The Boltzmann-Grad Limit for the Periodic Lorentz Gas

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In 1905, Lorentz proposed to describe the motion of electrons in metals by the methods of kinetic theory

•Gas of electrons described by its phase-space density $f \equiv f(t, x, v)$

(At time t, there are $f(t, x, v)dxdv$ electrons in a phase-space volume $dxdv$ centered at (x, v), $x =$ position, $v =$ velocity)

•Electron-electron collisions neglected (unlike in the kinetic theory of gases)

•Only the collisions between electrons and metallic atoms are considered

$$\boxed{\Rightarrow \text{LINEAR KINETIC EQUATION}}$$

$\neq$ the Boltzmann equation in the kinetic theory of gases is NONLINEAR

The Lorentz kinetic model

•Equation for the phase-space density of electrons $f \equiv f(t,x,v)$:

$$(\partial_t + v \cdot \nabla_x + \frac{1}{m} F(t,x) \cdot \nabla_v) f(t,x,v) = N_{at} r_{at}^2 |v| \mathcal{C}(f(t,x,\cdot))(v)$$

where $\mathcal{C}$ is the Lorentz collision integral

$$\mathcal{C}(\phi)(v) = \int_{\substack{|\omega|=1 \\ \omega\cdot v>0}} \Big(\phi(\mathcal{R}_\omega v) - \phi(v)\Big) \cos(v,\omega) d\omega$$

with $\mathcal{R}_\omega$ denoting the specular reflection: $\mathcal{R}_\omega(v) = v - 2(v\cdot\omega)\omega$

Notation: m =mass of the electron;

•$F \equiv F(t,x)$ is the electric force (known);

•N_{at}, r_{at} density, radius of metallic atoms.

It is a mesoscopic model (between microscopic and macroscopic);

- it is a *single-particle* phase-space equation; but

- a *statistical* description and $\boxed{\text{not a first principle}}$

$\boxed{\text{Probabilistic interpretation}}$

Direction of each particle jumps at *exponentially distributed times*, so that

- jump times, and jumps in direction are independent;

$$\boxed{\Rightarrow \text{ the Lorentz collision integral } \mathcal{C}}$$

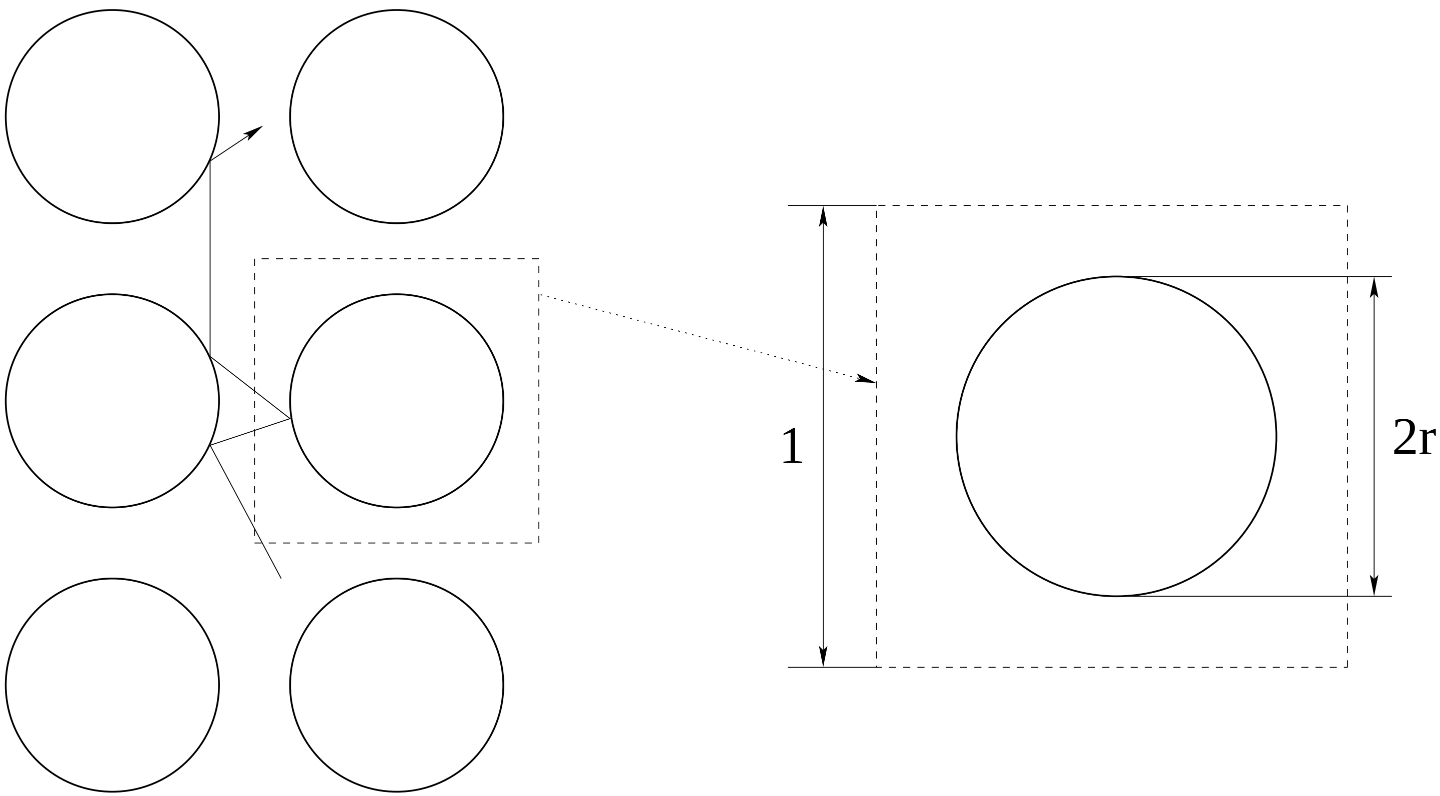
- between two jumps, each particle is driven by the electric force field F

