Semi-lagrangian schemes for first and second order Hamilton-Jacobi equations

joint work with M.Falcone, R.Ferretti, G.Russo

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APPLICATION FOR HJ EQUATION

- Image Analysis (contour detection, shape from shading, ...)
- Front Propagation (level set method)
- Optimal Control Process

EXISTENCE and UNICITY

in the class of viscosity solution:

• Y.G. Chen, Y. Giga, S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, Journal Diff. Geom., **33** (1991), 749-786.

•M.G. Crandall, H. Ishii, P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations* Bull. Amer. Math. Soc., **27** (1992), 1-67.

•L.C. Evans, J. Spruck, *Motion of level sets by mean curvature motion I*, Journal Diff. Geom., **33** (1991), 635-681.

OUTLINE

first order: $u_t + H(Du) = 0$

- SL scheme
- High order scheme with Weno reconstruction
- Numerical Test

second order: $u_t + H(D^2u, Du) = 0$

- SL scheme for Mean Curvature Motion codimension 1
- SL scheme for Mean Curvature Motion codimension 2
- Numerical Test

FIRST ORDER EQUATION

$$\begin{cases} u_t(x,t) + H(\nabla u(x,t)) = 0 & x \in \mathbb{R}^N, t \in [0,T] \\ u(x,0) = u_0(x) \end{cases}$$

Legendre Transform : $H^*(\alpha) = \sup_{p \in \mathbb{R}^N} \{\alpha \cdot p - H(p)\} \qquad \alpha \in \mathbb{R}^N.$

$$p \ \to H(p) \quad \text{convex} \ \Rightarrow \ H^{**} = H$$

$$H(\nabla u) = \sup_{\alpha \in \mathbb{R}^N} \{\alpha \cdot \nabla u - H^*(\alpha)\}$$

Hopf-Lax formula :

$$u(x,t) = \inf_{\alpha \in \mathbb{R}^N} \{t \ H^*(\alpha) + u_0 \left(x - \alpha t\right)\}$$

NUMERICAL SOLUTION

Dynamic Programming Principle

$$u(x, t+\Delta t) = \inf_{\alpha \in \mathbb{R}^N} \{\Delta t H^*(\alpha) + u(x - \alpha \Delta t, t)\}$$

SL scheme
$$\begin{cases} u_i^{n+1} = \min_{a \in \mathbb{R}^N} \{\Delta t H^*(a) + I[u^n](x_i - \Delta t a)\} \\ u_i^0 = u(x_i, 0). \end{cases}$$

We need to compute:

- $I[u^n](x_i - \Delta t a)$ by an INTERPOLATION procedure $\Rightarrow$ WENO
- Legendre trasform $H^*(a)$
- minimum for $a \in \mathbb{R}^N$ by PRAXIS algorithm (ftp.netlib.org)

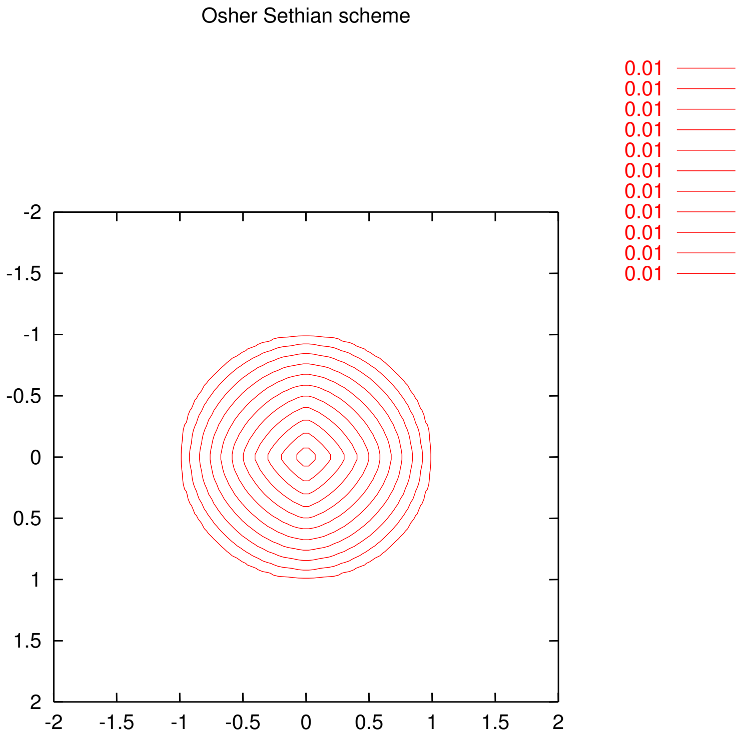
TEST 1: two dimensional circle collapse

$$\begin{cases} v_t(x,t) + |Dv(x,t)| = 0 \\ v(x,0) = \max(1 - |x|, 0), \end{cases}$$

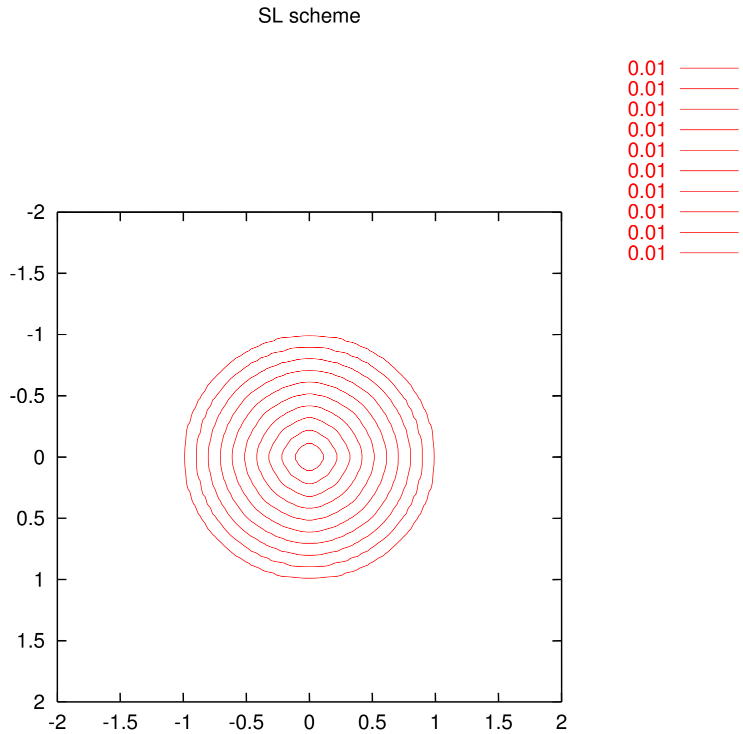
OSHER-SETHIAN scheme					
$n.it$	Δx	Δt	relativeL^∞-order	relative L^1-error	CPU time
10	0.08	0.01	$3.35 \cdot 10^{-2}$	$2.74 \cdot 10^{-2}$	$< 1s$
2	0.08	0.05	$2.14 \cdot 10^{-2}$	$1.63 \cdot 10^{-2}$	$< 1s$
50	0.04	0.02	$1.85 \cdot 10^{-2}$	0.38	

