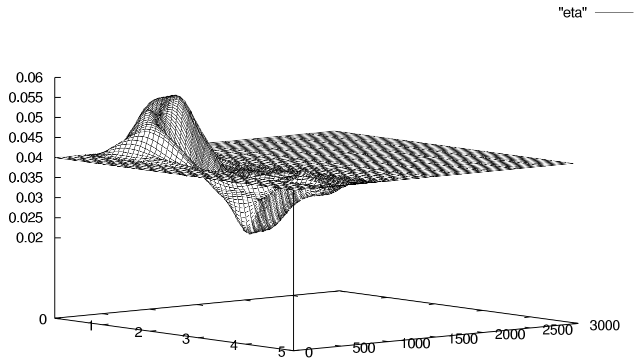
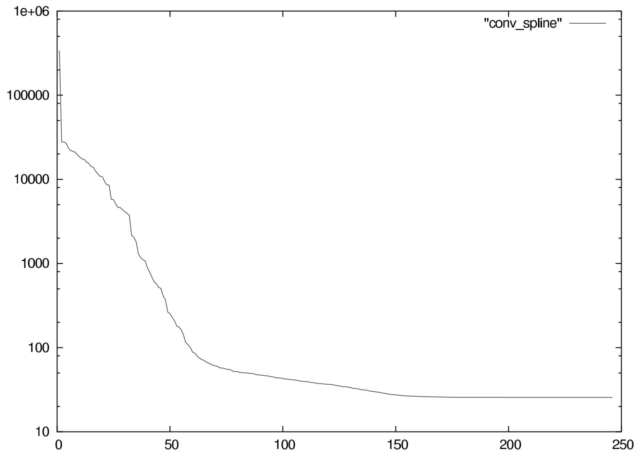
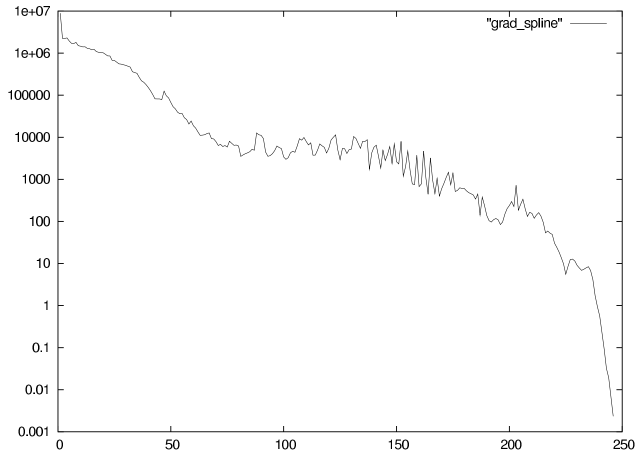
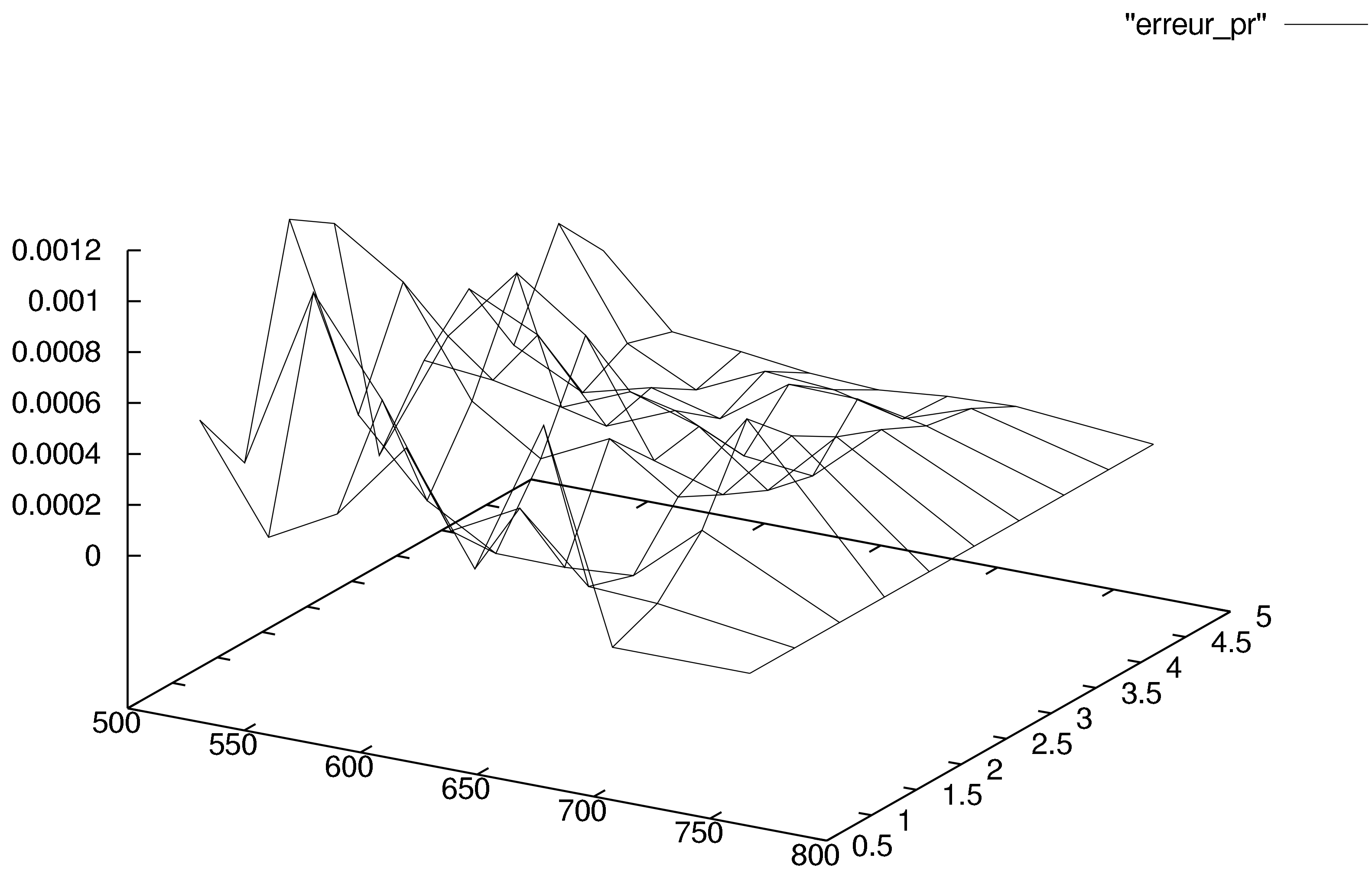
With BiCubic Splines







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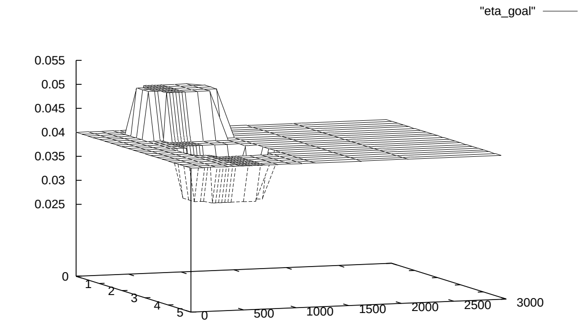