$$\boxed{\Rightarrow \text{ the streaming operator } \partial_t + v \cdot \nabla_x + \frac{1}{m} F \cdot \nabla_v}$$

The microscopic model (Lorentz gas)

- Periodic configuration of spherical obstacles

$$Z_r = \{x \in \mathbf{R}^D \mid \mathsf{dist}(x, \mathbf{Z}^D) > r\}, \quad Y_r = Z_r/\mathbf{Z}^D$$



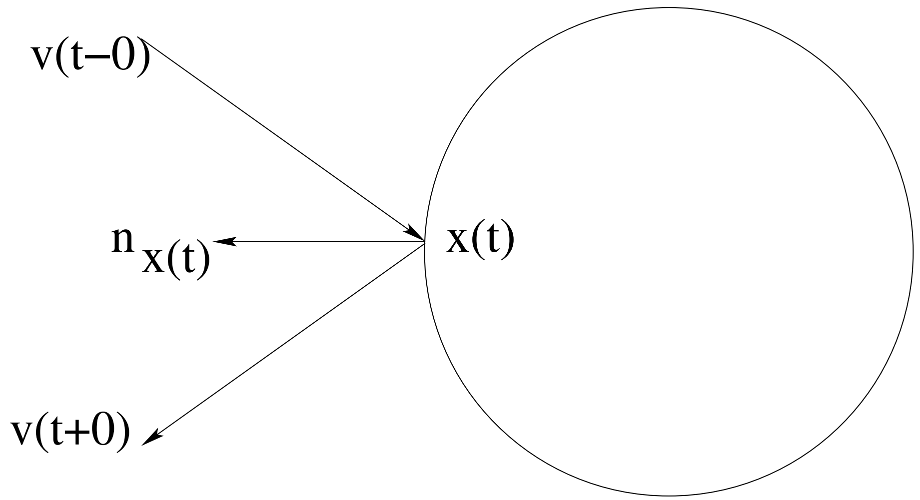
•Particles move freely between the obstacles

$$\dot{x}(t) = v(t)\,, \quad \dot{v}(t) = 0\,, \quad \text{if } x(t) \in Z_r$$

and are reflected upon impinging on the surface of the obstacles

$$v(t^+) = \mathcal{R}_{n_{x(t)}} v(t^-)\,, \quad \text{whenever } x(t) \in \partial Z_r$$

(with n_x the inward unit normal at $x \in \partial Z_r$).



•The prescription above define a (broken) flow

$$(x,v) \mapsto (X_t^r(x,v), V_t^r(x,v))$$

•Define then a phase-space density (propagated by the flow above)

$$f_\epsilon(t,x,v) \equiv f^{in}(\epsilon X^r_{t/\epsilon}(x/\epsilon,v), V_t^r(x/\epsilon,v))\,, \quad \text{with } r = \epsilon^{\frac{1}{D-1}}$$

Question

Does $f_\epsilon \rightharpoonup f$, the solution of the Lorentz kinetic equation, as $\epsilon \to 0$?

