SPDEs and Level Sets

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Plan

1. Level Sets Method (LSM)
   1. Overview
   2. Main advantages
   3. Applications to Computer Vision
2. Adding Stochastic Perturbations to LSM-based Shape Optimization Algorithms
   1. Results
   2. Mathematical Elements

Level Sets Method

$\Gamma(t) \subset \mathbb{R}^n$ interface

$\Omega(t) \subset \mathbb{R}^n$ such that $\Gamma(t) = \partial\Omega(t)$

Consider $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\Gamma(t) = \{x \in \mathbb{R}^n : u(t,x) = 0\}$$

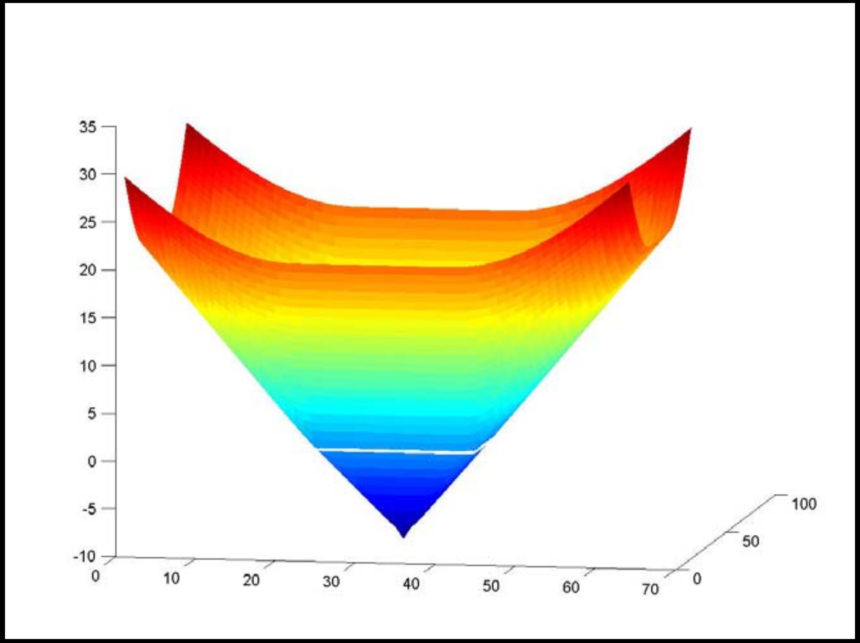
$$u(t,x) < 0 \qquad \forall\, x \in \Omega(t)$$

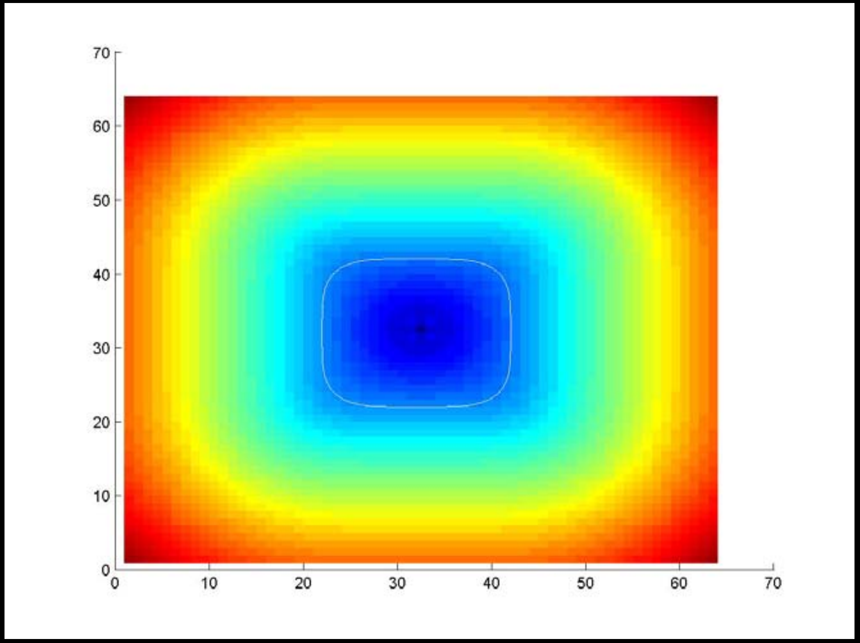
$$u(t,x) > 0 \qquad \forall\, x \notin \overline{\Omega(t)}$$

$$\frac{\partial\Gamma}{\partial t} = \beta\mathbf{n} \qquad \Leftrightarrow \qquad du = \beta|Du| \qquad \beta \text{ intrinsic}$$

Example of Level Sets Evolution

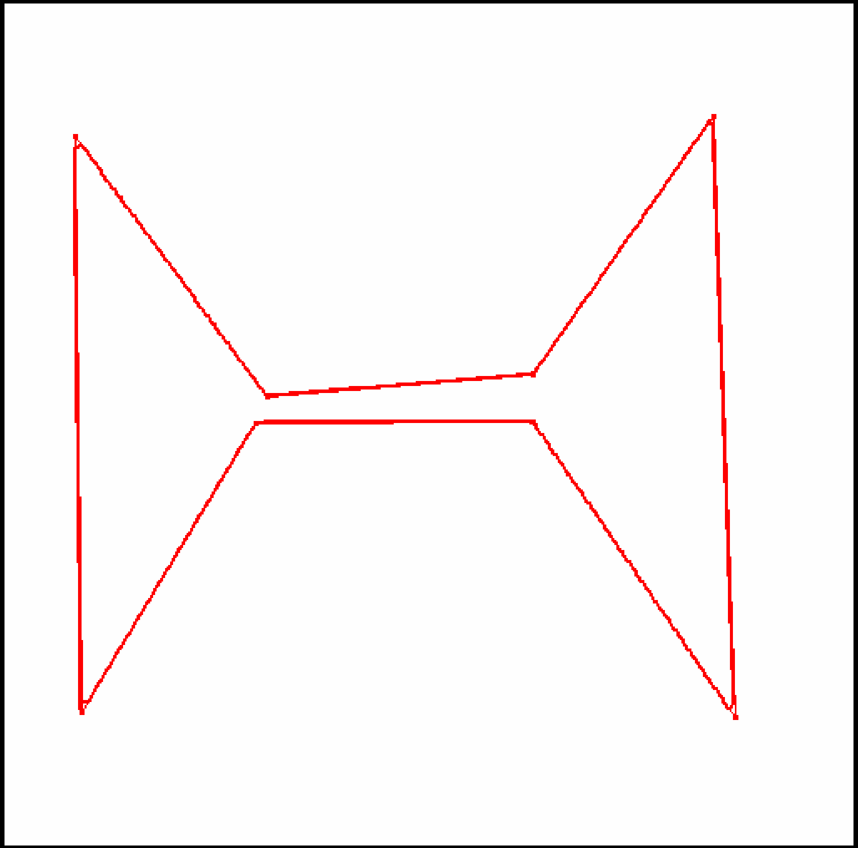
$$\frac{\partial \Gamma}{\partial t} = k\mathbf{n} \qquad du = |Du| \, \mathrm{div}\left(\frac{Du}{|Du|}\right)$$





