American Option Pricing: a Variance Reduction Technique

Nicola Moreni

CERMICS/ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2

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PLAN OF THE TALK

1. The price of an American option: Dynamic Programming and the Longstaff-Schwartz algorithm
2. Reducing Variance by changing the drift
3. Numerical results

PRICE OF AN AMERICAN OPTION

Two equivalent formulations

Optimal Stopping Time Problem

- compact and "clean" mathematical formulation
- easy to handle when studying theory (*)
- untreatable from a pratical point of view: "impossible" to have a straightforward numerical implementation.

$$U_0 = \mathbb{E} f(\tau_0, X_{\tau_0})$$

Dynamic Programming Problem

- no closed formula
- may reveal "uncomfortable" for explicit calculus
- easy numerical implementation (*)

$$\begin{cases} U_M = f(T, X_T) \\ U_i = \max\{f(t_i, X_{t_i}), \mathbb{E}(U_{i+1} | X_{t_i})\} \end{cases}$$

THE LONGSTAFF-SCHWARTZ ALGORITHM ('01)

$$\mathbb{E}(U_{j+1}|X_{t_j}) = \phi_j(X_{t_j})$$

(Markov Property)

L.-S. approximations —— L^2 basis ——>

$$\begin{cases} \phi_j(\cdot) \approx \alpha_j \cdot e(\cdot) \\ \alpha_j \doteq \arg\min_{a \in \mathbb{R}^m} \mathbb{E}\left[U_{j+1} - a \cdot e(X_{t_j})\right]^2 \end{cases}$$

↓ Monte Carlo

$$\alpha_i^N = \arg\min_{a \in \mathbb{R}^m} \sum_{n=1}^{N} \frac{1}{N} \left[U_{i+1}^{(n)} - a \cdot e\left(X_{t_i}^{(n)}\right)\right]^2$$

$\phi_j(x)=\mathbb{E}(U_{j+1}|X_{t_j}=x)$

THE LONGSTAFF-SCHWARTZ ALGORITHM ('01)

$$\begin{cases} U_M = f(T, X_T) \\ U_i = \max\{f(t_i, X_{t_i}), \mathbb{E}(U_{i+1} | X_{t_i})\} \end{cases}$$

↓ L^2

$$\begin{cases} U_M^m = f(T, X_T) \\ U_i^m = \max\{f(t_i, X_{t_i}), \alpha_i \cdot e\big(X_{t_i}\big)\} \end{cases}$$

↓ M.C.

$$\begin{cases} U_M^{(m,N,n)} = f(T, X_T^{(n)}) \\ U_i^{(m,N,n)} = \max\{f(t_i, X_{t_i}^{(n)}), \alpha_i^N \cdot e\big(X_{t_i}^{(n)}\big)\} \end{cases}$$