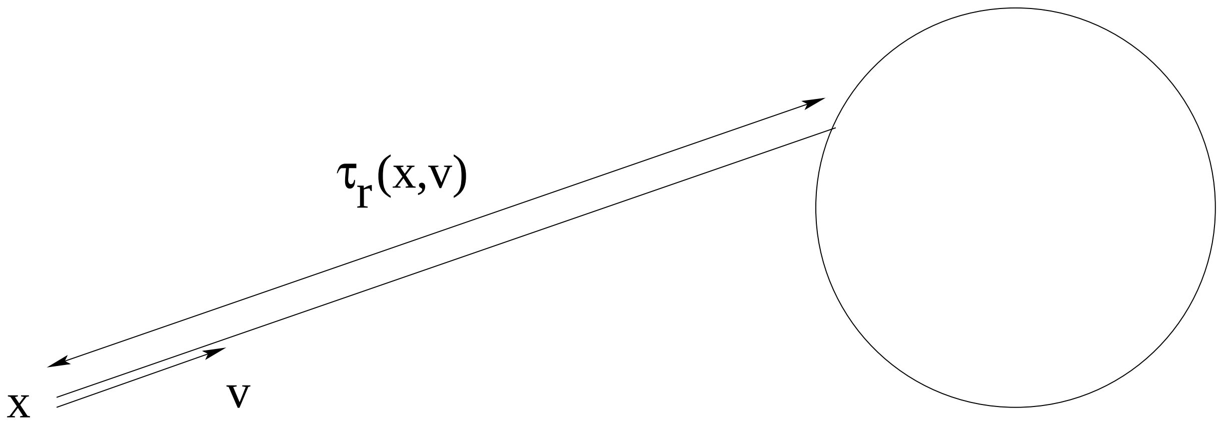
•Proved for a Poisson distribution of obstacles (Gallavotti 1972)

•See also Spohn (CMP 1978), Boldrighini-Bunimovich-Sinai (JSP 83)

Distribution of free path lengths

- Free path length (i.e. exit time)

$$\tau_r(x,v) = \inf\{t > 0 \,|\, x + tv \in \partial Z_r\}$$



- For (x,v) uniformly distributed on $Z_r \times \mathbf{S}^{D-1}$

$$\Phi_r(t) = \mathrm{Prob}\{(x,v) \,|\, \tau_r(x,v) > t\}$$

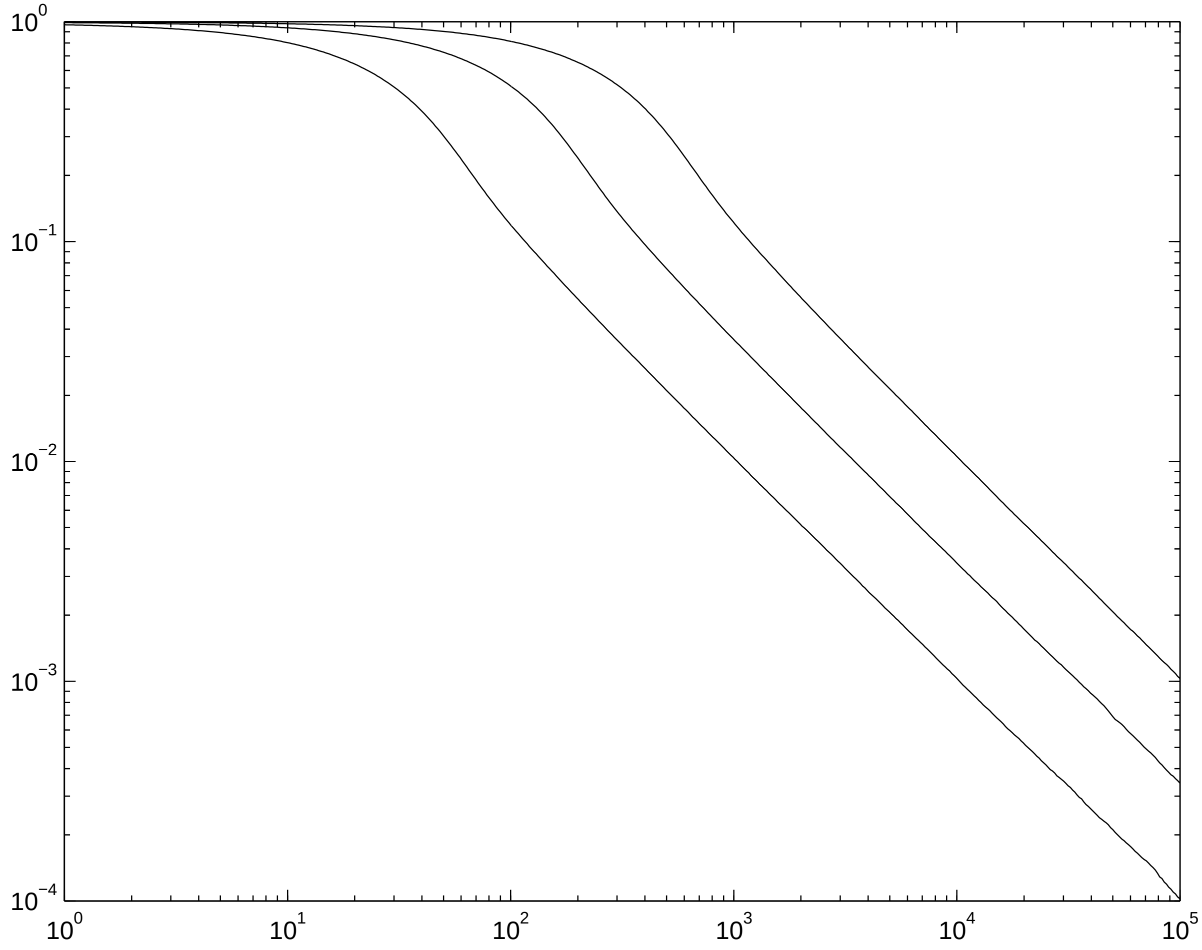
Theorem. *There exists* $0 < C_- < C_+$ *such that, for all* $t > 1/r^{D-1}$