Semi-Lagrangian scheme					
$n.it$	Δx	Δt	relativeL^∞-order	relative L^1-error	CPU time
10	0.08	0.01	$3.19 \cdot 10^{-2}$	$2.61 \cdot 10^{-2}$	$25.69s$
2	0.08	0.05	$2.00 \cdot 10^{-2}$	$1.00 \cdot 10^{-2}$	$2.48s$
10	0.04	0.1	$4.57 \cdot 10^{-3}$	$1.88 \cdot 10^{-2}$	

Osher Sethian scheme



SL scheme



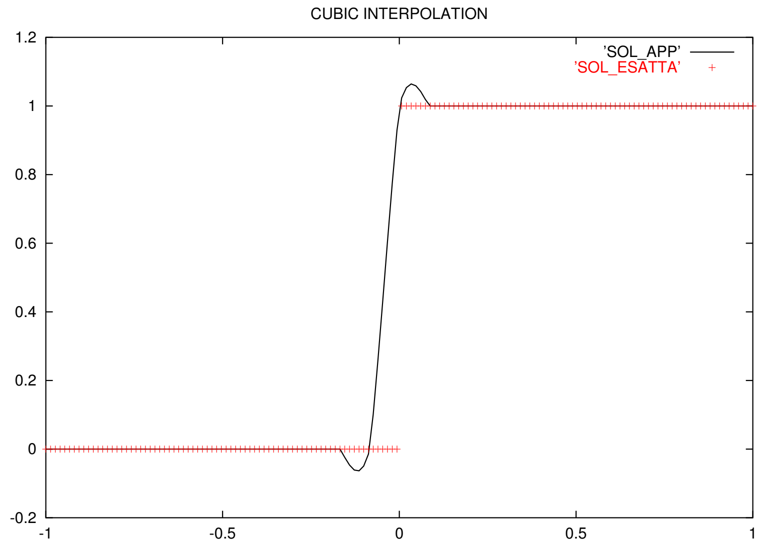
WENO RECOSTRUCTION

Idea: convex combination of lower order reconstruction.

$$I[V](x) = \sum_{k=1}^{n} \omega_k P_k(x).$$

P_k are lower order Lagrange polynomials interpolating V

w_k are non linear weights, which measure how smooth the functions P_k are in x


CUBIC INTERPOLATION
'SOL_APP'
'SOL_ESATTA'
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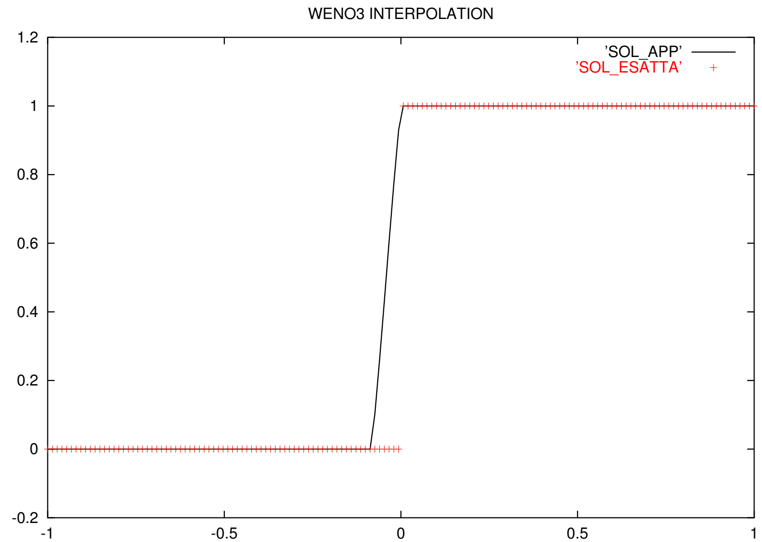
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.5
0
0.5
1


WENO3 INTERPOLATION
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1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.5
0
0.5
1

ENO-WENO REFERENCES

S.Osher, C.W.Shu *High order essentially non oscillatory schemes for Hamilton Jacobi equation*, SIAM J.Num.Anal., 28, (1991), 907-922.

X.D.Liu, S.Osher,T.Chan *Weighted essentially non oscillatory schemes*, J. of Computational Physics, V115, (1994), 220-212.

C.Jiang, C.W.Shu *Efficient Implementation of WENO Schemes*, J. of Computational Physics, V126, (1996), 202-228.

S.Bryson, D.Levy *Central Schemes for Multi-Dimensional Hamilton Jacobi Equations*, submitted to SICS.

$2n-1$ WENO RECONSTRUCTION $I[V](x)$

Assume we have an interval 1D with a mesh structure, we'd like to reconstruct V in x

-V is known on the mesh nodes
-$x \in I_j = [x_j, x_{j+1}]$

1. $S_k = \{x_{j-n+k}, ..., x_{j+k}\}$ $k = 1, ...n,$
   n stencils s.t. x belong to each one

2. $\Gamma = \bigcup_{k=1}^{n} S_k$

3. $P_k(x)$ polynomials of degree n interpolating V on S_k

4. $Q(x)$ higher order $2n-1$ polynomial on the larger stencil Γ

5. $C_k(x)$ linear weight s.t. $Q(x) = \sum_{k=1}^{n} C_k(x) P_k(x)$

6. β_k SMOOTHNESS INDICATOR ,

$$\beta_k = \sum_{1 \le l \le n} \int_{I_j} h^{2l-1} (D^l P_k(x))^2 dx \quad k = 1, ..., n \quad h \text{ size of } I_j$$