↑ Convergence L²

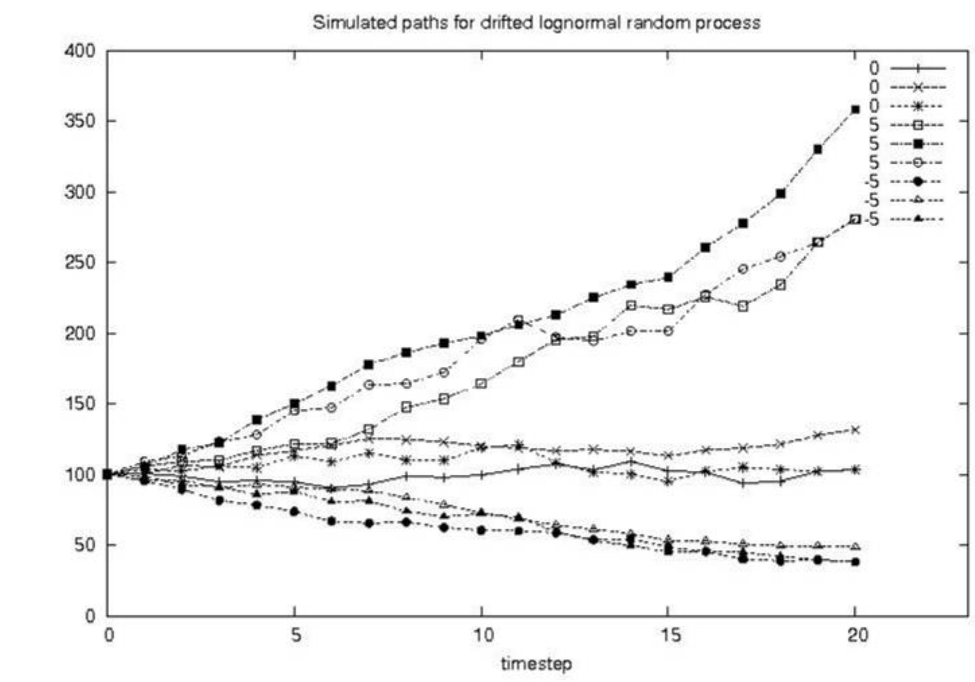
↑ T.C.L.
N→ +∞

REDUCING VARIANCE BY CHANGING THE DRIFT

Original Model → Drifted model

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t$$

$$\frac{dX_t^\theta}{X_t^\theta} = \mu dt + \sigma\theta dt + \sigma dW_t$$



REDUCING VARIANCE BY CHANGING THE DRIFT

Thanks to Girsanov's Theorem, we have, $\forall\, \theta \in \mathbb{R}^D$

$$\mathbb{E} f(\tau_0, X_{\tau_0}) = \mathbb{E} e^{-\frac{1}{2}||\theta||^2 \tau_0 - \theta \cdot W_{\tau_0}} \cdot f(\tau_0, X^\theta_{\tau_0}) \doteq \mathbb{E} f^\theta(\tau_0, X^\theta_{\tau_0})$$

that is, we dispose of a set of equivalent pricing problems.

We want to find the optimal drift i.e. the drift which speeds up the convergence of the L.S. algorithm
Central Limit Theorem

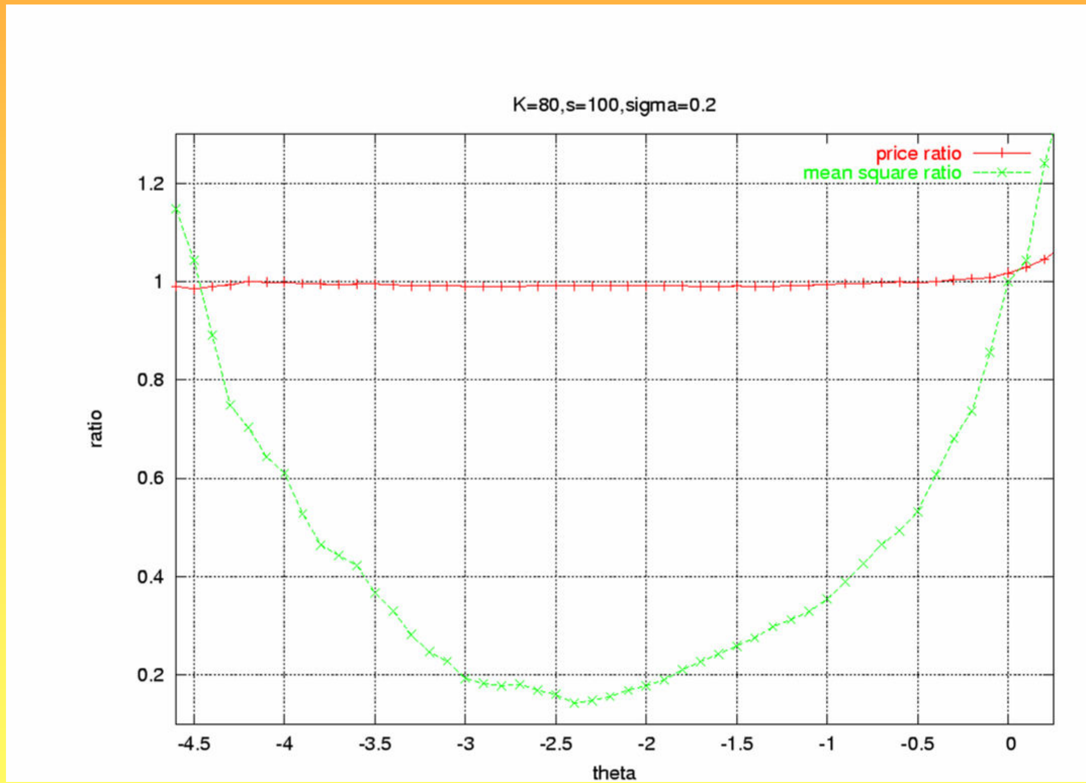
$$U_0^{(m,N)} = \max\{f(t_0, X_{t_0}), \sum_{n=1}^{N} U_1^{(m,N,n,\theta)}/N\}$$