Main advantages

- Handles automatically topological changes
- Robust mathematical theory behind – viscosity solutions
- Stable numerical schemes
- Natural extensions to higher dimensions



Applications to Computer Vision

- Mean Curvature Motion
- Shape Optimisation
  - Active Contours
  - Active Regions
  - Adaptative Active Regions

Mean Curvature Motion

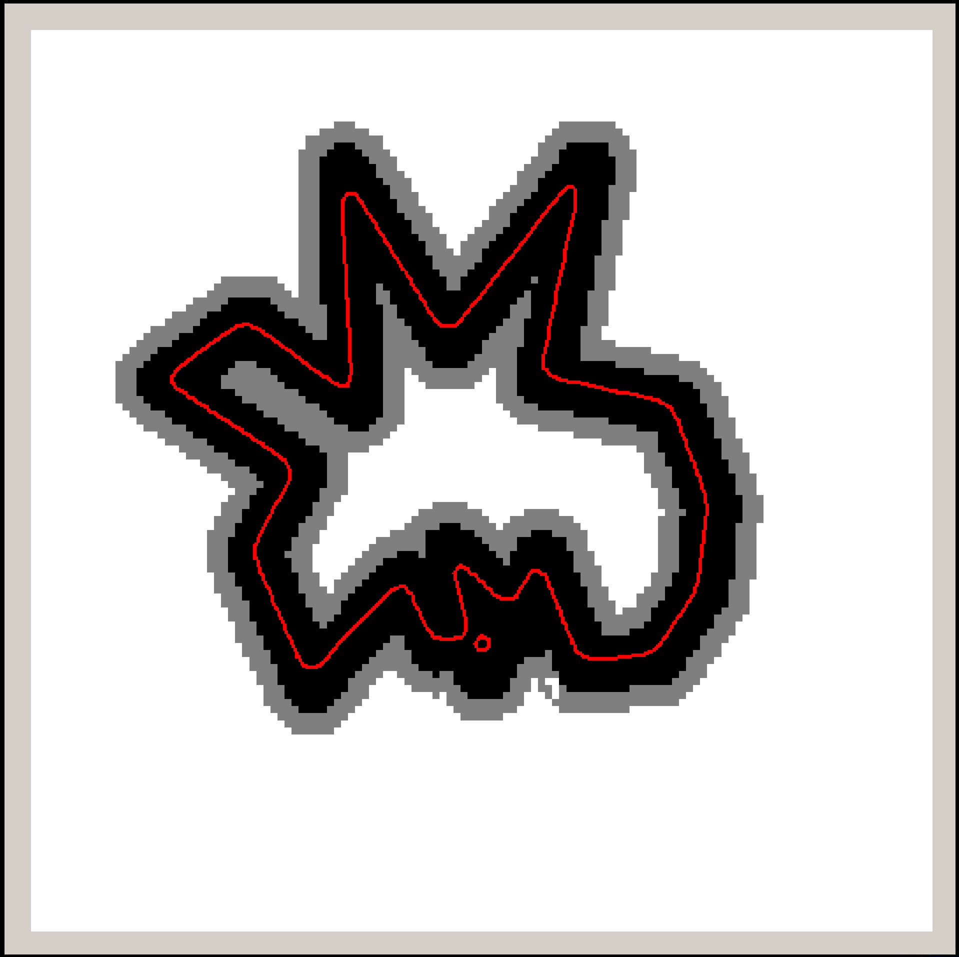
Isotropic smoothing of a curve in the Euclidian plane

Interface Evolution

$$\frac{\partial \Gamma}{\partial t} = \frac{\partial^2 \Gamma}{\partial v^2}\mathbf{n} = k\mathbf{n}$$

Corresponding implicit function evolution

$$du = |Du| \,\mathrm{div}\left(\frac{Du}{|Du|}\right)$$



Active Contours

Energy to minimize $$\int_{\Gamma(t)} g(|DI(p)|)dp$$

Euler-Lagrange equation $$\frac{\partial \Gamma}{\partial t} = g(I)k\mathbf{n} - (Dg \cdot \mathbf{n})\mathbf{n}$$

Corresponding evolution of the implicit function $$du = g(I)\,|Du|\,k + Dg(I) \cdot Du$$

Active Regions

Paragios Deriche 1999

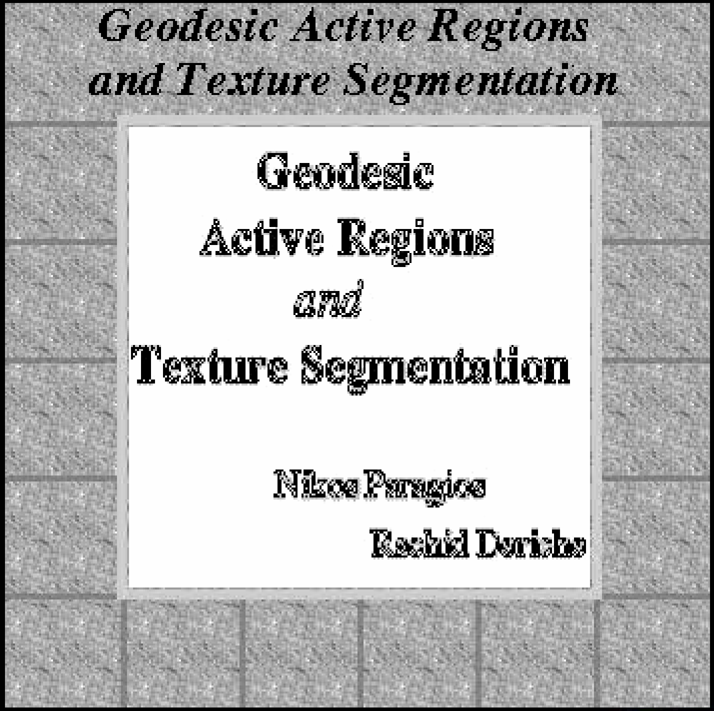
Feature extraction step – supervised

Energy to minimize

$$(1-\alpha)\int_{\Gamma(t)} g\left(p\left(I(m)\right)\right)dm$$

$$+\alpha\Big[\int_{\Omega(t)}\log(p_A(I(m)))dm$$

$$+\int_{\Omega^C(t)}\log(p_B(I(m)))dm\Big]$$



Unsupervised Active Regions

Rousson Brox Deriche 2003

Segment an image in 2 regions, called generically the interior and the exterior, based on a single Gaussian distribution assumption both of the inside and the outside.