7. ω_k NON LINEAR WEIGHT

$$\omega_k = \frac{\tilde{\omega}_k}{\sum_{k=1}^{n} \tilde{\omega}_k}, \quad \tilde{\omega}_k = \frac{C_k}{(\epsilon + \beta_k)^2}$$

The final WENO reconstruction is

$$I[V](x) = \sum_{k=1}^{n} \omega_k P_k(x).$$

POSITIVITY of the LINEAR WEIGHTS $C_k(x)$

$$Q(x) = \sum_{k=1}^{n} C_k(x) P_k(x) \quad (1)$$

should interpolate V on the larger stencil $\Gamma \Rightarrow$

$$C_k(x) = \gamma_k \prod_{x_l \in \Gamma \setminus S_k} (x - x_l) \quad (2)$$

with γ_k unknowns to be determined.
Using (2) in (1):

$$C_1(x_{j-n+1}) = 1$$
$$C_1(x_{j-n+2}) + C_2(x_{j-n+2}) = 1$$
$$\vdots$$
$$\sum_{i=1}^{k} C_i(x_{j-n+k}) = 1 \quad k = 1, .., n$$

More explicitly we obtain the relation

$$\sum_{i=1}^{k} \gamma_i \prod_{x_l \in \Gamma \setminus S_k} (x - x_l) = 1 \quad k = 1, .., n$$

Using some tools from combinatorics we proved that

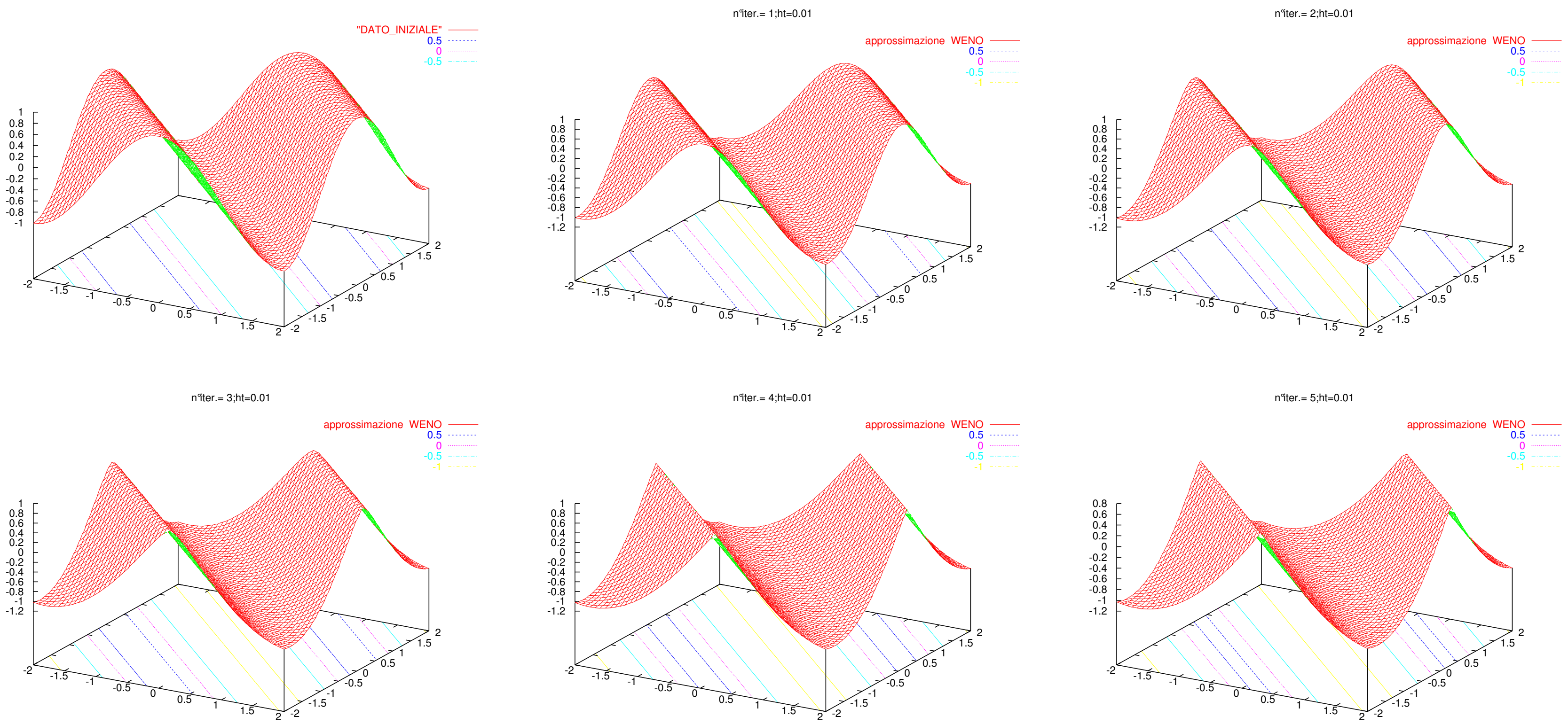
$C_k(x)$ are **positive** for each k

Therefore $I[V](x) = \sum_{k=1}^{n} \omega_k P_k(x)$ is a convex combination and

$$\min_k P_k(x) \leq I[V](x) \leq \max_k P_k(x)$$

TEST 1: two dimensional periodic solution

$$\begin{cases} v_t(x,t) + \frac{(v_{x_1}(x,t)+v_{x_2}(x,t)+1)^2}{2} = 0 \\ v(x,0) = v_0(x) = -\cos\frac{\pi(x_1+x_2)}{2}, \end{cases}$$



