A HYPERVISCOSITY SPECTRAL LARGE EDDY SIMULATION MODEL FOR TURBULENT FLOWS

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THE NAVIER–STOKES EQUATIONS

Overview

BASIC FACTS ABOUT THE NAVIER–STOKES EQUATIONS

THE NAVIER–STOKES EQUATIONS

- u: velocity, p: pressure
- Ω is the fluid domain

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \nu \nabla^2 u = \mathrm{f} \quad \text{in } \Omega \\ \nabla \cdot u = 0 \quad \text{in } \Omega, \\ u|_\Gamma = 0 \quad \text{or } u \text{ is periodic,} \\ u|_{t=0} = u_0, \end{cases}$$

- u_0 is the initial data
- f a source term.
- ρ is chosen equal to unity.

EXISTENCE AND UNIQUENESS

- **J. Leray (1934): introduces the notion of turbulent solution. A turbulent solution is a weak solution in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.**
- **J. Leray uses mollification to prove existence: $\psi \in D(\mathbb{R}^3)$, $\psi \geq 0$, $\int_{\mathbb{R}^3} \psi = 1$, $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon}\right)$.**

$$\partial_t u_\epsilon + (\psi_\epsilon * u_\epsilon) \cdot \nabla u_\epsilon + \nabla p_\epsilon - \nu \nabla^2 u_\epsilon = \mathrm{f}$$

- **E. Hopf (1951) *et al.* uses the Galerkin technique to prove existence.**

The question of the existence of classical solutions for long times is open. This question is linked to the question of uniqueness of turbulent solutions.

$\Rightarrow$ Clay Institute 1M$ prize.

SUITABLE WEAK SOLUTION

V. Sheffer (1976) introduces suitable weak solutions:

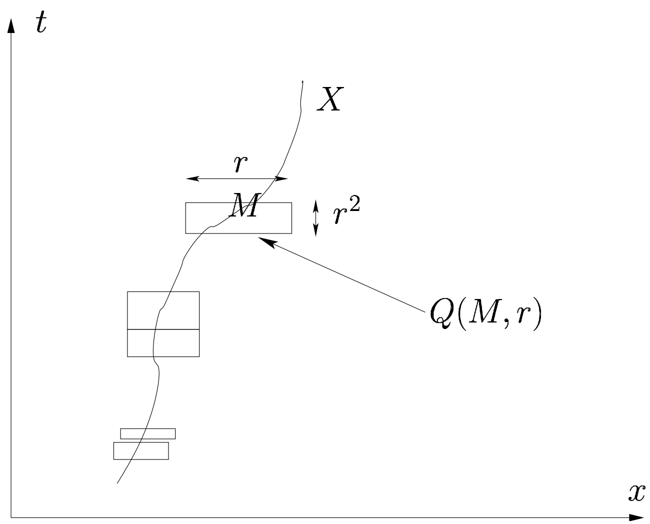
DEFINITION: ***A NS weak solution is said to be suitable (or dissipative) iff***

$$\partial_t(\tfrac{1}{2}u^2) + \nabla\cdot(u(\tfrac{1}{2}u^2 + p)) - \nu\nabla^2(\tfrac{1}{2}u^2) + \nu(\nabla u)^2 - \mathrm{f}\cdot u \leq 0.$$

THEOREM: (Caffarelli-Kohn-Nirenberg (1982)) ***The "dimension" of the set of singular points of suitable weak solutions in $\Omega \times [0, +\infty]$ is less than 1.***

Best partial regularity theorem to date.

$$\mathcal{P}^k(X) = \lim_{\delta\to 0^+} \inf\{\sum r_i^k,\ X \subset \cup Q(M_i, r_i),\ r_i < \delta\}$$



Singular set: S

$$S = \{(x,t) \in \Omega\times]0,T[,\ u \notin L^\infty(V), \forall V \text{ s.t. } (x,t) \in V\}$$

Caffarelli-Kohn-Nirenberg $\Rightarrow \mathcal{P}^1(S) = 0$

THEOREM: **(Duchon-Robert (2000))** ***The limits (as $\varepsilon \to 0^+$) of Leray-regularized NS solutions are dissipative*.**

The question is open for Galerkin weak NS solutions.