$$\frac{C_-}{tr^{D-1}} \leq \Phi_r(t) \leq \frac{C_+}{tr^{D-1}}$$

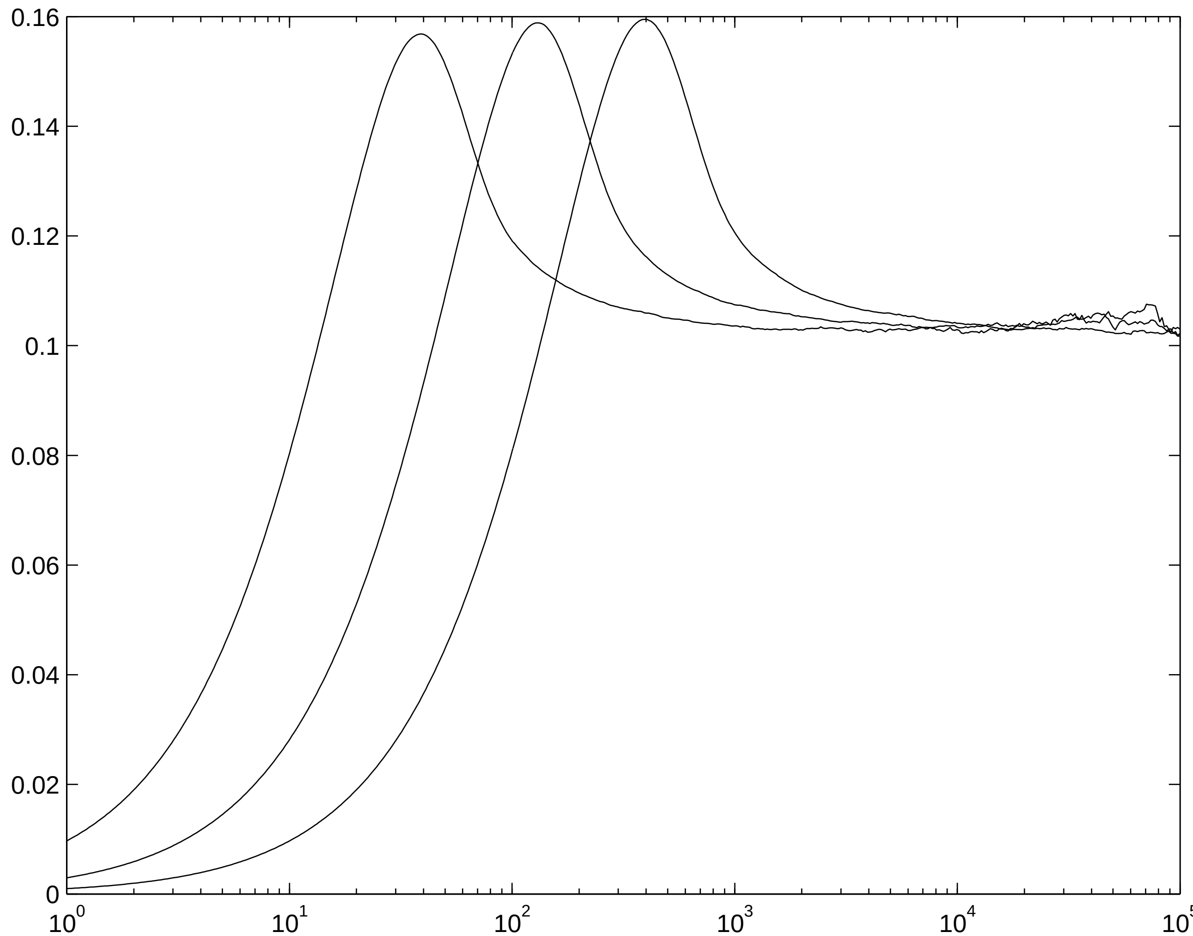
Upper bound + lower bound for $D = 2$: Bourgain-G-Wennberg CMP 1998