WENO 3rd order				
Δx	relative L^∞-error	L^∞-order	relative L^1-error	L^1-order
0.16	$3.00 \cdot 10^{-3}$		$7.43 \cdot 10^{-4}$	
0.08	$4.66 \cdot 10^{-4}$	2.6	$4.79 \cdot 10^{-5}$	3.9
0.04	$2.68 \cdot 10^{-5}$	4.1	$2.44 \cdot 10^{-6}$	4.2
0.02	$1.48 \cdot 10^{-6}$	4.1	$1.62 \cdot 10^{-7}$	3.9

$$\Delta t = 0.02, T = \frac{0.8}{\pi^2}$$

WENO 5th order				
Δx	relative L^∞-error	L^∞-order	relative L^1-error	L^1-order
0.16	$1.28 \cdot 10^{-3}$		$1.51 \cdot 10^{-4}$	
0.08	$4.89 \cdot 10^{-5}$	4.3	$3.40 \cdot 10^{-6}$	4.8
0.04	$2.07 \cdot 10^{-6}$	4.5	$1.01 \cdot 10^{-7}$	5
0.02	$2.37 \cdot 10^{-8}$	6.4	$1.09 \cdot 10^{-9}$	6.5

$$\Delta t = 0.02, T = \frac{0.8}{\pi^2}$$

WENO 3rd order				
Δx	relative L^∞-error	L^∞-order	relative L^1-error	L^1-order
0.16	$1.00 \cdot 10^{-2}$		$1.09 \cdot 10^{-3}$	
0.08	$8.68 \cdot 10^{-5}$	6.8	$3.50 \cdot 10^{-5}$	5
0.04	$9.35 \cdot 10^{-6}$	3.2	$1.97 \cdot 10^{-6}$	4.1
0.02	$2.97 \cdot 10^{-7}$	4.4	$1.09 \cdot 10^{-7}$	4.1

$$\Delta t = 0.03, T = \frac{1.5}{\pi^2}$$

WENO 5th order				
Δx	relative L^∞-error	L^∞-order	relative L^1-error	L^1-order
0.16	$8.55 \cdot 10^{-3}$		$7.00 \cdot 10^{-4}$	
0.08	$8.53 \cdot 10^{-4}$	3.6	$9.30 \cdot 10^{-5}$	4.3
0.04	$2.08 \cdot 10^{-6}$	8.6	$4.68 \cdot 10^{-8}$	9.4
0.02	$5.34 \cdot 10^{-9}$	8.6	$1.69 \cdot 10^{-10}$	8.1

$$\Delta t = 0.03, T = \frac{1.5}{\pi^2}$$

SL SCHEME for MEAN CURVATURE MOTION

$$\begin{cases} u_t(x,t) & = div\left(\frac{Du(x,t)}{|Du(x,t)|}\right)|Du(x,t)| \\ u(x,0) & = u_0(x) \\ & \end{cases}$$

representation formula by Soner and Touzi:

$$u(x,t) = E\{u_0(y(x,t,t))\}, \qquad Du \neq 0$$

$$\begin{cases} dy(x,t,s) = \sqrt{2}P(y,t,s)dW(s) \\ y(x,t,0) = x \end{cases}$$

$$P(y,t,s) = \frac{1}{|Du|^2}\begin{pmatrix} u_{x_2}^2 & -u_{x_1}u_{x_2} \\ -u_{x_1}u_{x_2} & u_{x_1}^2 \end{pmatrix}$$

H.M. Soner, N. Touzi, *A stochastic representation for mean curvature type geometric flows*, Ann. Probab. 31 (2003), no. 3, 1145–1165.

CONSTRUCTION OF THE SCHEME

Soner Touzi formula between t and $t + \Delta t$:

$$u(x, t + \Delta t) = E\{u(y(x, t + \Delta t, \Delta t), t)\}$$

Brownian dimension reduction :

$$\sqrt{2} P dW = \frac{\sqrt{2}}{|Du|} \begin{pmatrix} u_{x_2} \\ -u_{x_1} \end{pmatrix} \left(\frac{u_{x_2} dW_1}{|Du|} - \frac{u_{x_1} dW_2}{|Du|} \right) = \sigma d\widehat{W}$$

we can replace the Stochastic Initial problem by

$$\begin{cases} dy(x, t, s) = \sigma(y, t, s) d\widehat{W}(s) \\ y(x, t, 0) = x. \end{cases}$$

Weak Euler :

$$\begin{cases} y_{k+1} = y_k + \sqrt{2}\sigma(y, t_k, 0)\Delta\widehat{W}_k \\ y_0 = x. \end{cases}$$

with

$$P(\Delta\widehat{W}_k = \pm\sqrt{\Delta t}) = \frac{1}{2}.$$

Time-discretization

$$\begin{aligned} u_{\Delta t}(x, t_{n+1}) = \ & \tfrac{1}{2}u_{\Delta t}(x + \sqrt{2}\sigma(x, t_n, 0)\sqrt{\Delta t}, t_n) + \\ & + \tfrac{1}{2}u_{\Delta t}(x - \sqrt{2}\sigma(x, t_n, 0)\sqrt{\Delta t}, t_n). \end{aligned}$$

Fully discrete scheme $Du \neq 0$

$$u_j^{n+1} = \frac{1}{2}\Big(I[u^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I[u^n](x_j - \sigma_j^n\sqrt{\Delta t})\Big)$$

MODIFIED SCHEME

$$\begin{cases} u_j^{n+1} = \frac{1}{2}\Big(I[u^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I[u^n](x_j - \sigma_j^n\sqrt{\Delta t})\Big) & \text{if } |D_j^n| > C\Delta x^s \\ u_j^{n+1} = \frac{1}{4}\sum\limits_{i\in\mathcal{D}(j)} u_i^n & \text{if } |D_j^n| \le C\Delta x^s \end{cases}$$

CONSISTENCY ERROR

(case $|D_j^n| > C\Delta x^s$)

$$L_{\Delta x,\Delta t} = O\left(\frac{\Delta x^r}{\Delta t}\right) + O\left(\frac{\Delta x^{q-s}}{\Delta t^{\frac{1}{2}}}\right) + O(\Delta t^{\frac{1}{2}}) + O\left(\frac{\Delta t}{\Delta x^s}\right)$$

MONOTONE SCHEME

$$\widehat{H}(u^n;j) = \frac{(1-\lambda)}{2}\Big[I_1[u^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I_1[u^n](x_j - \sigma_j^n\sqrt{\Delta t})\Big]$$
$$+\frac{\lambda}{4}\sum_{i\in\mathcal{D}(j)} u_i^n, \qquad\qquad \text{with } \lambda \in [0,1].$$

Theorem Assume $\lambda \geq \frac{1}{1+\delta}$, where $\delta \equiv \frac{C\Delta x^{s+1}}{16\sqrt{\Delta t}L_{I_1[u^n],\mathcal{G}}}$.