Remarks: - τ_0 is a r.v. This is more than a change in integration measure!
- τ_0 does NOT depend on θ!

NUMERICAL RESULTS

Put (Basket) American Option, 1D

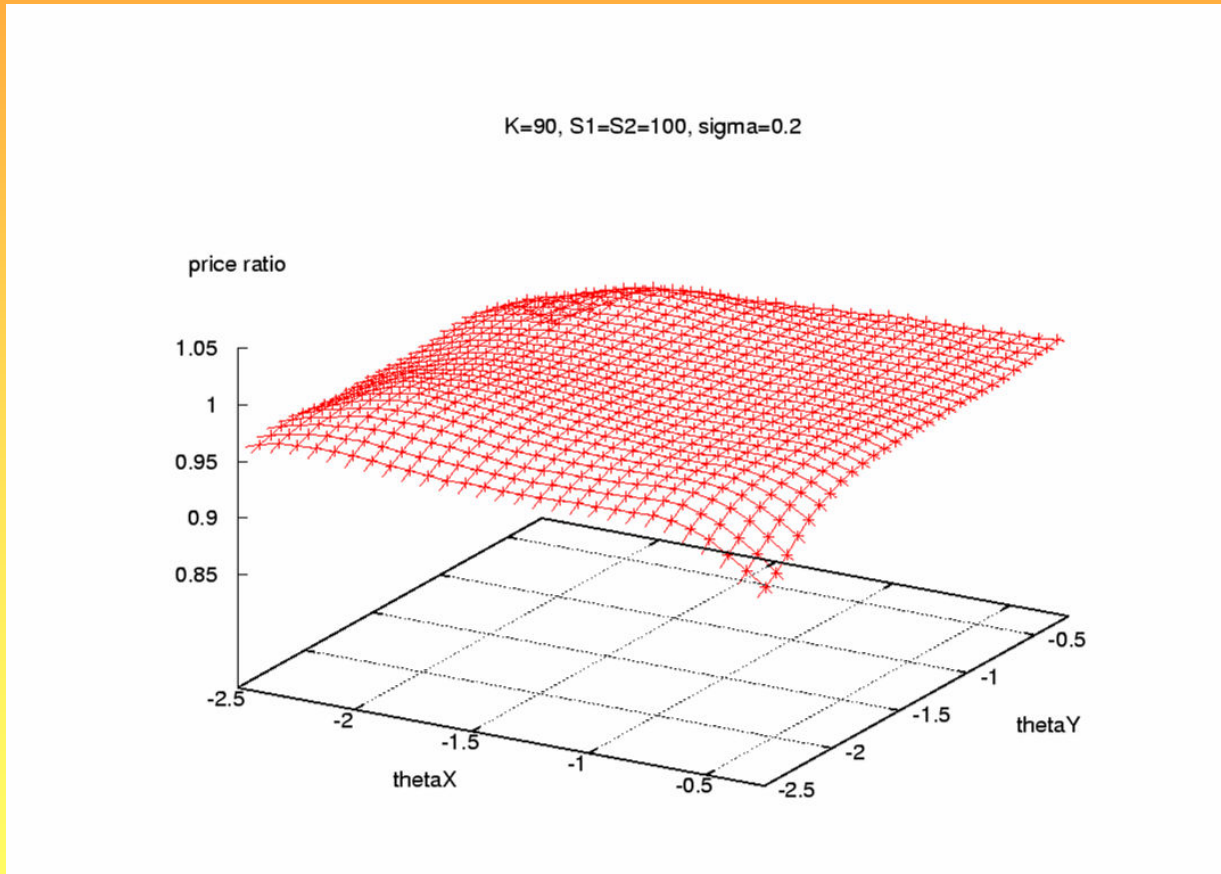
Ratios $\text{Price}(\theta)/\text{Price}(0)$ and $\sqrt{\text{Var}(\theta)/\text{Var}(0)}$