Proof: $u_\epsilon \rightharpoonup u$ **in** $L^2(H^1)$ **(up to subsequences) and**
$u_\epsilon \to u$ **in** $L^p(L^q)$, $1 \le p \le \frac{4q}{3(q-2)}$, $2 \le q < 6$.
Test with ϕu_ϵ, $\phi \in \mathcal{D}(Q)$ **(OK, since** u_ϵ **is smooth!).**

$$\int_Q -\frac{1}{2}u_\epsilon^2(\partial_t\phi + \nu\nabla^2\phi) - (\psi_\epsilon * u_\epsilon u_\epsilon^2 + p_\epsilon u_\epsilon)\nabla\phi + \nu(\nabla u_\epsilon)^2\phi - f u_\epsilon \phi = 0$$

$u_\epsilon^2 \to u^2$ **in** $L^p(L^1)$, $\forall p \ge 1$

$$\left.\begin{array}{r} \psi_\epsilon * u_\epsilon \to u \text{ in } L^4(L^3) \\ u_\epsilon^2 \to u^2 \text{ in } L^{\frac{4}{2}}(L^{\frac{3}{2}}) \end{array}\right\} \Rightarrow \psi_\epsilon * u_\epsilon u_\epsilon^2 \to u u^2 \text{ in } L^{\frac{4}{3}}(L^1)$$

Proof (continued):

Periodic domain and $\nabla \cdot f = 0$

$\nabla^2 p_\epsilon = -\nabla \cdot \nabla \cdot (\psi_\epsilon * u_\epsilon \otimes u_\epsilon) \Rightarrow$

$\|p_\epsilon\|_{L^2} \leq c\|\psi_\epsilon * u_\epsilon \otimes u_\epsilon\|_{L^2} \leq c\|u_\epsilon\|_{L^4} \Rightarrow p_\epsilon \longrightarrow p$ in $L^{\frac{4}{3}}(L^2)$.

$$\left.\begin{array}{r} p_\epsilon \longrightarrow p \text{ in } L^{\frac{4}{3}}(L^2) \\ u \longrightarrow u \text{ in } L^4(L^2) \end{array}\right\} \Rightarrow p_\epsilon u_\epsilon \longrightarrow pu \text{ in } L^1(L^1)$$

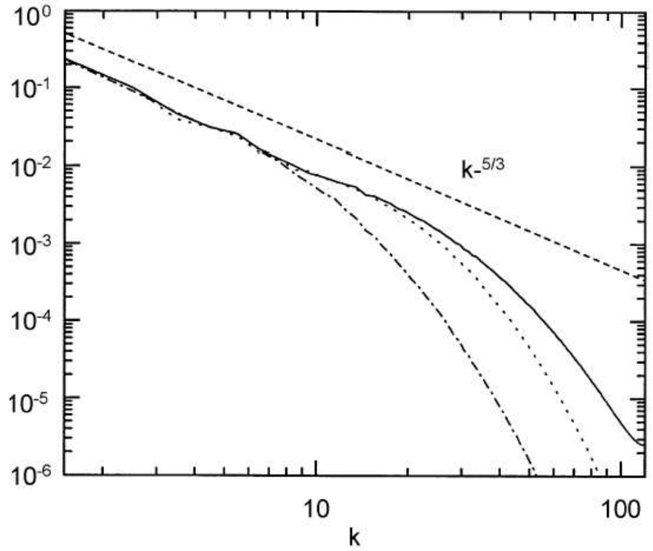
$\int_Q (\nabla u_\epsilon)^2 \phi \geq 2 \int_Q \phi \nabla (u_\epsilon - u) \nabla u + \int_Q \phi (\nabla u)^2 \Rightarrow$

$\liminf_{\epsilon \to 0} \int_Q (\nabla u_\epsilon)^2 \phi \geq \int_Q (\nabla u)^2 \phi$

LARGE EDDY SIMULATIONS

OBJECTIVES OF LES

- **Concept introduced by Leonard (1974)**
- **The objective is to modify the NS model so that the new model is amenable to numerical simulations.**



(From Foias, Holm and Titi, Physica D, 152–153 (2001))

- **Almost no reasonable mathematical theory for LES.**

THE FILTERING-EVERYTHING PARADIGM

Let $\overline{(\cdot)} : w \longmapsto \overline{w}$ be a regularizing operator acting on space- and time-dependent functions (we call it a filter).

Assume that the filter is linear and commutes with differential operators.