Lower bound for $D \geq 3$: G-Wennberg M2AN 2000.

$$\boxed{\Rightarrow \langle \tau_r \rangle = \int_{Y_r \times \mathbf{S}^{D-1}} \tau_r(x,v) \frac{dxdv}{|Y_r||\mathbf{S}^{D-1}|} = +\infty \quad \text{infinite mean free path}}$$



Log-log-plot of $\Phi_r(t)$ *vs.* t for $r = 0.01$, $r = 0.03$ and $r = 0.001$



Plot of $tr\Phi_r(t)$ *vs.* t

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Theorem. (Caglioti-G CMP 2003) *For* $D = 2$,

$$\lim_{r \to 0^+} \frac{1}{|\ln \epsilon|} \int_\epsilon^{1/4} \Phi_r(t/r) \frac{dr}{r} = \frac{2}{\pi^2 t} + O(1/t^2)$$

Non convergence to the Lorentz kinetic equation

For $n \geq 1$ and $r = n^{-\frac{1}{D-1}}$, and for an initial phase space density $\rho^{in} \equiv \rho^{in}(x) \geq 0$ independent of v, set

$$f_n(t,x,v) = \rho^{in}(\tfrac{1}{n}X^r_{nt}(nx,v))$$

Theorem. *For some $\rho^{in} \in L^\infty(\mathbf{T}^D)$, neither f_n nor any subsequence thereof converges in L^∞ weak-* to the solution of*

$$(\partial_t + v \cdot \nabla_x)f = \mathcal{C}(f) \text{ on } \mathbf{R}^D \times \mathbf{S}^{D-1}, \ f\Big|_{t=0} = \rho^{in}$$

where $\mathcal{C}$ is the Lorentz collision integral

$$\mathcal{C}(\phi)(v) = \int_{\substack{|\omega|=1 \\ \omega \cdot v > 0}} \Big(\phi(\mathcal{R}_\omega v) - \phi(v)\Big) \cos(v,\omega) d\omega$$

•The same is true if the Lorentz collision integral is replaced with any operator of the form

$$\mathcal{C}(f) = \sigma \int_{\mathbf{S}^{D-1}} p(v,v')(f(v') - f(v))dv'$$

where $\sigma > 0$ and the function p is the kernel of a compact operator on $L^2(\mathbf{S}^{D-1})$ that satisfies

$$p(v,v') = p(v',v) \geq 0\,, \quad \int_{\mathbf{S}^{D-1}} p(v,v')dv' = 1$$

Method of proof

- Spectral theory of transport operators (Ukai-Ghidouche-Point JMPA 1977)

$$\|f(t) - \langle\rho^{in}\rangle\|_{L^2(\mathbf{T}^D\times\mathbf{S}^{D-1})} \le ce^{-\gamma t}\|\rho^{in}\|_{L^2(\mathbf{T}^D)}$$

- Pointwise inequality

$$f_n(t,x,v) \ge \rho^{in}(x-tv)\mathbf{1}_{\tau_r(nx,v)\ge nt}$$

- If some subsequence $f_{n'} \rightharpoonup f$ in L^∞ weak-*,

$$\|f(t,\cdot,\cdot)\|_{L^2_{x,\omega}} \ge \|\rho^{in}\|_{L^2_x}\Phi_r(nt) \ge \frac{C_-\|\rho^{in}\|_{L^2_x}}{ntr^{D-1}} = \frac{C_-\|\rho^{in}\|_{L^2_x}}{t}$$

- Concentration argument: pick ρ^{in}_δ such that

$$\|\rho^{in}_\delta\|_{L^2(\mathbf{T}^D)} = 1 \text{ while } \langle\rho^{in}_\delta\rangle \to 0 \text{ as } \delta \to 0^+$$

