Congestion games with player-specific cost functions

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Abstract

We consider congestion games on graphs. In *nonatomic games*, we are given a set of infinitesimal players. Each player wants to go from one vertex to another by taking a route of minimal cost, the cost of a route depending on the number of players using it. In *atomic splittable games*, we are given a set of players with a non-negligible demand. Each player wants to ship his demand from one vertex to another by dividing it among different routes. In these games, we reach a Nash equilibrium when every player has chosen a minimal-cost strategy.

The existence of a Nash equilibrium is ensured under mild conditions. The main issues are the uniqueness, the computation, the efficiency and the sensitivity of the Nash equilibrium. Many results are known in the specific case where all players are impacted in the same way by the congestion. The goal of this thesis is to generalize these results in the case where we allow player-specific cost functions.

We obtain results on uniqueness of the equilibrium in nonatomic games. We give two algorithms able to compute a Nash equilibrium in nonatomic games when the cost functions are affine. We find a bound on the price of anarchy for some atomic splittable games, and prove that it is unbounded in general, even when the cost functions are affine. Finally we find results on the sensitivity of the equilibrium to the demand in atomic splittable games.

Résumé court

Nous considérons des jeux de congestion sur des graphes. Dans les jeux *non-atomiques*, nous considérons un ensemble de joueurs infinitésimaux. Chaque joueur veut aller d'un sommet à un autre en choisissant une route de coût minimal. Le coût de chaque route dépend du nombre de joueur la choisissant. Dans les jeux *atomiques divisibles*, nous considérons un ensemble de joueurs ayant chacun une demande à transférer d'un sommet à un autre, en la subdivisant éventuellement sur plusieurs routes. Dans ces jeux, un équilibre de Nash est atteint lorsque chaque joueur a choisi une stratégie de coût minimal.

L'existence d'un équilibre de Nash est assurée sous de faibles hypothèses. Les principaux sujets sont l'unicité, le calcul, l'efficacité et la sensibilité de l'équilibre de Nash. De nombreux résultats sont connus dans le cas où les joueurs sont tous affectés de la même façon par la congestion. Le but de cette thèse est de généraliser ces résultats au cas où les joueurs ont des fonctions de coût différentes.

Nous obtenons des résultats sur l'unicité de l'équilibre dans les jeux non-atomiques. Nous donnons deux algorithmes capables de calculer un équilibre dans les jeux non-atomiques lorsque les fonctions de coût sont affines. Nous obtenons une borne sur le prix de l'anarchie pour certains jeux atomiques divisibles et prouvons qu'il n'est pas borné en général, même lorsque les fonctions sont affines. Enfin, nous prouvons des résultats sur la sensibilité de l'équilibre par rapport à la demande dans les jeux atomiques divisibles.

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Résumé de la thèse

Les jeux de congestion forment un outil mathématique capable de modéliser de grands nombres de problèmes concrets. C'est le cas de toute situation dans laquelle plusieurs *joueurs* ont accès à des *ressources*, le prix d'une ressource dépendant du nombre de joueurs la choisissant. En pratique, les champs d'applications des jeux de congestion concernent essentiellement les transports, les télécommunications et les réseaux d'énergie. De nombreux résultats sont connus lorsque tous les joueurs sont affectés de la même façon par la congestion. Le but de cette thèse est de généraliser ces résultats au cas où les joueurs peuvent être affectés différemment par la congestion.

Dans la Section 0.1, nous présentons les principaux enjeux et expliquons comment ils peuvent être modélisés par des jeux de congestion. Nous introduisons trois exemples montrant des comportements qui peuvent être contre-intuitifs. Nous considérons des jeux sur des réseaux où chaque joueur veut aller d'un sommet, son *origine*, à un autre, sa *destination*. Pour cela, il choisit une *route* reliant ces deux sommets. Le coût de cette route est la somme du coût de chaque *arc* qui la constitue, et le coût d'un arc dépend du nombre de joueurs l'utilisant. On atteint un *équilibre de Nash* lorsque chaque joueur a choisi une route de coût minimal.