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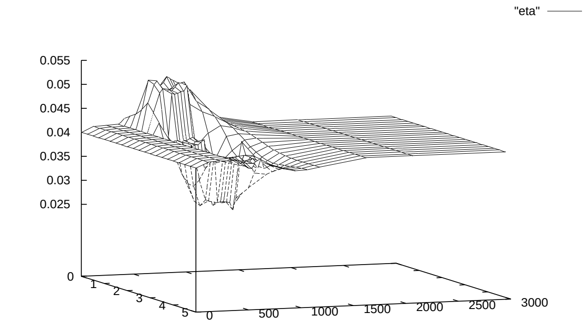
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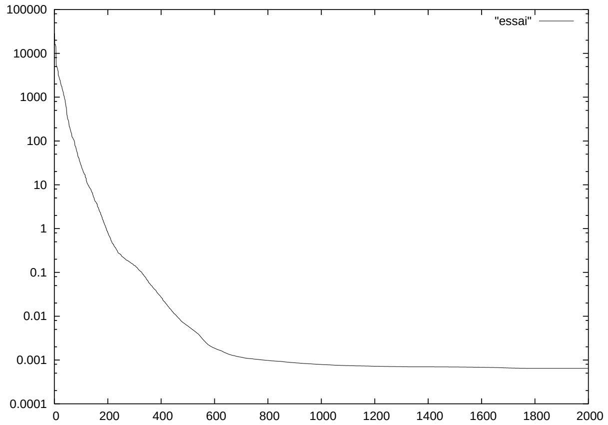
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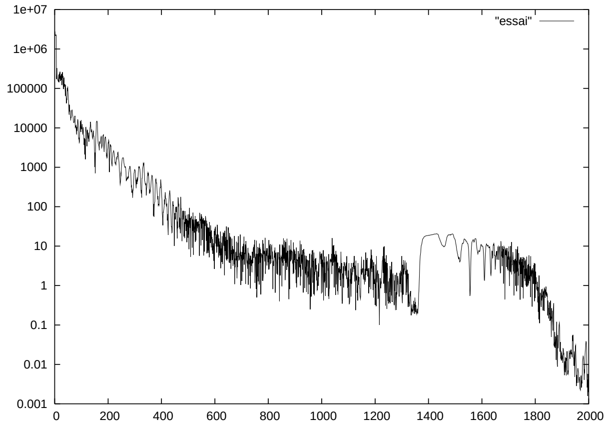
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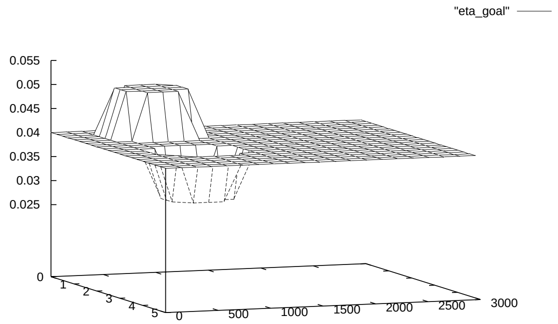