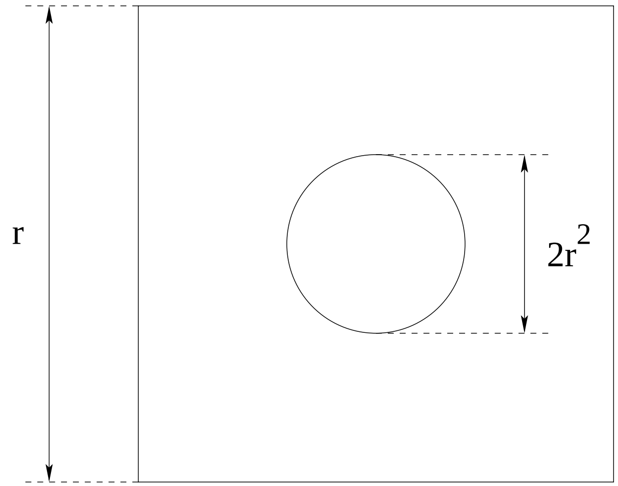
$\Rightarrow$ Contradiction with the spectral bound

$\boxed{\textbf{Case of absorbing obstacles, } D = 2}$

•**Pbm:** to find the limit as $r \to 0^+$ of g_r s.t.

$$\begin{aligned}(\partial_t + v \cdot \nabla_x) g_r &= 0\,, \quad x \in rZ_r\,, \qquad v \in \mathbf{S}^{D-1}\,,\\ g_r(t,x,v) &= 0\,, \quad x \in \partial(rZ_r)\,, \quad v \cdot n_x > 0\,,\\ g_r\Big|_{t=0} &= g^{in}\Big|_{rZ_r}\,.\end{aligned}$$

Fundamental domain of rZ_r



Theorem. (Caglioti-G CMP 2003) *Let* $g^{in} \geq 0$ *be in* $C_c^1(\mathbf{R}^2 \times \mathbf{S}^1)$. *For each* $\chi \in C_c^1(\mathbf{R}^2 \times \mathbf{S}^1)$,

$$\lim_{r \to 0^+} \frac{1}{|\ln r|} \int_r^{1/4} \langle g_r(t)\chi \rangle \frac{dr}{r} = \langle g(t)\chi \rangle + O(1/t^2)$$

where

$$g(t,x,v) = \frac{2g^{in}(x - tv, v)}{\pi^2 t}\,.$$

- This suggests that the limiting equation for the above model should be

$$\partial_t g + v \cdot \nabla_x g + \tfrac{1}{t} g = 0\,, \quad t > 0\,,\ x \in \mathbf{R}^2\,,\ |v| = 1\,.$$

This, however, holds $\boxed{\text{for large } t \text{ only.}}$

Method of proof

Idea no.1 Given a linear flow with irrational slope on a 2-torus with a disk removed, what is the longest orbit of this flow? (question raised by R. Thom in 1989).

Blank-Krikorian, IJM'93: On a 2-torus with a slit parallel to one of the coordinate axis, there are generically 3 classes of orbits (say A, B and C). All orbits in a given class have the same length: $l(A)$, $l(B)$ and $l(C)$. Each such length is determined by the size of the slit and the continued fraction expansion of the slope.

•This defines a three-term partition of the 2-torus: each term of this partition is the union of all orbits of type A (resp. B and C).

•In each term of this partition, the distribution of exit times (from the 2-torus with the slit removed) knowing the direction v is explicitly computed.

Gauss map $T: (0,1) \to (0,1)$ defined by $x \mapsto Tx = 1/x - [1/x]$

Continued fractions For $\alpha \in (0,1) \setminus \mathbf{Q}$, one has

$$\alpha = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \quad \text{with } a_k = \left[\frac{1}{T^{k-1}\alpha}\right]$$