- Variance is reduced up to 2.3% of its initial value!
- Prices are in very good accord with the Benchmark (Premia tree)

NUMERICAL RESULTS

Put Basket American Option, 2D,
uncorrelated symmetric asset

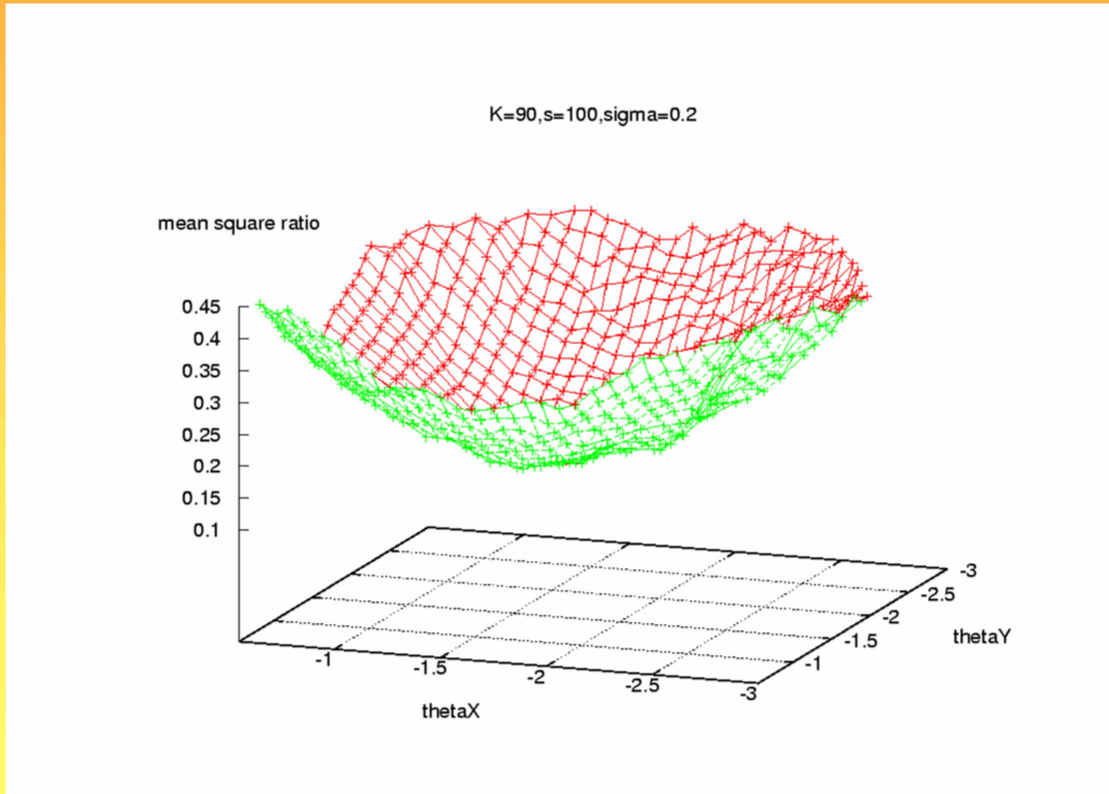


Very good accord with reference price!