Apply the filter operator to the Navier–Stokes equations:

$$\begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} - \nu \nabla^2 \overline{u} = \overline{\mathrm{f}} - \nabla \cdot \mathbb{T}, \\ \nabla \cdot \overline{u} = 0 \\ \overline{u}|_\Gamma = 0, \quad \text{or } \overline{u} \text{ is periodic}, \\ \overline{u}|_{t=0} = \overline{u}_0, \end{cases}$$

where have introduced the so-called subgrid-scale tensor:

$$\mathbb{T} = \overline{u \otimes u} - \overline{u} \otimes \overline{u}$$

THE FILTERING PARADOX

THE CLOSURE PROBLEM:
The Goal of LES (Leonard (1974)) is to model $\mathbb{T}$ in terms of $\overline{u}$ only, without resorting to u.

THEOREM: (Germano (1986))
***Exact closure is possible,* e.g., *take* $\overline{\mathrm{v}} := (I - \varepsilon^2\nabla^2)^{-1}\mathrm{v}$**

PARADOX: (Guermond-Oden-Prudhomme (2001))
If the filter induces an isomorphism, the solution set of NS is isomorphic to the solution set of the filtered equations.

$\Rightarrow$ The two sets of weak solutions being isomorphic, one should expect to use the same number of degrees of freedom for approximating the NS solution as for approximating the filtered solution.

THE FILTERING PARADOX

CONCLUSION:

- **Filtering and achieving exact closure may not reduce the number of degrees of freedom.**
- **Filtering the Navier–Stokes equations is an efficient approach only if inexact closure is performed.**
- **The filtering-everything paradigm is probably nonsensical.**

PROPOSITION FOR A NEW PARADIGM FOR LES

DEFINITION: A sequence $(u_\epsilon, p_\epsilon)_{\epsilon>0}$ is said to be a LES solution of NS if

- For all $\epsilon > 0$, (u_ϵ, p_ϵ) is uniquely defined for all times (i.e. regularization technique s.t there is existence and uniqueness for u_ϵ and p_ϵ for all times).
- $u_\epsilon \longrightarrow u$, $p_\epsilon \longrightarrow p$ (up to subsequences) and (u, p) is a suitable weak solution of NS.

EXAMPLE 0: Galerkin approximations are regularizing techniques (so-called DNS).
Does a Galerkin solution converge to a suitable weak solution? (500k$ question?).

EXAMPLES OF LES

EXAMPLE 1: Leray's regularization is a LES technique (1934)! However the "model" is not frame invariant (is it important?).

EXAMPLE 2: NS-α model (Holmes–Marsden–Ratius (1998), Holmes *et al.* (1999)). Theoretical mechanics $\Rightarrow$

$$\partial_t u_\epsilon + (\overline{u_\epsilon}) \cdot \nabla u_\epsilon - (\nabla u_\epsilon)^T \overline{u_\epsilon} + \nabla p_\epsilon - \nu \nabla^2 u_\epsilon = \mathrm{f}$$

Alternative interpretation

$$\partial_t u_\epsilon + (\nabla \times u_\epsilon) \times \overline{u_\epsilon} + \nabla \pi_\epsilon - \nu \nabla^2 u_\epsilon = \mathrm{f}.$$

- **It is a Leray regularization where the nonlinear term is $(\nabla \times u) \times u + \nabla(\frac{u^2}{2})$.**
- **The equation is frame-indifferent.**
- **$u_\epsilon \to u$ (up to subsequences) and u is suitable**

EXAMPLES OF LES

EXAMPLE 3: **Ladyzenskaja (1967) proposed**

$$\begin{cases} \partial_t u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon + \nabla p_\epsilon - \nabla \cdot (\nu \nabla u_\epsilon + \varepsilon \mathrm{T}(\mathrm{D})) = \mathrm{f}, \\ \nabla \cdot u_\epsilon = 0 \\ u_\epsilon|_\Gamma = 0, \quad \text{or } u \text{ is periodic,} \\ u_\epsilon|_{t=0} = u_0. \end{cases}$$

where operator T is nonlinear, and $D = \frac{1}{2}(\nabla u_\epsilon + (\nabla u_\epsilon)^T)$.