Convergents Truncated continued fractions give rational approximants

$$\alpha \simeq [a_1, \dots, a_{n-1}] =: p_n/q_n$$

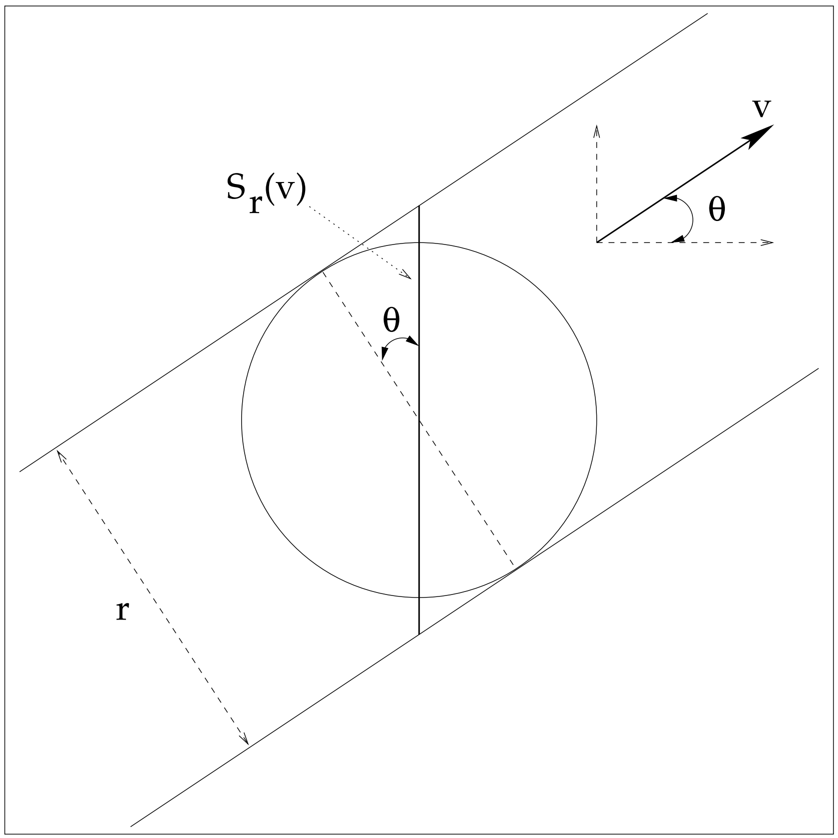
This defines recursively two sequences of integers p_n and q_n

$$p_{n+1} = a_n p_n + p_{n-1}\,, \quad p_0 = 1\,,\ p_1 = 0$$
$$q_{n+1} = a_n q_n + q_{n-1}\,, \quad q_0 = 0\,,\ q_1 = 1$$

Error $d_n = (-1)^{n-1}(q_n\alpha - p_n) > 0$ (signs of $q_n\alpha - p_n$ alternate)

Distribution of exit times for a given direction

$\bullet$Let $\theta \in (0, \frac{\pi}{4})$, $\alpha = \tan\theta$ and $v = (\cos\theta, \sin\theta)$; set $R = 2r/\cos\theta$;



•Let $p_n/q_n = n$th convergent and $d_n = n$th error in the continued fraction expansion of α.

Partition of $(0,1)$: For $I_{n,k} = [\max(d_n, d_{n-1} - kd_n), d_{n-1} - (k-1)d_n)$

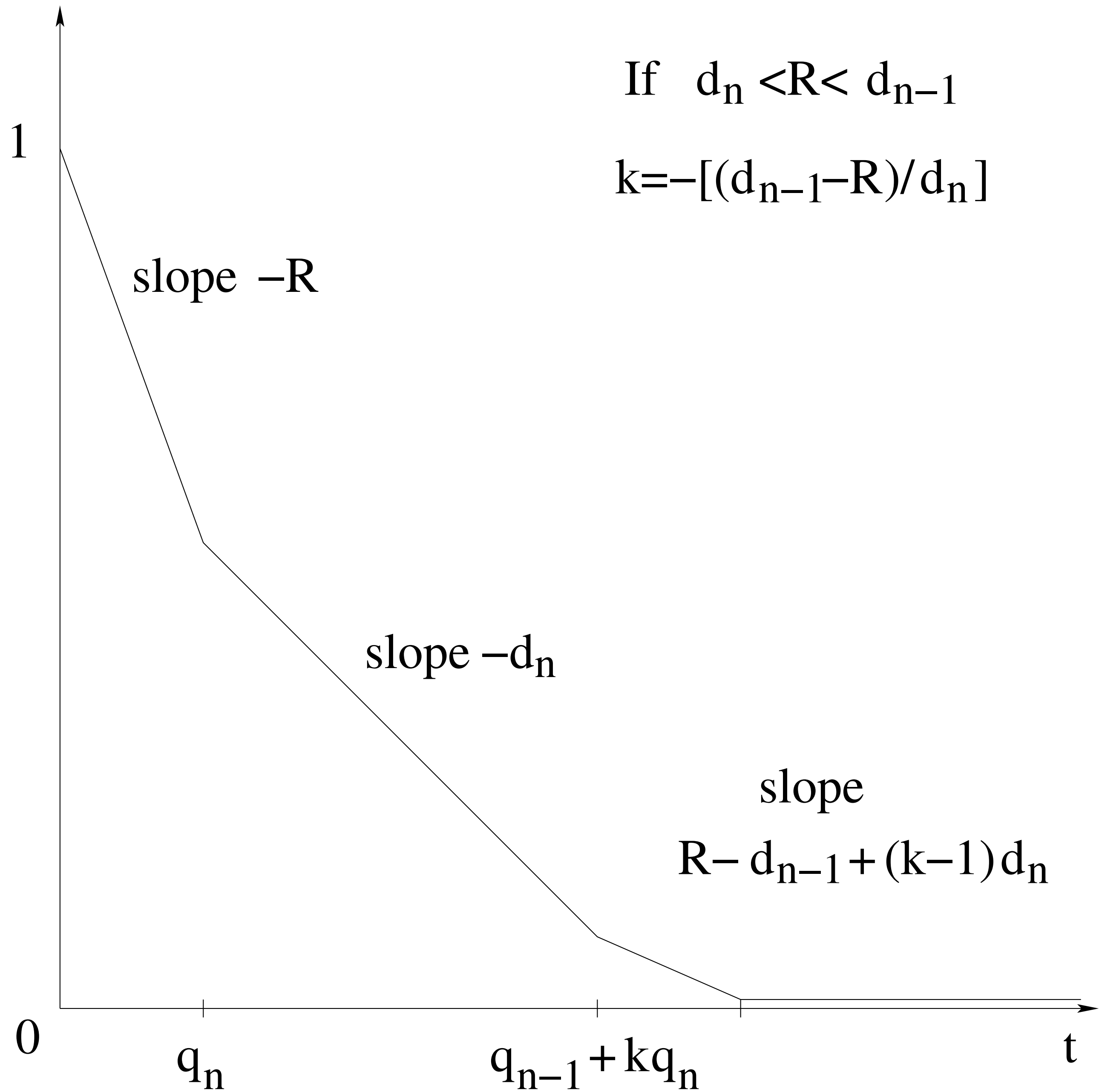
$$(0,1) = \bigcup_{n\geq 1} \bigcup_{1\leq k\leq a_n} I_{n,k}$$