NUMERICAL RESULTS

Put Basket American Option, 2D, uncorrelated symmetric asset

Ratios $\sqrt{Var(\theta)/Var(0)}$



Min=0.15 (Min Var < 2.3%)

NUMERICAL RESULTS

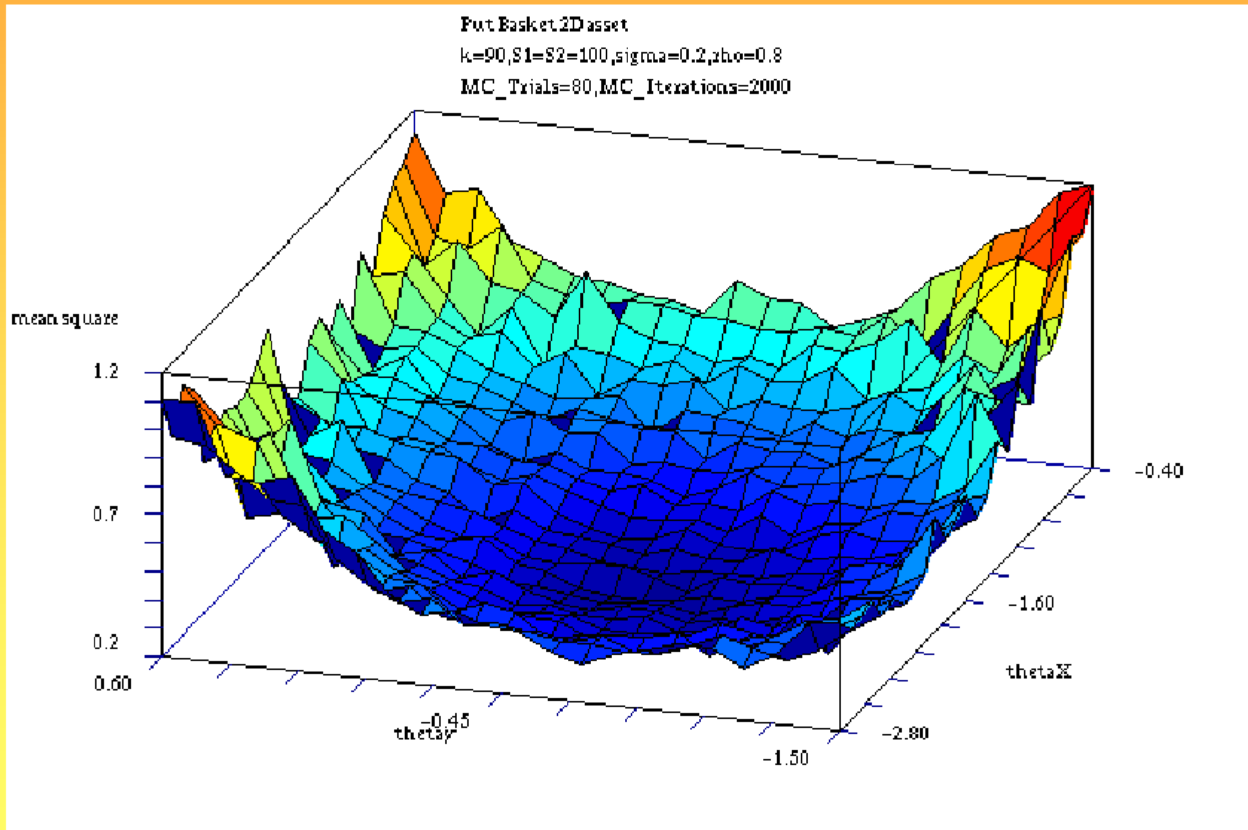
Put Basket American Option, 5D, uncorrelated asset

θ_1	θ_2	θ_3	θ_4	θ_5	Price(θ)/Price(0)	$\sqrt{\text{Var}(\theta)/\text{Var}(0)}$.
-2	-2	-2	-2	-2	0.897768	1.08737
...	...	...	...	...	...	...
-1	-1	-1.5	-0.5	-0.5	1.01046	0.443951
-1	-1	-1	-0.5	-0.5	1.01677	0.345993
-1.0	-1.0	-1.0	1.0	-1.0	1.01038	**0.208331**
-1	-1	-1	-0.5	-1.5	0.956187	0.337243
-1	-1	-1	-0.5	-0.5	1.01677	0.345993
...	...	...	...	...	...	...
-1	-0.5	-2	-0.5	-2	0.920403	0.626063