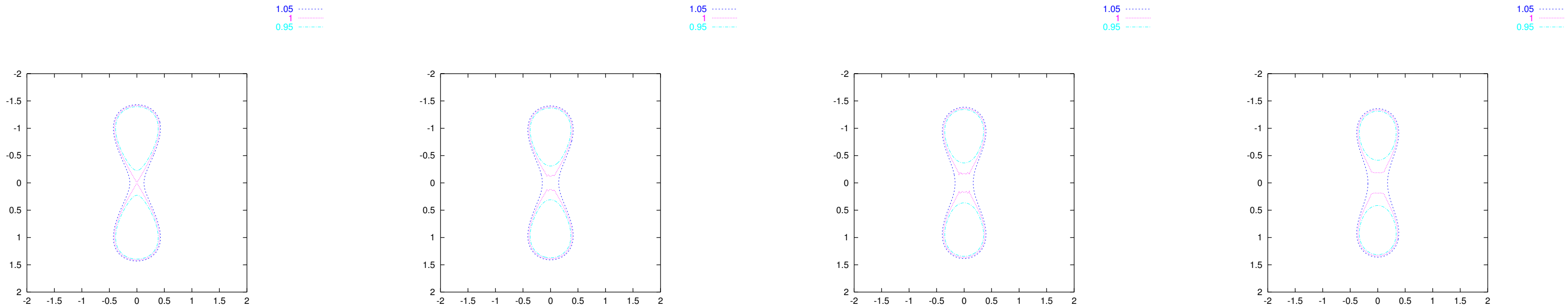
Then $\widehat{H}(u^n;j)$ is monotone.

TEST: two dimensional circle evolution

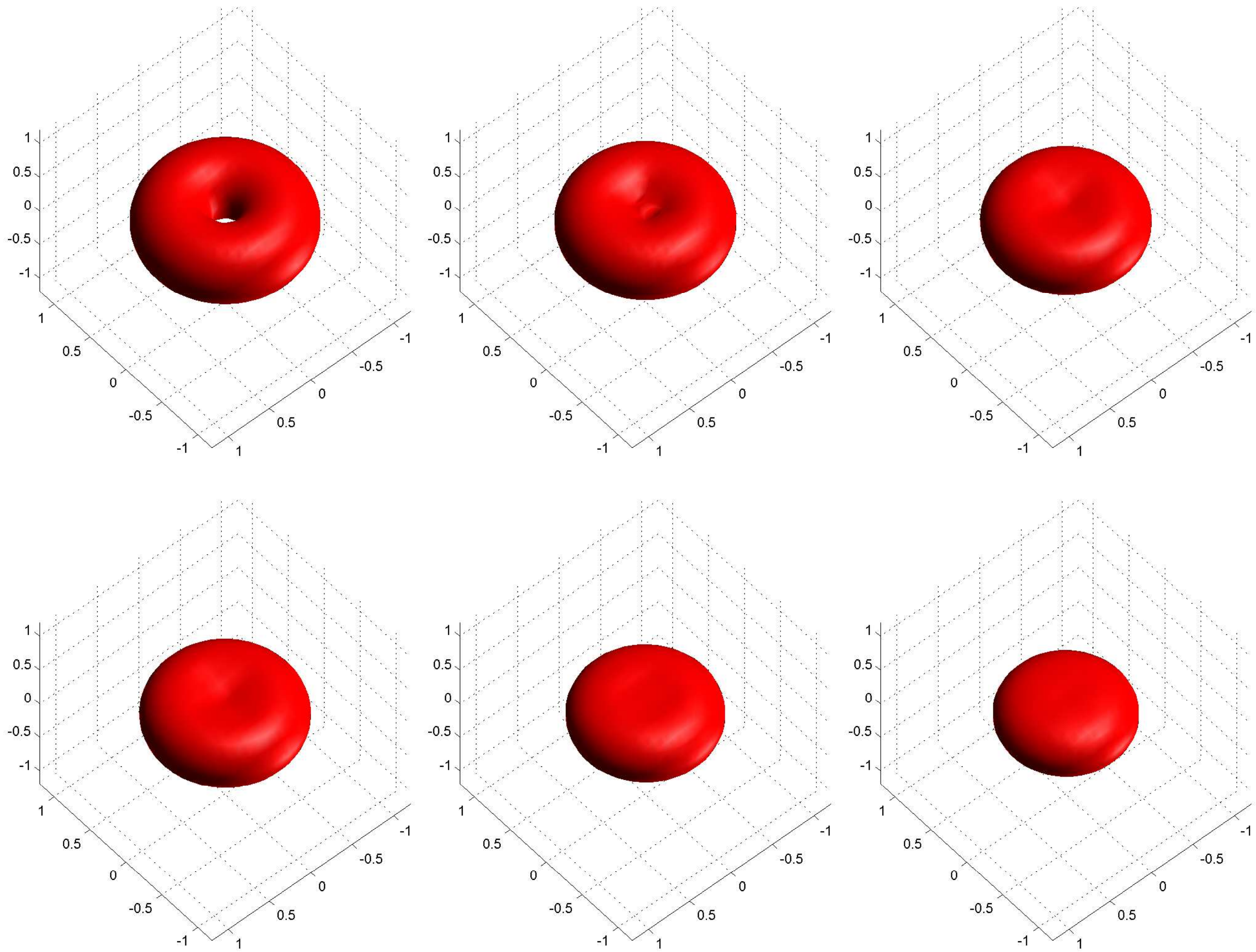
$$\Delta t = O(\Delta x^{\frac{3}{4}}),\ s = \frac{1}{4}.$$

Δx	Δt	$\|\cdot\|_\infty$	$\|\cdot\|_1$	$L^\infty - order$	$L^1 - order$
0.04	0.08	$3.04 \cdot 10^{-4}$	$6.50 \cdot 10^{-6}$		
0.02	0.053	$1.25 \cdot 10^{-4}$	$3.42 \cdot 10^{-6}$	1.2	0.9
0.01	0.032	$5.22 \cdot 10^{-5}$	$1.82 \cdot 10^{-6}$	1.2	1.6
0.005	0.02	$2.09 \cdot 10^{-5}$	$7.75 \cdot 10^{-7}$	1.3	1.2