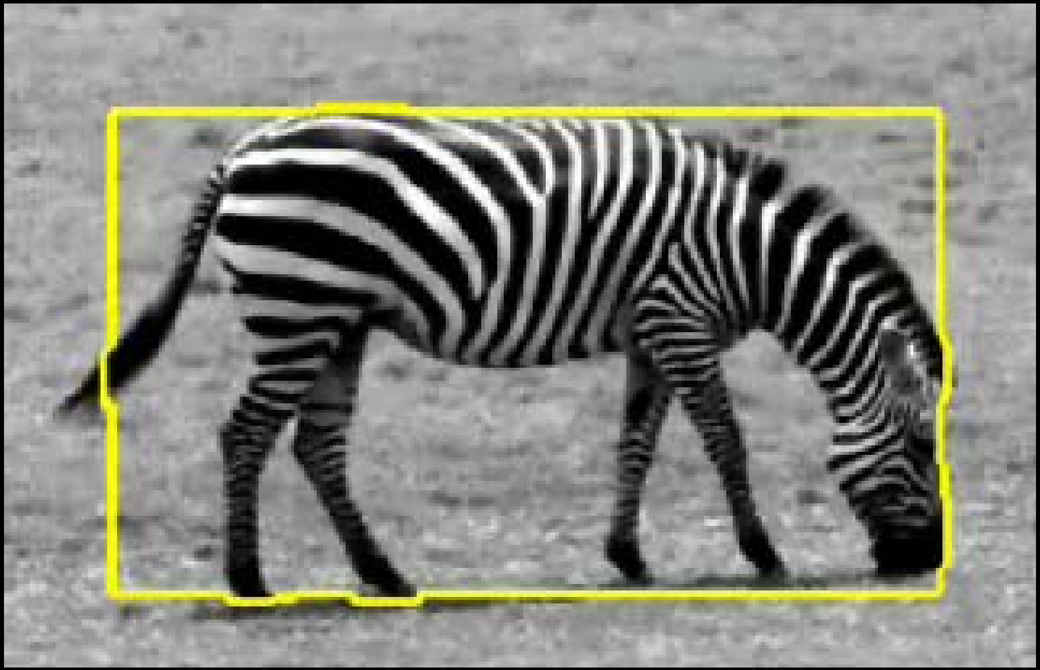
Interior $\mu_1, \Sigma_1 \quad \Gamma$

Exterior $\mu_2, \Sigma_2 \quad D\backslash\Gamma$

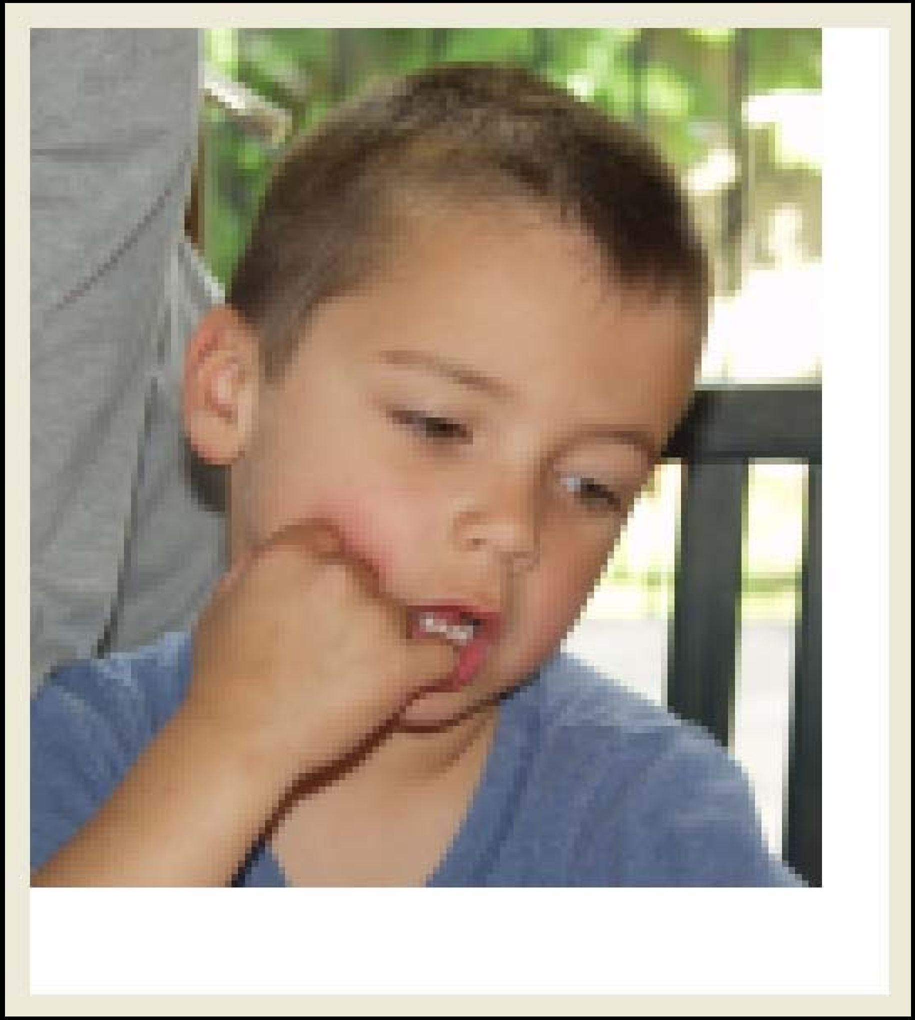
$$E(\Gamma, \mu_1, \Sigma_1, \mu_2, \Sigma_2) = \int_\Gamma e_1(x) + \int_{D\backslash\Gamma} e_2(x) + \nu\, length(\partial\Gamma)$$

$$e_i(x) = -\log p_{\mu_i, \Sigma_i}(I(x))$$

$$p_{\mu_i, \Sigma_i} = C|\Sigma_i|^{-\frac{1}{2}} e^{-(I(x)-\mu_i)^T \Sigma_i^{-1} (I(x)-\mu_i)/2}$$



Evolution sometimes gets stuck in local minima



Adding Stochastic Perturbations to Shape Optimization Algorithms

- SAC (Stochastic Active Contours)
  - Single Gaussian Model
  - Gaussian Mixtures
- Mathematical Elements
  - General Theory
  - Noise
  - Implementation

Shape Optimization problems through Simulated Annealing

Stochastic Active Contours (SAC)

Drawbacks of classical Active Contours/Regions methods

- Sometimes get stuck in local minima;
- Euler-Lagrance equations do not always provide explicit gradients.

Single Gaussian Model SAC

Adaptative Segmentation [Rousson Deriche 2002]

+

Simulated Annealing through

Stochastic Mean Curvature Motion (SMCM)

Segment an image in 2 regions, called generically the interior and the exterior, based on a single Gaussian distribution assumption both of the inside and the outside.