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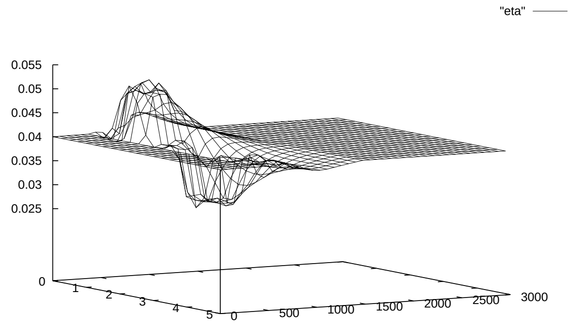
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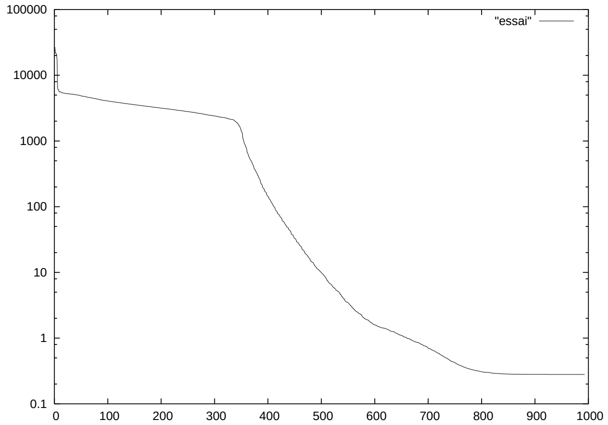
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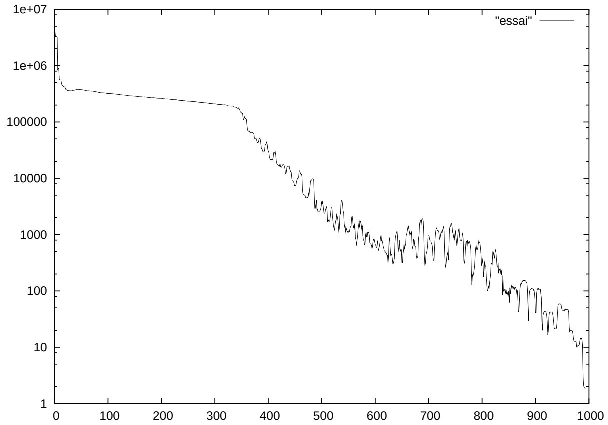
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