For instance $\mathrm{T}(\xi) = \beta(|\xi|^2)\xi$ with $c\tau^\mu \leq \beta(\tau) \leq c'\tau^\mu$, with $\mu \geq \frac{1}{4}$ is possible. $\beta(\tau) = \tau^{1/2}$ yields Smagorinsky's model.

THEOREM: **(Ladyzenskaja (1967))** ***The modified NS equations have a unique weak solution for all $t > 0$.***

Moreover $u_\epsilon \to u$ (up to subsequences) and u is suitable.

EXAMPLE 4: HYPERVISCOSITY

HYPERVISCOSITY

EXAMPLE 4: **Lions (1959) proposed to use hyperviscosity. Ω is the d-torus, where d is the space dimension.**

$$\begin{cases} \partial_t u_\epsilon + u_\epsilon \cdot \nabla u_\epsilon + \nabla p_\epsilon - \nu \nabla^2 u_\epsilon + \varepsilon(-\nabla^2)^\alpha u_\epsilon = \mathrm{f}, \\ \nabla \cdot u_\epsilon = 0 \\ u_\epsilon \text{ is periodic,} \\ u_\epsilon|_{t=0} = u_0. \end{cases}$$

THEOREM: **(Lions (1959))** ***The modified NS equations have a unique weak solution for all $t > 0$ if $\alpha > \frac{d+2}{4}$.***

Moreover $u_\epsilon \to u$ (up to subsequences) and u is suitable.

QUESTION: **How this technique can be implemented numerically?**

HYPERVISCOSITY: A PRIORI ESTIMATES

Testing the momentum equation by u_ϵ yields

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{L^2}^2 + \nu\|\nabla u_\epsilon\|_{L^2}^2 + \epsilon\|(-\nabla^2)^{\frac{\alpha}{2}} u_\epsilon\|_{L^2}^2 = (f, u_\epsilon).$$

Integration in time, gives the following estimate:

$$\|u_\epsilon\|_{L^2}^2 + \int_0^T \|u_\epsilon\|_{H^\alpha}^2 dt \lesssim \|u_0\|_{L^2}^2 + \|f\|_{L^2(L^2)}^2.$$

In a similar manner, testing by $u_{\epsilon,t}$ yields

$$\|u_{\epsilon,t}\|_{L^2(L^2)}^2 + \|u_\epsilon\|_{H^\alpha}^2 \lesssim \|u_0\|_{H^\alpha}^2 + \int_0^T dt \int_\Omega |u_\epsilon|^2 |\nabla u_\epsilon|^2\, dx + \|f\|_{L^2(L^2)}^2.$$

Using Hölder's inequality, we write

$$\int_\Omega |u_\epsilon|^2 |\nabla u_\epsilon|^2\, dx \lesssim \|u_\epsilon\|_{L^{2p}}^2 \|\nabla u_\epsilon\|_{L^{2p'}}^2, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

HYPERVISCOSITY: A PRIORI ESTIMATES

Owing to Sobolev inequalities we also have

$$\|u_\epsilon\|_{L^{2p}} \lesssim \|\nabla u_\epsilon\|_{L^{2p'}} \qquad \text{if} \quad \frac{1}{2p} \geq \frac{1}{2p'} - \frac{1}{d},$$

$$\|\nabla u_\epsilon\|_{L^{2p'}} \lesssim \|u_\epsilon\|_{H^\alpha} \qquad \text{if} \quad \frac{1}{2p'} \geq \frac{1}{2} - \frac{\alpha-1}{d}.$$

These two conditions yield

$$\alpha \geq \frac{d+2}{4} \quad \text{and} \quad p' \geq \frac{2d}{d+2},$$

and

$$\|u_{\epsilon,t}\|^2_{L^2(L^2)} + \|u_\epsilon\|^2_{H^\alpha} \lesssim \|u_0\|^2_{H^\alpha} + \int_0^T \|u_\epsilon\|^2_{H^\alpha} \|u_\epsilon\|^2_{H^\alpha}\, dt + \|f\|^2_{L^2(L^2)}$$

and since $\int_0^T \|u_\epsilon\|^2_{H^\alpha}\, dt$ is bounded, Gronwall's lemma yields

$$\|u_{\epsilon,t}\|_{L^2(L^2)} + \|u_\epsilon\|_{L^\infty(H^\alpha)} \lesssim c(\nu, u_0, f, \epsilon).$$

Existence of solutions in $L^\infty(0,T;H^\alpha)$, $\forall T > 0$, is proved by means of the Galerkin technique using the *a priori* estimates.