Interior	μ_1, Σ_1	Γ
Exterior	μ_2, Σ_2	$D\backslash\Gamma$

$$E(\Gamma, \mu_1, \Sigma_1, \mu_2, \Sigma_2) = \int_\Gamma e_1(x) + \int_{D\backslash\Gamma} e_2(x) + \nu\, length(\partial\Gamma)$$

$$e_i(x) = -\log\, p_{\mu_i, \Sigma_i}(I(x))$$

$$p_{\mu_i, \Sigma_i} = C|\Sigma_i|^{-\frac{1}{2}} e^{-(I(x)-\mu_i)^T \Sigma_i^{-1}(I(x)-\mu_i)/2}$$

Euler-Lagrange simplifies to [Rousson Deriche]

$$du = \left(e_2(x) - e_1(x) + \nu \,\mathrm{div}\left(\frac{Du}{|Du|}\right)\right)|Du|dt \boxed{+noise}$$

Standard approach sometimes gets stuck in local minima, while SMCM does not!

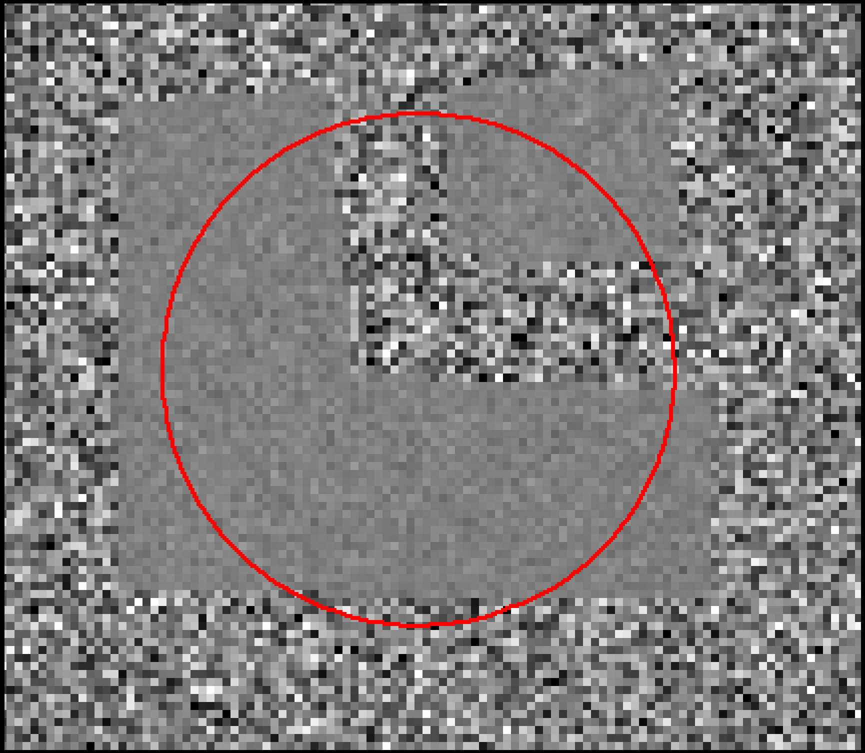
Empirical evidence shows that SMCM is more robust wrt to interface initialization

Test Image:

2 regions modeled by 2 unknown Gaussian distributions with

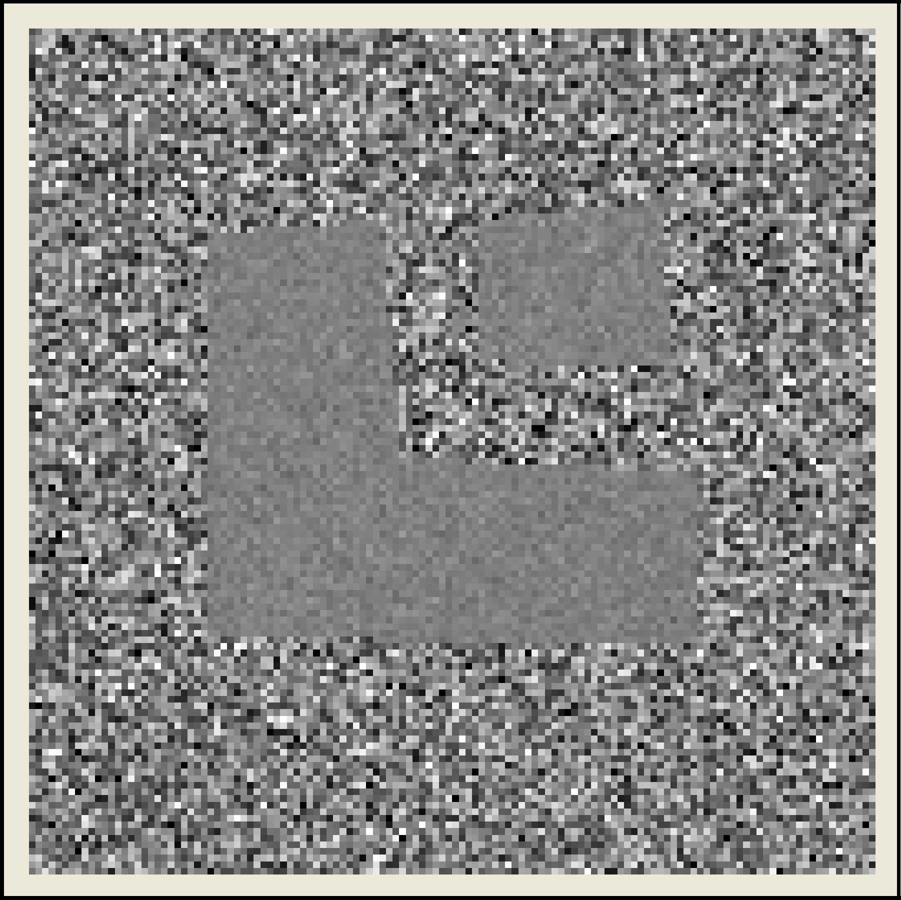
- Same mean
- Different variances

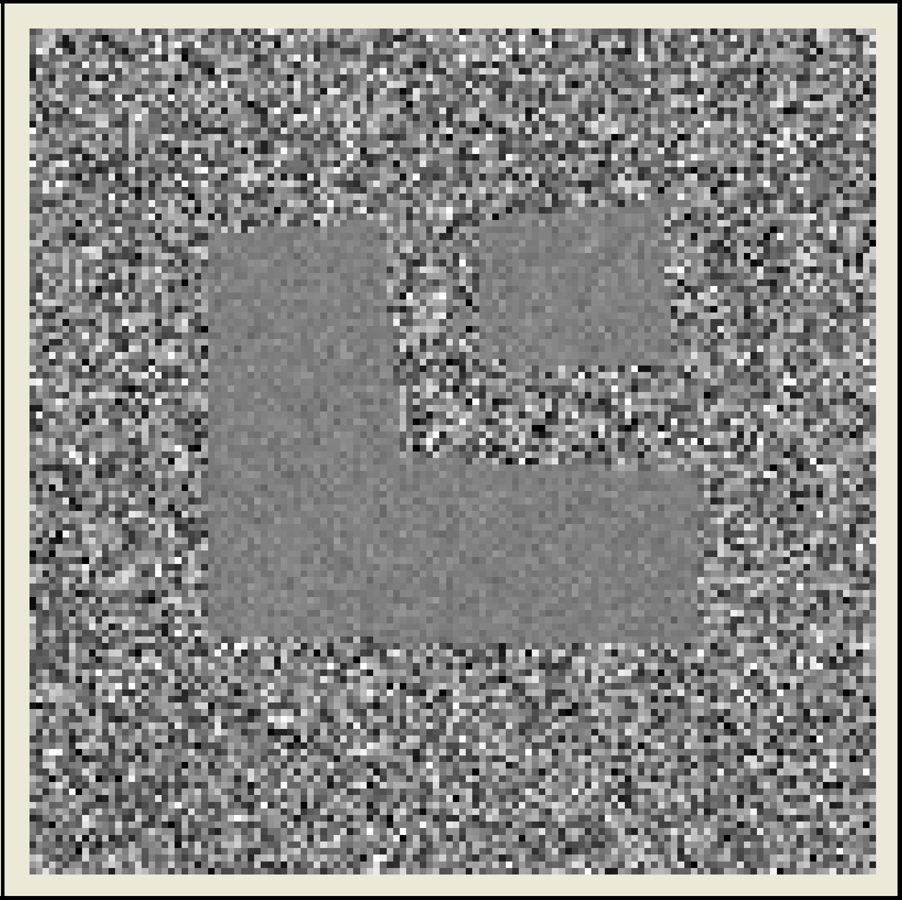
Test Image with Initial Contour



Standard algorithm

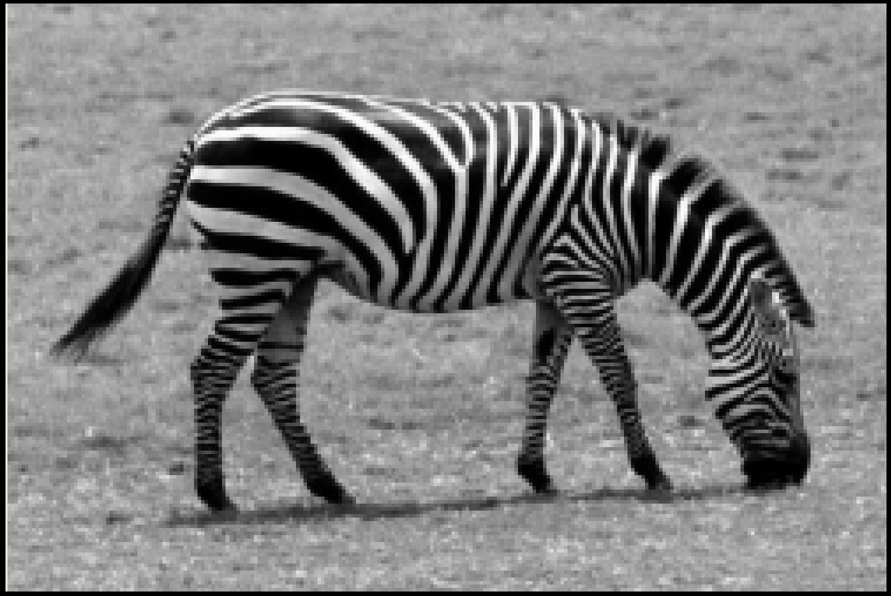
SAC

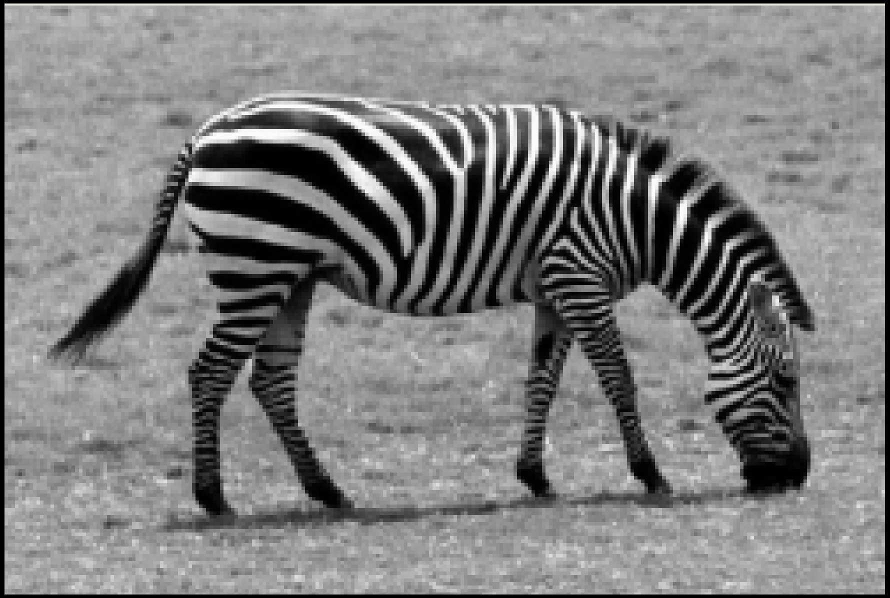




Deterministic Contour Evolution

SAC Evolution





Gaussian Mixtures SAC

- Extend the previous algorithm for the case when region statistics are modeled by a mixture of Gaussian distributions with parameters

$$\Theta_i = \left(\pi_i^1, \mu_i^1, \Sigma_i^1 \ldots \pi_i^{n_i}, \mu_i^{n_i}, \Sigma_i^{n_i}\right)$$