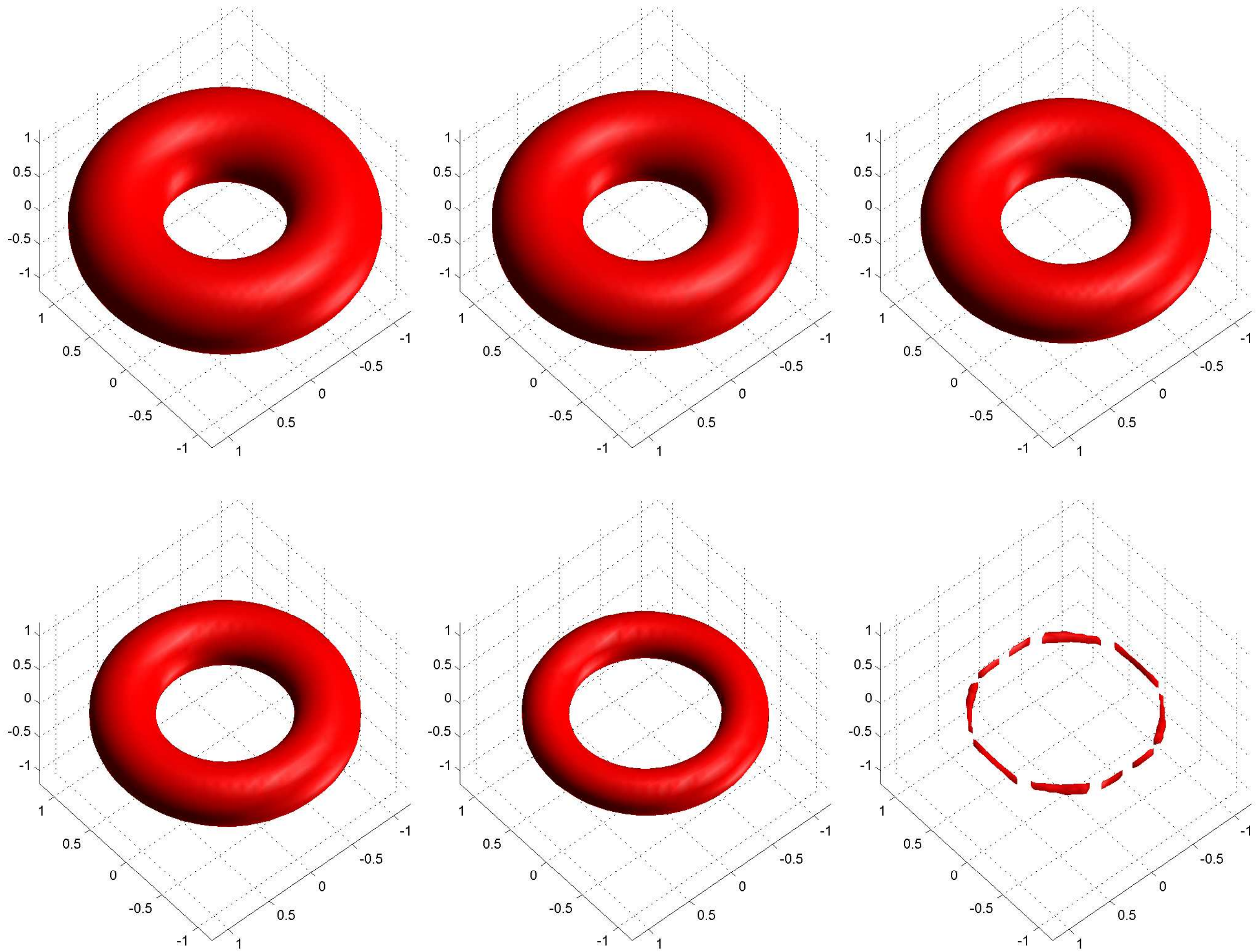
DEVELOPMENT of a NONEMPTY INTERIOR



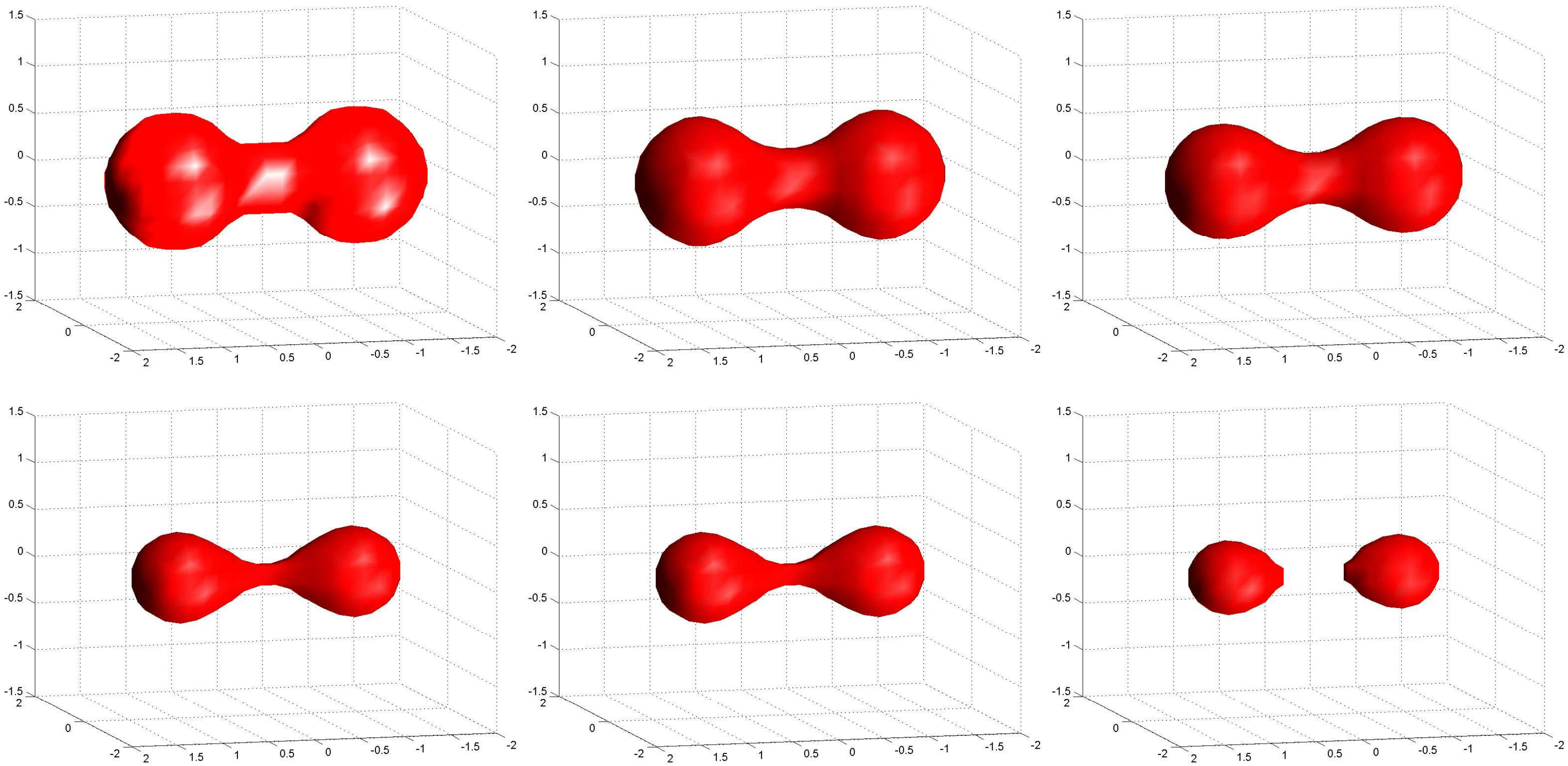
Fattening: evolution of the level curves $u = 0.095, 1, 1.05$