SPECTRAL HYPERVISCOSITY

FOURIER APPROXIMATION

DEFINITION:

$$H^s(\Omega) = \{u = \sum_{k\in\mathbb{Z}^3} u_k e^{ik\cdot x},\ u_k = \overline{u}_{-k},\ \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |u_k|^2 < +\infty\}.$$

$$(u, v) = (2\pi)^{-3} \int_\Omega u\overline{v} = \sum_{k\in\mathbb{Z}^3} u_k \overline{v}_k.$$

$$\mathbb{P}_N = \left\{p(x) = \sum_{|k|_\infty \le N} c_k e^{ik\cdot x},\ c_k = \overline{c}_{-k}\right\},$$

Velocity space $\longrightarrow$ $X_N = \dot{\mathbb{P}}^3_N$,

Pressure space $\longrightarrow$ $M_N = \dot{\mathbb{P}}_N$.

FOURIER APPROXIMATION

$$H^s(\Omega) \ni \sum_{k\in\mathbb{Z}^3} v_k e^{ik\cdot x} = v \longmapsto P_N v = \sum_{|k|_\infty \le N} v_k e^{ik\cdot x} \in \mathbb{P}_N$$

LEMMA: P_N **satisfies the following properties:**

1) P_N **is the restriction on** $H^s(\Omega)$ **of the** L^2 **projection onto** $\mathbb{P}_N$**.**

2) $\forall s \ge 0$, $\|P_N\|_{\mathcal{L}(H^s(\Omega);H^s(\Omega))} \le 1$.

3) P_N **commutes with differentiation operators.**

4) $\forall v \in H^s(\Omega)$, $\forall \mu$, $0 \le \mu \le s$, $\|v - P_N v\|_{H^\mu} \lesssim N^{\mu-s}\|v\|_{H^s}$.

5) $\forall v \in \mathbb{P}_N$, $\forall \mu, s$, $s \le \mu$, $\|P_N v\|_{H^\mu} \lesssim N^{\mu-s}\|v\|_{H^s}$.

A NAIVE HYPERVISCOSITY MODEL

DEFINITION: $Q(x) = \frac{1}{(2\pi)^3} \sum_{1\le|k|_\infty\le N} |k|^{2\alpha} e^{ik\cdot x}$.

$$Q*u_N(x) = \int_\Omega u_N(y) Q(x-y)\,dy \;=\; \sum_{1\le|k|_\infty\le N} |k|^{2\alpha} u_k e^{ik\cdot x}$$

If α **is an integer** $Q*u_N(x) = (-\nabla^2)^\alpha u_N$

Let $\epsilon > 0$

$$\begin{cases}\text{Find } u_N \in \mathcal{C}^1([0,T];X_N) \text{ and } p_N \in \mathcal{C}^0([0,T];M_N) \text{ such that} \\ (\partial_t u_N, v) + (u_N \cdot \nabla u_N, v) - (p_N, \nabla\cdot v) + \nu(\nabla u_N, \nabla v) \\ \qquad + \epsilon\,(Q*u_N, v) = (f,v), \qquad \forall v \in X_N, \forall t \in (0,T], \\ (\nabla\cdot u_N, q) = 0, \qquad \forall q \in M_N, \forall t \in (0,T], \\ u_N|_{t=0} = P_N u_0. \end{cases}$$

A NAIVE HYPERVISCOSITY MODEL

Consistency error in L^2 : $\epsilon \|Q * u_N\|_{L^2} \leq \epsilon \|u_N\|_{H^{2\alpha}} = \mathcal{O}(\epsilon^{\frac{1}{2}})$.
But ϵ cannot be too small to play the regularizing effect we expect.
$\Rightarrow$ Consistency error cannot be arbitrarily small.

But if $u \in H^s$, where s may be arbitrarily large if u is smooth,
$\|u - P_N u\|_{L^2} \leq cN^{-s}\|u\|_{H^s} = \mathcal{O}(N^{-s})$.
$\Rightarrow$ The interpolation error can be arbitrarily small.

$\Rightarrow$ The hyperviscosity regularization spoils the consistency.

A SPECTRAL HYPERVISCOSITY MODEL

KEY: Actually it is not necessary to stabilize the low wave numbers since they should be controlled by means of the L^2 *a priori* estimate.