NUMERICAL RESULTS

Put Basket American Option, 2D, correlated asset

Ratio $\sqrt{Var(\theta)/Var(0)}$

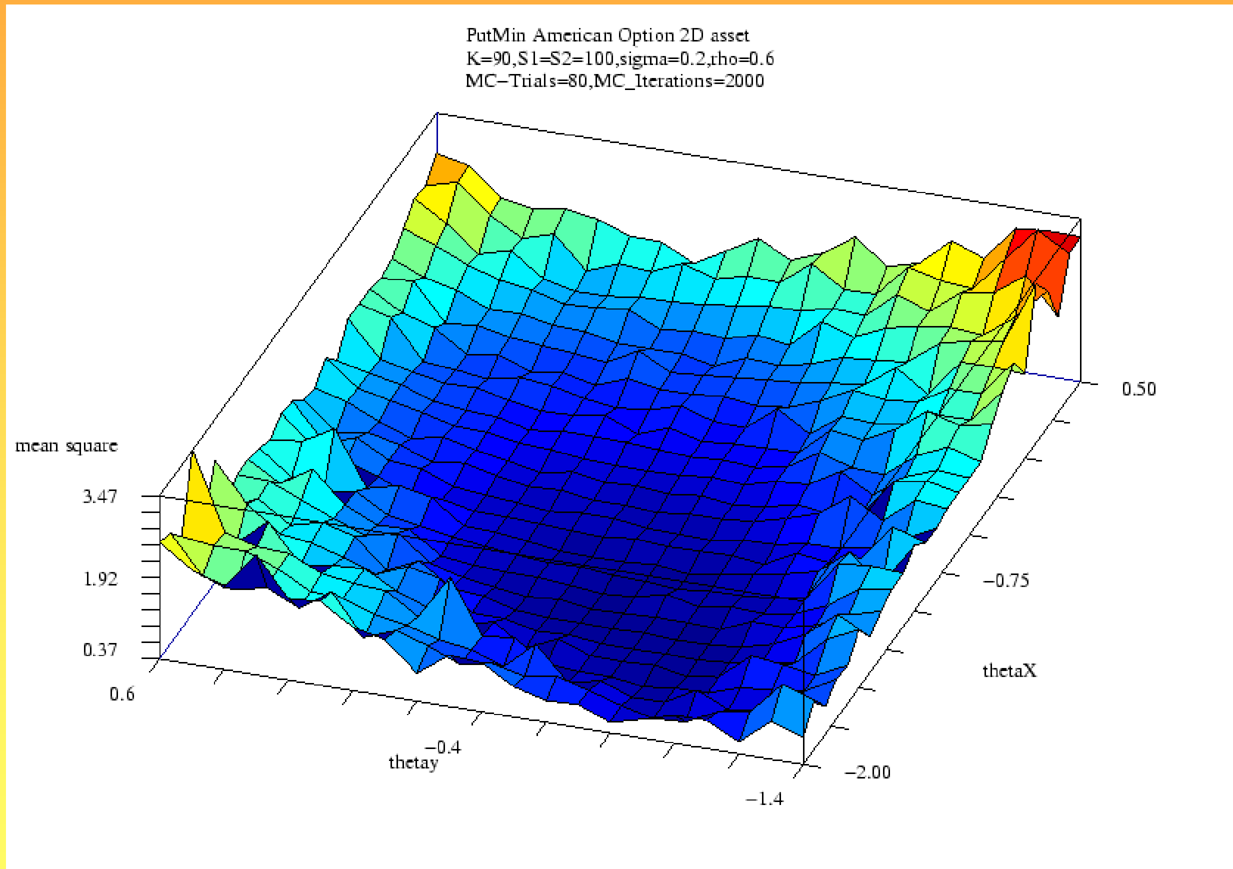


Min 0.2 (MinVar 4%)