The torus evolving into a sphere



The torus collapsing in a circle



Dumb-bell: topology change in $\mathbb{R}^3$

MCM codimension 2 $\begin{cases} u_t = F(D^2u, Du) & \mathbb{R}^3 \times [0,\infty) \\ u(x,0) = \frac{1}{2}d(x,\mathcal{C})^2 \end{cases}$

$F(A,p) = \inf_{\nu \in \mathcal{N}(p)} \{\mathrm{trace}[AP^\nu]\}$ $\mathcal{N}(p) = \{\nu \in S^2 : P^\nu p = 0\}$

$$P^\nu = \nu\nu^T$$

Soner Touzi formula:

$$u(x,t) = \inf_{\nu \in \mathcal{U}} \{E\{u(y_\nu(x,t,t),0)\}\}$$

with $\mathcal{U} = \{\nu : [0,T] \to S^2 \subset \mathbb{R}^3 : \nu \cdot Du(x,t) = 0\}$

$$\begin{cases} dy_\nu(x,t,s) = \sqrt{2}\nu(s)d\widehat{W}(s) \\ y_\nu(x,t,0) = x \end{cases}$$

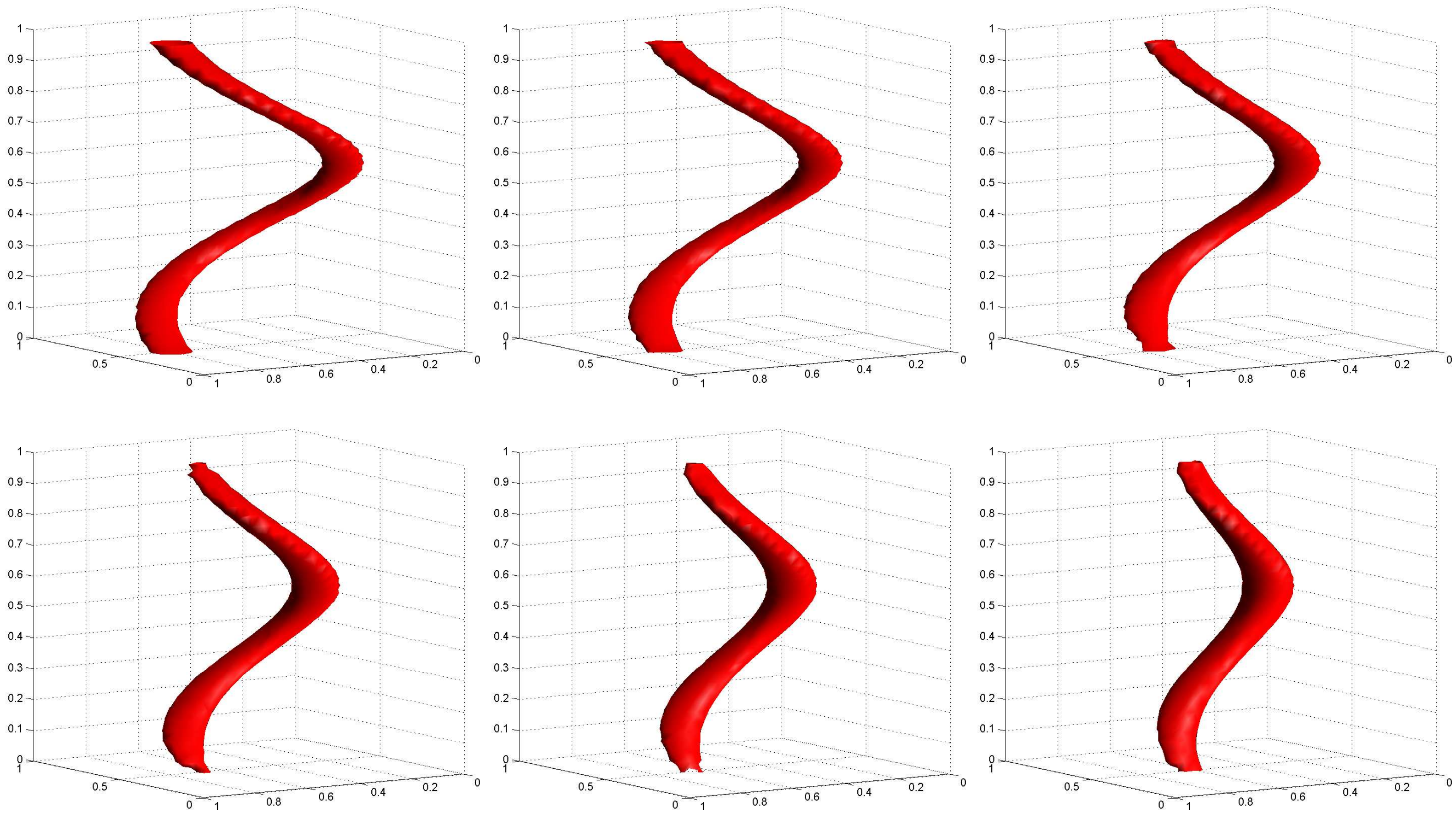
L.Ambrosio, M.Soner, *Level Set Approach to Mean Curvature Flow in Arbitrary Codimension*, J.Differential Geometry, **43** (1996), 693-737.