DEFINITION: Let $\alpha > \frac{5}{4}$ and $\beta > 0$ (two parameters).

$$\begin{cases} \epsilon_N = N^{-\beta}, \text{ with } \quad 0 < \beta < \left\{ \begin{matrix} \frac{4\alpha-5}{2} & \text{If } \alpha \le \frac{3}{2}, \\ \frac{4\alpha(\alpha-1)}{2\alpha+3} & \text{Otherwise.} \end{matrix} \right\} < 2\alpha \\ N_i = N^{\theta}, \quad \text{with} \quad \theta = \dfrac{\beta}{2\alpha}, \\ Q(x) = (2\pi)^{-3} \displaystyle\sum_{N_i \le |k|_\infty \le N} |k|^{2\alpha} e^{ik\cdot x} \end{cases}$$

A SPECTRAL HYPERVISCOSITY MODEL

One can also define the hyperviscosity kernel as follows:

$$Q(x) = \frac{1}{(2\pi)^3} \sum_{N_i \le |k|_\infty \le N} \hat{Q}_{|k|} |k|^{2\alpha} e^{ik\cdot x},$$

where the viscosity coefficients $\hat{Q}_{|k|}$ are such that

$$|1 - \hat{Q}_{|k|}| \lesssim \frac{N_i^{2\alpha}}{|k|^{2\alpha}}, \qquad \forall |k|_\infty \ge N_i.$$

This definition has the practical advantage of ensuring a smooth transition of the viscosity coefficients across the threshold N_i.

All the results stated hereafter hold also with this definition.

A SPECTRAL HYPERVISCOSITY MODEL

We'd like β large so that $\epsilon_N = N^{-\beta}$ is small.

But the condition $u_N \longrightarrow u$ where u is dissipative enforces

$$\beta < \left\{ \begin{array}{ll} \frac{4\alpha-5}{2} & \text{If } \alpha \leq \frac{3}{2}, \\ \frac{4\alpha(\alpha-1)}{2\alpha+3} & \text{Otherwise.} \end{array} \right\} < 2\alpha$$

α	$\frac{3}{2}$	2	3	4	5
β	$< \frac{1}{2}$	$< \frac{8}{7}$	$< \frac{8}{3}$	$< \frac{48}{11}$	$< \frac{80}{13}$
θ	$< \frac{1}{6}$	$< \frac{2}{7}$	$< \frac{4}{9}$	$< \frac{6}{11}$	$< \frac{8}{13}$

Admissible values of the parameters α, β, and θ.

A SPECTRAL HYPERVISCOSITY MODEL

Note that $\boxed{\epsilon_N = N_i^{-2\alpha}}$

$$\begin{cases} \text{Find } u_N \in \mathcal{C}^1([0,T];X_N) \text{ and } p_N \in \mathcal{C}^0([0,T];M_N) \text{ such that} \\ (\partial_t u_N, v) + (u_N \cdot \nabla u_N, v) - (p_N, \nabla\cdot v) + \nu(\nabla u_N, \nabla v) \\ \qquad + \epsilon_N(Q * u_N, v) = (f, v), \qquad \forall v \in X_N, \forall t \in (0,T], \\ (\nabla\cdot u_N, q) = 0, \qquad \forall q \in M_N, \forall t \in (0,T], \\ u_N|_{t=0} = P_N u_0. \end{cases}$$

Note that $\boxed{\nabla\cdot u_N = 0}$ **since** $\nabla\cdot u_n \in M_N$**, i.e. the approximate velocity field is solenoidal.**

PROPOSITION: **The hyperviscosity perturbation is spectrally small in the sense that**

$$\epsilon_N \|Q * v_N\|_{L^2} \le c\, N^{-\theta s} \|v_N\|_{H^s}, \qquad \forall v_N \in H^s(\Omega),\ \forall s \ge 2\alpha.$$

PROOF: **Note that**

$$\|Q * u_N\|_{L^2}^2 = \sum_{N_i \le |k|_\infty \le N} |k|^{4\alpha} |u_k|^2.$$

Using the fact that $2\alpha \le s$ and that $N_i \le |k|_\infty \le |k|$, we have:

$$|k|^{4\alpha} = N_i^{4\alpha} \left(\frac{|k|}{N_i}\right)^{4\alpha} \le N_i^{4\alpha} \left(\frac{|k|}{N_i}\right)^{2s} = N_i^{4\alpha - 2s} |k|^{2s}$$