•Assume $R \in I_{n,k}$ — this defines a unique pair (n,k) — and let $t > 2$.

•Let $\psi_r(t,v)$ be the distribution of exit times for a particle moving at speed 1 in the direction v on a 2-torus punctured with a vertical slit of length R.

Then

$$\left|\psi_r(t,v) - \left(1 - \frac{R}{d_{n-1}} - t\frac{d_n}{R}\right)_+\right| \leq \frac{4}{k}\mathbf{1}_{k\geq t-2}\,.$$


If $d_n < R < d_{n-1}$
$k = -[(d_{n-1} - R)/d_n]$
1
slope $-R$
slope $-d_n$
slope
$R - d_{n-1} + (k-1)\, d_n$
0
q_n
$q_{n-1} + k q_n$
t

Idea no.2 Given the direction $v = (\cos\theta, \sin\theta)$, the distribution of exit times in a 2-torus with a disk of radius r removed is obtained by comparing $2r$ with the errors in the continued fraction expansion of $\alpha = \tan\theta$.

•Observe that $d_n(\alpha) = \alpha d_{n-1}(T\alpha)$.

•Renormalization replace the problem defined by a slope α and a disk of radius r with the analogous problem with slope $T\alpha$ and a disk of radius αr.

•This suggests seeking a fixed point of this transformation, by using some ergodic theorem where

$$\boxed{\text{TIME } = \ln(\text{ DISK SIZE })}$$

$\Rightarrow$ this explains why the Cesarò mean involves the invariant measure of the multiplicative group $\mathbf{R}_+^*$, i.e. $\frac{dr}{r}$.

•The Gauss map T is ergodic on $(0,1)$ with invariant measure

$$dg(\alpha) = \tfrac{1}{\ln 2}\frac{d\alpha}{1+\alpha}$$

•Define $N(\alpha,\epsilon) = \inf\{n \in \mathbf{N} \mid d_n(\alpha) < \epsilon\}$; for $j = 0, 1$, we define

$$\Delta_j(\alpha, x) = -x - \ln d_{N(\alpha, e^{-x})+1-j}(\alpha)$$

Lemma. *Define* $F(\theta) = \int_0^{|\ln\theta|} f(|\ln\theta| - y, -y)dy$*; then, one has*

$$\lim_{\epsilon\to 0^+} \frac{1}{|\ln \epsilon|}\int_{x^*}^{|\ln\epsilon|} f(\Delta_0(\alpha,x), \Delta_1(\alpha,x))dx = \frac{12}{\pi^2}\int_0^1 \frac{F(\theta)d\theta}{1+\theta}$$

Conclude by applying the Lemma to $f(z_1, z_2) = (1 - e^{z_2} - te^{-z_1})_+$