NUMERICAL RESULTS

Put Min American Option, 2D, correlated asset

Ratio $\sqrt{Var(\theta)/Var(0)}$

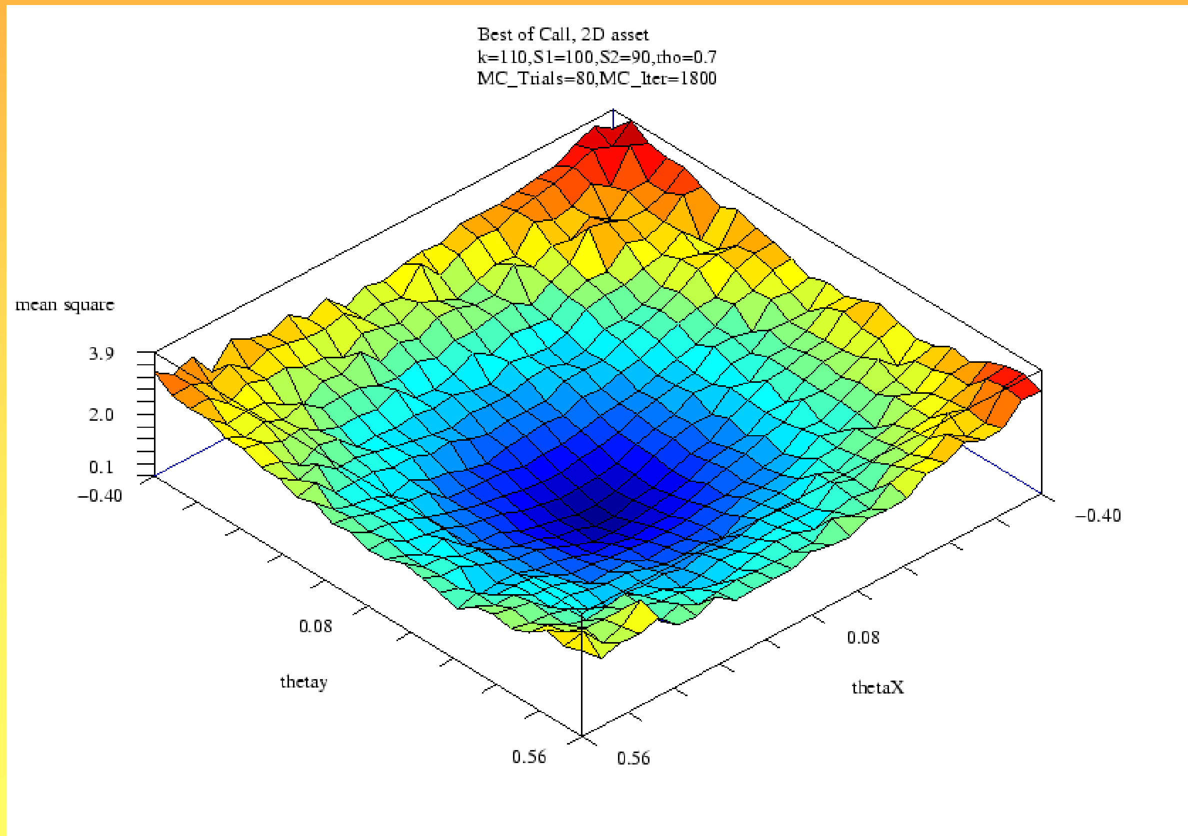


Min 0.37 (MinVar 14%)

NUMERICAL RESULTS

Best of Call American Option, 2D, correlated asset

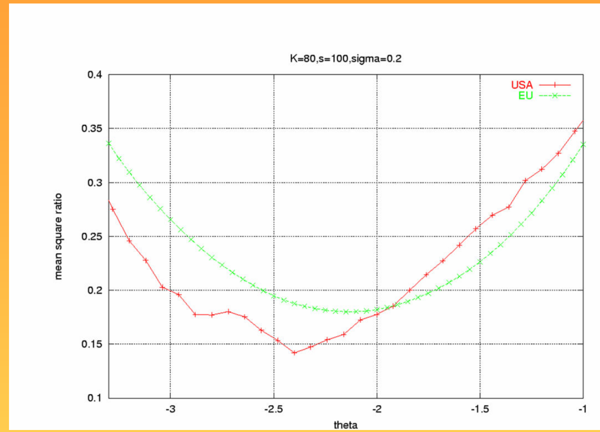
Ratio $\sqrt{Var(\theta)/Var(0)}$



Min 0.11 (MinVar 1.2%)

HOW COULD WE ESTIMATE THE MINIMUM?