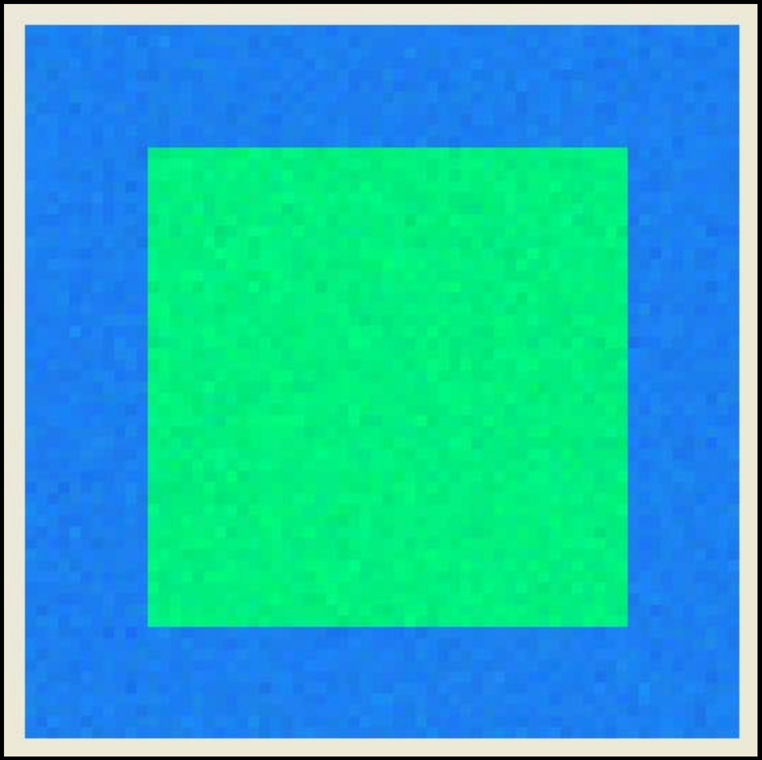
$$\sum_j \pi_i^j = 1$$

- The model dynamically calculates the optimal number of Gaussian distributions and then tries to fit the weights of those distributions using some algorithm (e.g. k-means).
- In this case, the k-means algorithm acts like a black box, due to the complex dependency $\Gamma \to \Theta_i(\Gamma)$

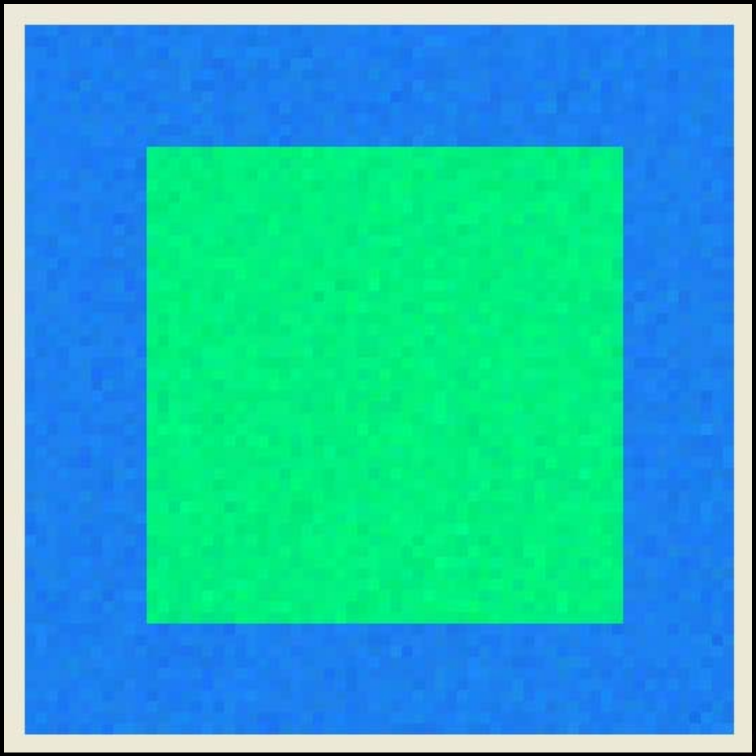
- **Cannot obtain an explicit form of the EL equation, but only the derivative of the energy wrt the shape at constant parameters.**