Moreover, from the definition of N_i, ϵ_N, and θ, we have

$$N_i^{4\alpha - 2s} = \epsilon_N^{-2} N^{-2\theta s}$$

so that

$$\epsilon_N^2 \|Q * u_N\|_{L^2}^2 \le \epsilon_N^2 \epsilon_N^{-2} N^{-2\theta s} \sum_{N_i \le |k|_\infty \le N} |k|^{2s} |u_k|^2 \le N^{-2\theta s} \|u_N\|_{H^s}^2$$

LEMMA: We have the *a priori* estimates

$$\|u_N\|_{L^\infty(L^2)} + \nu^{1/2}\|u_N\|_{L^2(H^1)} + \epsilon_N^{1/2}\|u_N\|_{L^2(H^\alpha)} \lesssim c.$$

PROOF: Observe that

$$\epsilon_N \|u_N\|^2_{H^\alpha} = \epsilon_N \|u_N\|^2_{\dot{H}^\alpha} = \epsilon_N (Q * u_N, u_N) + \boxed{\epsilon_N \sum_{1\le |k|_\infty < N_i} |k|^{2\alpha} |u_k|^2}$$

To estimate the last term in the above inequality. Use the fact that $\epsilon_N = N_i^{-2\alpha}$ and that $|k| \le \sqrt{3}|k|_\infty$.

$$\epsilon_N \sum_{1\le |k|_\infty < N_i} |k|^{2\alpha}|u_k|^2 \le 3^\alpha N_i^{-2\alpha} N_i^{2\alpha} \sum_{1\le |k|_\infty < N_i} |u_k|^2 \le 3^\alpha \|u_N\|^2_{L^2}.$$

Since u_N is solenoidal and using the above bound, we obtain

$$\|u_N\|^2_{L^2} + \nu\|u_N\|^2_{L^2(H^1)} + \epsilon_N \|u_N\|^2_{L^2(H^\alpha)}$$
$$\lesssim \|u_0\|^2_{L^2} + \frac{1}{2}\|f\|^2_{L^2(L^2)} + (3^\alpha + \frac{1}{2})\|u_N\|^2_{L^2(L^2)}.$$

The result is a consequence of Gronwall's lemma.

LEMMA: We have

$$\|u_N\|_{L^p(H^{2/p})} + \|u_N\|_{L^p(L^q)} \le c, \quad \text{with} \quad 1 \le p \le \frac{4q}{3(q-2)}, \quad 2 \le q \le 6,$$
$$\|p_N\|_{L^{4/3}(L^2)} \le c.$$

PROOF: First, we observe that $\nabla^2 : M_N \longrightarrow M_N$ is bijective.
Then, we multiply the momentum equation by $\nabla(\nabla^2)^{-1}p_N$ (note that $\nabla(\nabla^2)^{-1}p_N \in X_N$ is an admissible test function).
By using several integrations by parts, we obtain

$$\begin{aligned}
\|p_N\|_{L^2}^2 &= (\nabla p_N, \nabla(\nabla^2)^{-1}p_N) \\
&= (-\partial_t u_N + \nu\nabla^2 u_N - u_N \cdot \nabla u_N + f, \nabla(\nabla^2)^{-1}p_N) \\
&= -(u_N \cdot \nabla u_N, \nabla(\nabla^2)^{-1}p_N), \quad \text{since } u_N \text{ and } f \text{ are solenoidal} \\
&= -(\nabla\cdot(u_N \otimes u_N), \nabla(\nabla^2)^{-1}p_N) \\
&= (u_N \otimes u_N, \nabla\nabla(\nabla^2)^{-1}p_N) \\
&\lesssim \|u_N\|_{L^4}^2 \|p_N\|_{L^2},
\end{aligned}$$

i.e. $\|p_N\|_{L^2} \lesssim \|u_N\|_{L^4}^2$. Use the bound on u_N, for $q = 4,\ p = \frac{8}{3}$.

COROLLARY: Let s be a real number such that $s \geq 2\alpha - 1 > \frac{3}{2}$, we have

$$\|\partial_t u_N\|_{L^{4/3}(H^{-s})} \lesssim c.$$

THEOREM: $u_N \to u$ and u is a **weak suitable** solution to NS.