Time-discretization:

$$u_{\Delta t}(x,t_{n+1}) = \inf_{\nu^n \in \widehat{\mathcal{U}}} \left\{ \frac{1}{2} u_{\Delta t}(x+\sqrt{2}\nu^n\sqrt{\Delta t}, t_n) + \frac{1}{2} u_{\Delta t}(x-\sqrt{2}\nu^n\sqrt{\Delta t}, t_n) \right\}.$$

Full-discrete scheme:

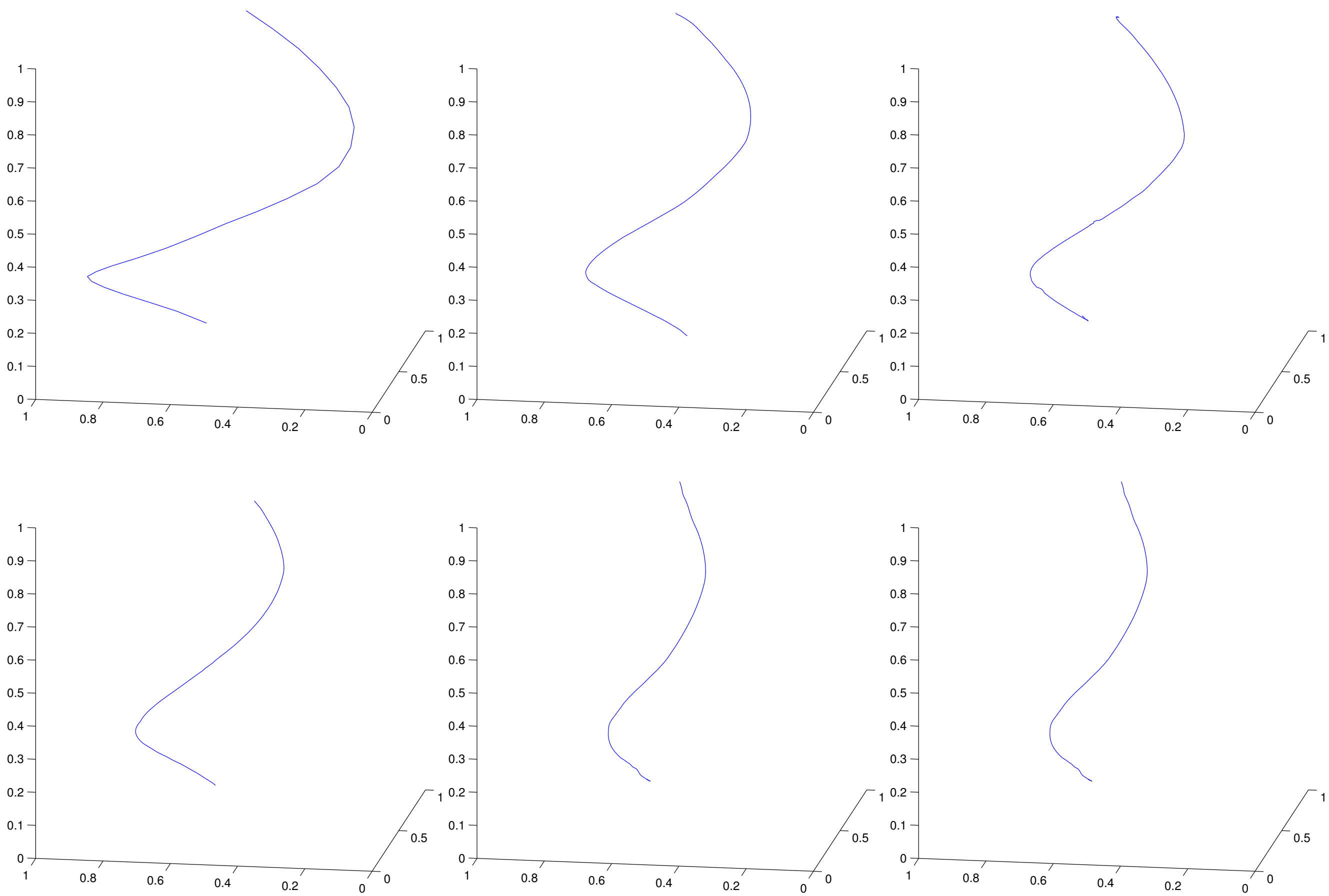
$$u_j^n \;= \min_{\nu^n \in \mathbb{R}^3} \{ \tfrac{1}{2} I[u^n](x+\sqrt{2}\nu^n\sqrt{\Delta t}) + \tfrac{1}{2} I[u^n](x-\sqrt{2}\nu^n\sqrt{\Delta t}) + \\ + \tfrac{(|D_j^n \nu^n|)^2}{\epsilon_1} + \tfrac{(|\nu^n|-1)^2}{\epsilon_2} \}.$$



Evolution of ϵ helical surface, $\epsilon = 0.008$.

OPTIMAL TRAJECTORY ALGORITHM

- $s_0 := x_{jmin}$ point of the curve
- $\nu_j^{n*} = \underset{\nu_j^n \in \hat{\mathcal{U}}}{\text{argmin}} \{u^n(s_j + \sqrt{2\Delta s}\nu_j^n)\} \quad j = 0, ..., \hat{j}$
- $s_{j+1} = s_j + \sqrt{2\Delta s}\nu_j^{n*} \quad j = 0, ..., \hat{j}$
- $|s_0 - s_{\hat{j}}| \leq \epsilon$



Evolution of a helix in $\mathbb{R}^3$.

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•M.Falcone, R.Ferretti, *Consistency of a large time-step scheme for mean curvature motion*, In F.Brezzi, A.Buffa, S.Corsaro and A.Murli (eds), Numerical Analysis and Advanced Applications-Proceedings on ENUMATH 2001, Ischia,(2001).