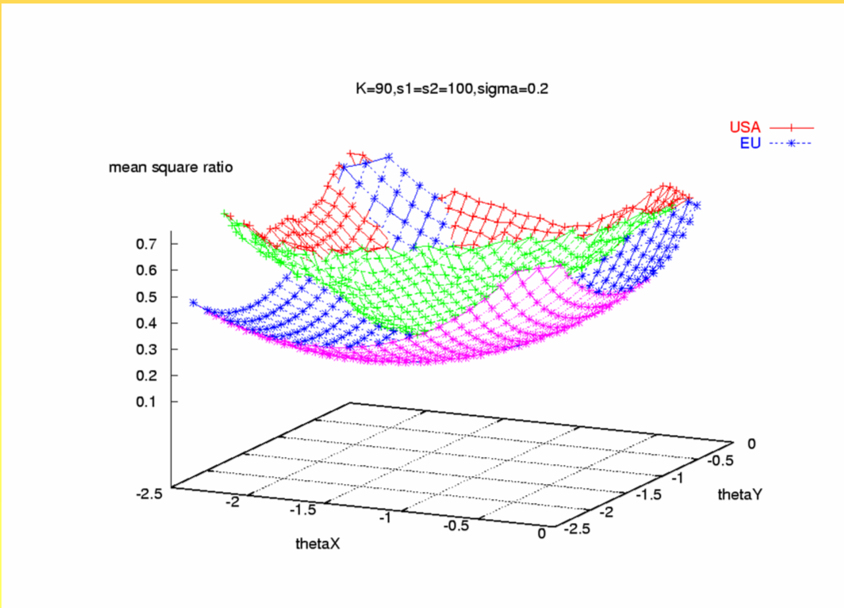
Let us compare our problem to to the corresponding European one:



- are the minima *"quite near"*?

- what about setting $\theta^{USA}_{min} \approx \theta^{EU}_{min}$

- whenever worthwhile, could we estimate θ^{EU}_{min}



HOW COULD WE ESTIMATE THE MINIMUM?

Model	$\sqrt{\text{Var}(\theta^{min}_{USA})/\text{Var}(0)}$	$\sqrt{\text{Var}(\theta^{min}_{EU})/\text{Var}(0)}$
PB $1D$, $K = 80$, $X_0 = 100$	0.14	0.15
PB $2D$, $K = 90$, $X_0 = (100, 100)$	0.15	0.18
PB $2D$, $K = 90$, $X_0 = (105, 70)$	0.23	0.37
PB $2D$, $K = 90$, $X_0 = (100, 100)$,$\rho = 0.8$	0.20	0.22
BC $2D$, $K = 90$, $X_0 = (110, 90)$,$\rho = 0.7$	0.11	0.30
PM $2D$, $K = 90$, $X_0 = (100, 100)$,$\rho = 0.6$	0.37	0.43
PB $3D$, $K = 95$, $X_0 = (100, 100, 100)$	0.20	0.31
PB $5D$, $K = 100$,$X_0 = (100, 100, 100, 100, 100)$	0.21	0.31

1. θ^{EU}_{min} gives good *sub-optimal* results

2. θ^{EU}_{min} can be estimated by fast and precise stochastic algorithms as the Robbins-Monro ones

CONCLUSIONS

Changing diffusion drift $\longrightarrow$ Monte Carlo Variance Reduction for the L.-S. Algorithm

Main features of our method:

1. Generality: it works for ALL
   - $L^{2+\xi}$ payoff function
   - Diffusion Markov process $\rightarrow$ extendable to stochastic volatility models (price to pay= supplementary dimensions)

2. Efficiency: reduction of variance up to 2%

3. Versatility: we have some further *sub-optimal* approximations