PROOF: Test the momentum equation by $P_N(u_N\phi)$:

$$(\partial_t u_N, P_N(u_N\phi)) + \boxed{(u_N \cdot \nabla u_N, P_N(u_N\phi))} - (p_N, \nabla \cdot P_N(u_N\phi))$$
$$+\nu(\nabla u_N, \nabla P_N(u_N\phi)) + \epsilon_N(Q * u_N, P_N(u_N\phi)) = (f, P_N(u_N\phi))$$

Define $R = (u_N \cdot \nabla u_N, P_N(u_N\phi) - u_N\phi)$ and because u_N is solenoidal, we have

$$(u_N \cdot \nabla u_N, P_N(u_N\phi)) = (u_N \cdot \nabla u_N, u_N\phi) + R = -(\tfrac{1}{2}|u_N|^2 u_N, \nabla\phi) + R.$$

For the remainder R we have

$$\begin{aligned} |R| &= |(u_N \cdot \nabla u_N, P_N(u_N\phi) - u_N\phi)|, \\ &\lesssim \|u_N\|_{L^4}^2 \|\nabla (P_N(u_N\phi) - u_N\phi)\|_{L^2}, \\ &\lesssim N^{1-\alpha}\|u_N\|_{L^4}^2 \|u_N\phi\|_{H^\alpha}, && \textbf{Approximation property,} \\ &\lesssim N^{1-\alpha}\|u_N\|_{L^4}^2 \|u_N\|_{H^\alpha}\|\phi\|_{H^s}. && \textbf{Leibniz-like rule, } s > \alpha + \tfrac{3}{2}. \end{aligned}$$

To bound $\|u_N\|_{L^4}$ we proceed as follows.

$$\begin{aligned} \|u_N\|_{L^4}^2 &\lesssim \|u_N\|_{H^r}^2, && \textbf{Sobolev emb., } \frac{1}{4} = \frac{1}{2} - \frac{r}{3}, r = \frac{3}{4}, \\ &\lesssim \|u_N\|_{L^2}^{2(1-\gamma)} \|u_N\|_{H^\alpha}^{2\gamma}, && \textbf{Interp. inq., } \gamma\alpha = r, \gamma = \frac{3}{4\alpha}, \\ &\lesssim \|u_N\|_{H^\alpha}^{\frac{3}{2\alpha}}. \end{aligned}$$

At this point, there are two possibilities: either $\frac{3}{2\alpha} \le 1$ or $\frac{3}{2\alpha} > 1$.

If $\alpha < \frac{3}{2}$, then

$$\begin{aligned} \|u_N\|_{L^4}^2 &\lesssim \|u_N\|_{H^\alpha}\, \|u_N\|_{H^\alpha}^{\frac{3}{2\alpha}-1}, \\ &\lesssim N^{\frac{3}{2}-\alpha}\|u_N\|_{H^\alpha}\, \|u_N\|_{L^2}^{\frac{3}{2\alpha}-1}, \qquad \textbf{Inverse inequality.} \end{aligned}$$

Then, owing to $\boxed{\epsilon_N = N^{-\beta}}$

$$|R| \lesssim N^{\frac{5}{2}-2\alpha}\|u_N\|_{H^\alpha}^2\|\phi\|_{H^s} = N^{\frac{5}{2}-2\alpha+\beta}\epsilon_N\|u_N\|_{H^\alpha}^2\|\phi\|_{H^s}.$$

That is to say, owing to the hypothesis $\boxed{\beta \leq \frac{4\alpha-5}{2}}$ **we have** $\int_0^T |R| \longrightarrow 0$ **as** $N \longrightarrow +\infty$**.**

CONCLUSIONS

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- **The new LES paradigm is constructive: i.e. enforcing the regularized solution to converge to a suitable weak solution implies strong constraints on the numerical methods.**
- **Extension to non periodic domains and finite elements is likely to be highly technical.**

$\Rightarrow$ **There is no easy *a priori* estimate on the pressure.**

$\Rightarrow$ **There is no easy *a priori* estimate on $\partial_t u_h$ (Fourier analysis gives $u_h \in H^{\frac{3}{8}-\epsilon}(L^2)$, this is not enough!)**

$\Rightarrow$ **The nonlinear term does not pose difficulties!**

$\Rightarrow$ **The trouble maker is the product $p_h u_h$!**