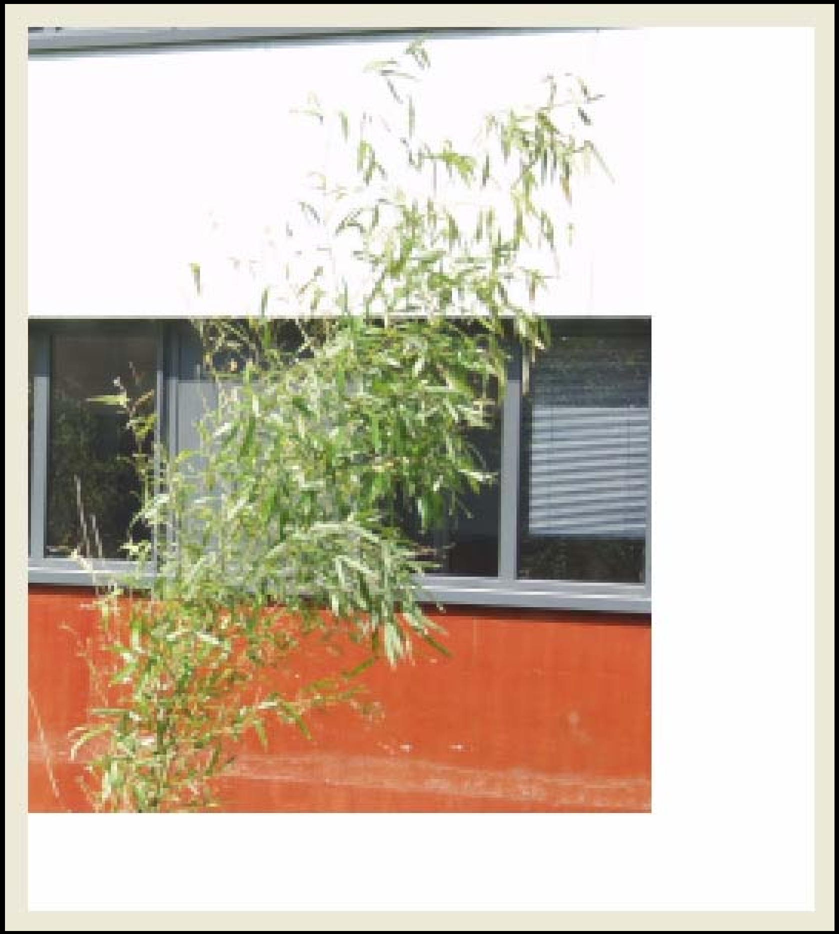
Gaussian Mixtures – SAC

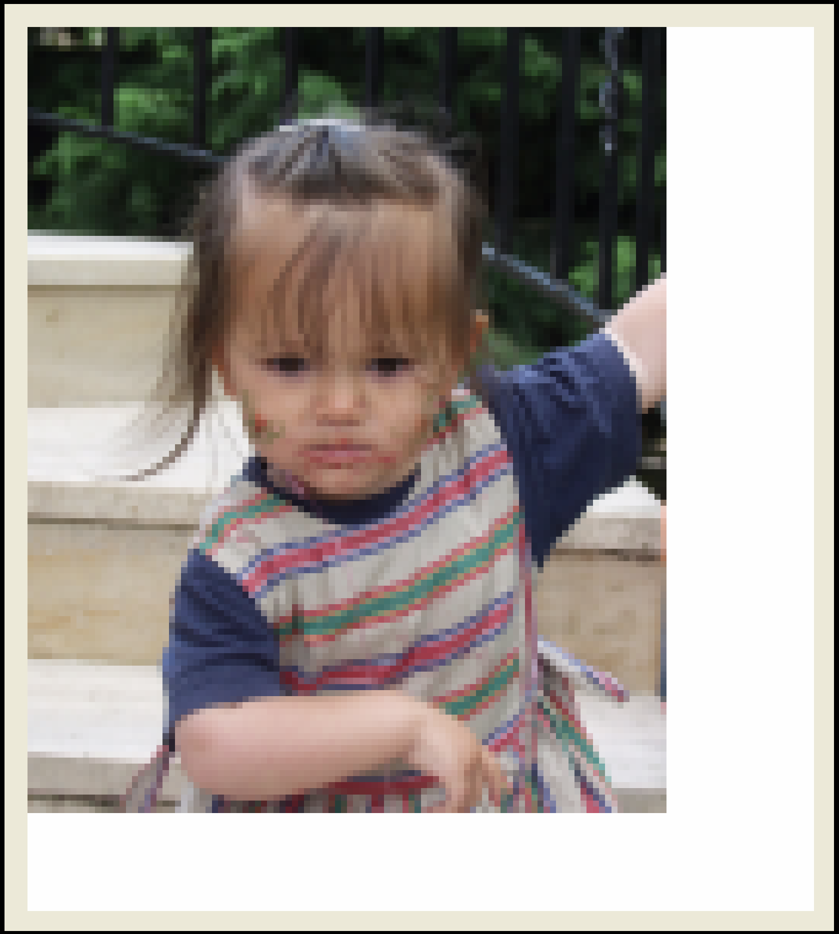
Deterministic Evolution with Approximated Gradient

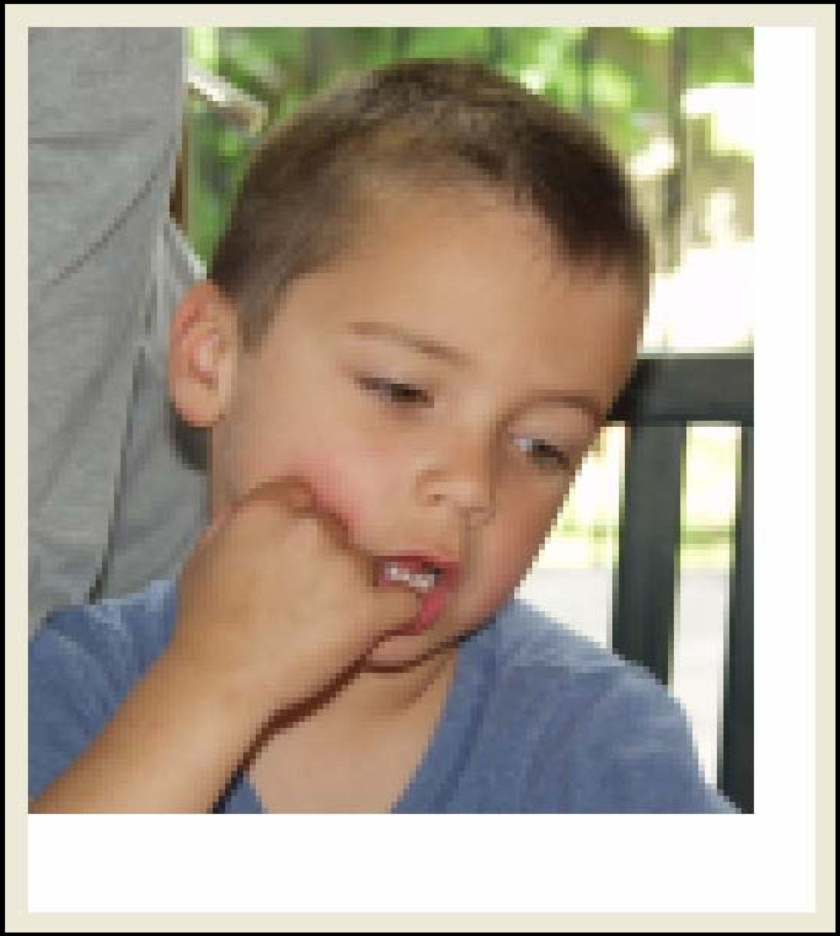


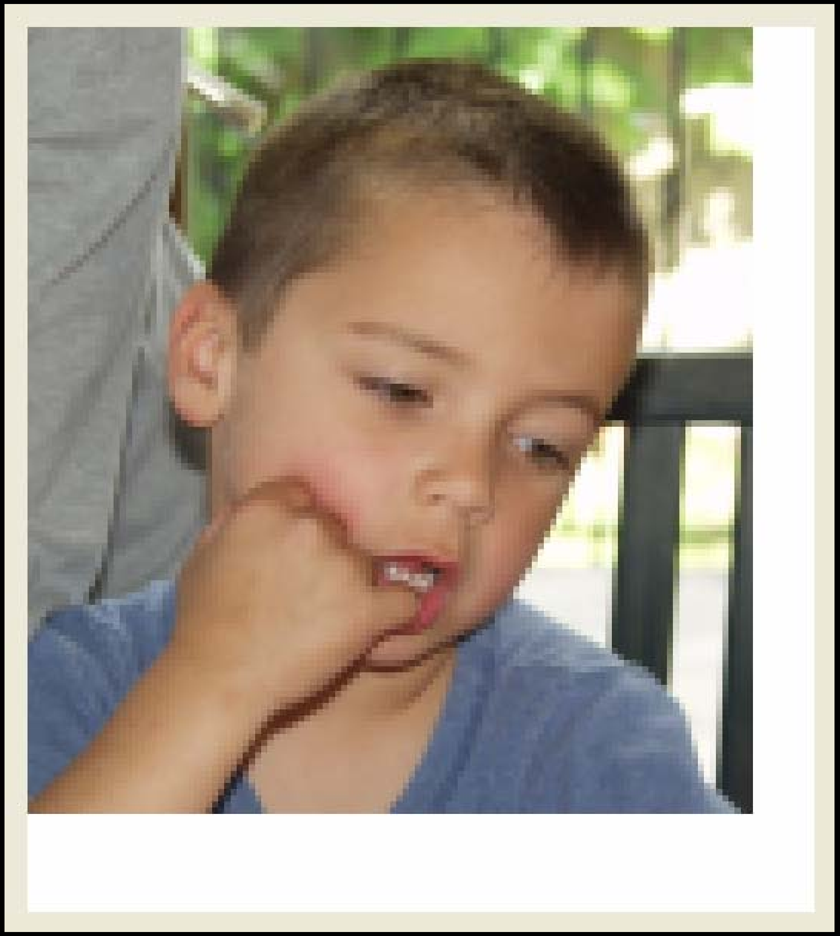
SAC Evolution with Approximated Gradient











Why do we need maths now that we have results?

- Well posedness …
- Geometric properties of stochastic evol.

Mathematical Theory

- Stochastic Mean Curvature Motion
- Viscosity Solutions for SPDEs
- Numerical Scheme used (Ito and Stratonovitch)
- Geometric properties
- Open Questions

Stochastic Mean Curvature Motion - SMCM

Notation

Domain $\Omega \in \mathbb{R}^2$

Curve $\Gamma = \partial\Omega$

Stochastic
Mean Curvature Motion

$$\frac{\partial\Gamma}{\partial t} = \kappa\mathbf{n} \boxed{+W(dt,x)\mathbf{n}}$$

White Noise $W(t,x)$

Intrinsic property

SMCM $$\frac{\partial \Gamma}{\partial t} = (\kappa + W(dt, x))\mathbf{n}$$

Level Sets SPDE $$du = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) dt + |Du| W(dt, x)$$

The curve evolution should be invariant
wrt the choice of the implicit function.

Simplified equation

$$du = |Du| dW(t) \qquad (EQ)$$

$$du = |Du| dW(t) \qquad\qquad (EQ)$$

$\alpha : \mathbb{R} \to \mathbb{R}$ smooth strictly increasing function

If u is solution of (EQ), then $\alpha(u)$ should be a solution of the same equation (EQ)

$$d\ [\alpha(u)] = \alpha'(u)|Du|dW + \frac{1}{2}\alpha''(u)|Du|^2 dt$$

$$= |D[\alpha(u)]|dW + \boxed{\frac{1}{2}\alpha''(u)|Du|^2 dt} \text{ Not intrinsic!}$$

The Itô form of the level sets SPDE is not intrinsic!

- Level Sets (Stratonovich) $\quad du = |Du| \circ dW(t)$

α - same as before

$$d[\alpha(u)] = \alpha'(u) \circ du = \alpha'(u)|Du| \circ dW(t)$$

$$= |D[\alpha(u)]| \circ dW(t)$$

The Stratonovich form of the SPDE satisfies the intrinsic property!

Well Posedness for Space-Independent Stochastic Hamiltonians

- Based on a series of articles of P.L. Lions and Souganidis

$$du = F(D^2u, Du, x, t)dt + \sum_i H_i(Du) \circ dW_i(t) \qquad (SPDE)$$

Theorem The equation (SPDE) admits an a.s. unique stochastic viscosity solution.

$$u_t^\epsilon = F(D^2u^\epsilon, Du^\epsilon) + \sum_i H_i(Du)\dot{\xi}_i^\epsilon(t)$$

$$\xi^\epsilon \to W \text{ uniformly on } (0,T) \text{ and a.s.}$$

Theorem The solutions of the approximated PDE converge a.s. locally uniformly on $\mathbb{R}^n \times [0,T]$ to the solution of (SPDE).

Noise $W(t,x)$

- Theoretical difficulties when working with white noise in space.
- Colored Noise in space : distribute noise on a discrete grid x_i at each moment in time

$$W(t,x) = \sum_{i=1}^{m} \phi_i(x) W_i(t)$$

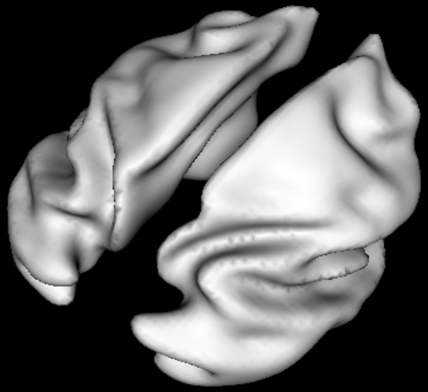
Noise – Scale defined by the distance between the x_i's

Implementation

- Explicit scheme for the Ito evolution

$$u(t+\Delta t) = u(t) + |Du|(t)\,\mathrm{div}\left(\frac{Du}{|Du|}\right)(t)\,\Delta t + |Du|\mathcal{N}(0,1)\sqrt{\Delta t}$$

- Narrow Band method
- The theory applies without problems in 3D



Implementation Details

Stratonovitch Drift

$$du = F(D^2u, Du)dt + H(x, Du) \circ dW(t)$$

$$d\langle H(x, Du), W\rangle_t = \left[\left(D^2u\frac{\partial H}{\partial p}\right) \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial x}\right] dt$$

Adding the above drift to the scheme before yields convergence towards the Stratonovitch equation

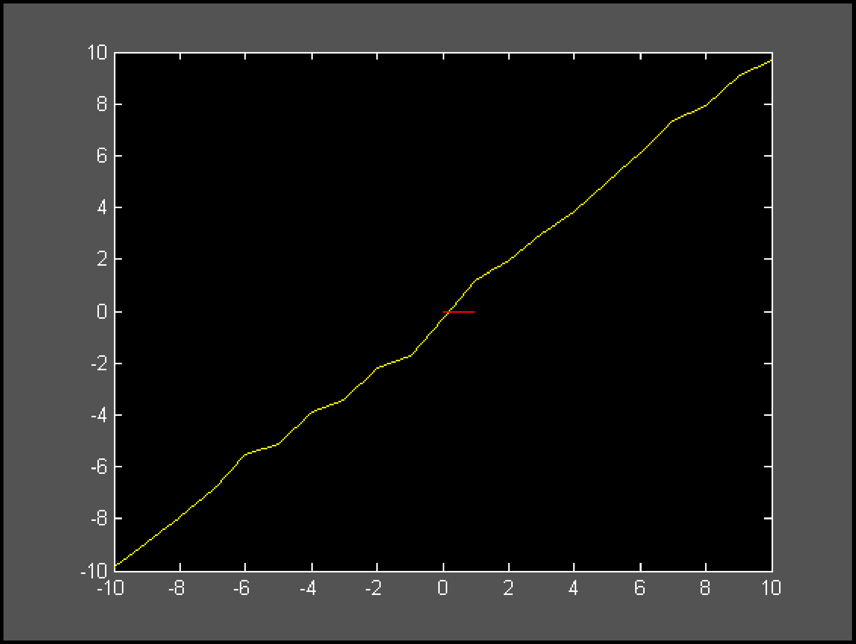
Geometric Properties

Page under construction!...

Open Questions

- Do not have a theorem on the time-convergence of the scheme (Ito or Stratonovitch) when the stochastic Hamiltonian depends on x
- Presence of artifacts in the evolution due to the presence of noise (when not colored enough)? (implementation dependent)

Example of artifacts



Artifacts : implementation details

- Narrow Band Method
- Implicit function re-initialization
- Distance-function preserving schemes