L'exemple 0.1 est celui de Pigou (1924). Nous considérons un ensemble de joueurs qui doivent choisir parmi deux ressources. Ce jeu est un jeu *non-atomique*, où chaque joueur est de taille infinitésimale et n'influence pas la congestion. Ce type de jeu représente une vision macroscopique : c'est la *proportion* de joueurs faisant un choix qui influence la congestion. Les joueurs ont les mêmes fonctions de coût : la première ressource a un coût constant égal à 1, alors que le coût de la seconde ressource est égal à la proportion de joueurs la choisissant. Ce jeu peut être modélisé par un jeu de congestion sur un graphe à deux sommets o et d avec deux arcs parallèles a et b, chaque arc reliant le sommet o au sommet d. Chaque joueur veut aller de l'origine o à la destination d et doit donc choisir entre ces deux arcs. Les fonctions de

coût sont $c_a(x) = 1$ et $c_b(x) = x$, voir Figure 1. A l'équilibre, tous les joueurs choisissent l'arc b, car le coût est toujours plus intéressant que l'arc a. Le coût à l'équilibre est donc égal à 1 pour chaque joueur, et le *coût social* à l'équilibre, i.e. le coût pour l'ensemble des joueurs est aussi égal à 1.
Cependant, le coût optimal est obtenu lorsque la moitié des joueurs choisit l'arc a est l'autre moitié l'arc b. Dans ce cas, la moitié des joueurs paye un coût de 1 alors que l'autre moitié paye un coût de $\frac{1}{2}$. Le coût social optimal est donc de $\frac{3}{4}$. Cet exemple montre que le coût social à l'équilibre n'est pas nécessairement optimal. La notion de *prix de l'anarchie* a été introduite afin de mesurer cette perte d'efficacité à l'équilibre.

L'exemple 0.2 est le paradoxe de Braess (1968). Comme dans l'exemple précédent, un ensemble de joueurs non-atomiques veut aller du sommet o au sommet d. Nous supposons dans un premier temps qu'il y a deux *routes* : la route "nord" oud qui passe par le sommet u, et la route "sud" ovd, qui passe par le sommet v. Chacune de ces deux routes est constituée de deux arcs, un "rapide" dont le coût est égal à la proportion de joueurs l'utilisant, et un "long" dont le coût est toujours égal à 1. Dans cet exemple les arcs ou et vd sont "rapides" et les arcs ud et ov sont "lents", voir Figure 2. A l'équilibre, la moitié des joueurs utilise la route nord et l'autre moitié la route sud. Le coût social à l'équilibre est donc de $\frac{3}{2}$.
Supposons maintenant que les deux arcs "rapides" sont reliés par un arc uv dont le coût est nul. On s'attend intuitivement à ce que cela améliore le coût social. Cependant, à l'équilibre, tous les joueurs utilisent désormais la nouvelle route $ouvd$. Le coût social à l'équilibre est de 2. Ainsi, ajouter un arc détériore le coût social à l'équilibre. Ce résultat est contre-intuitif, car cela revient à dire que construire une autoroute peut augmenter la congestion. De manière équivalente, cela veut dire qu'une augmentation du coût sur un arc peut avoir un effet bénéfique pour l'ensemble des joueurs.

Finalement, l'exemple 0.3 est celui de Fisk (1979) et considère des joueurs non-atomiques ayant des paires origine-destination différentes. Tous les joueurs ont les mêmes fonctions de coût. Le nombre total de joueurs ayant une paire origine-destination donnée est fixé. Dans cet exemple, en augmentant le nombre total de joueurs associé à une certaine paire origine-destination, nous trouvons un coût social à l'équilibre inférieur. Ce résultat est également contre-intuitif car augmenter le nombre total de joueurs sur un réseau peut être bénéfique.

Ces trois exemples illustrent des comportement parfois contre-intuitifs et montrent la nécessité d'effectuer une étude plus approfondie des jeux de congestion. La Section 0.2 introduit les principaux problèmes rencontrés : l'existence et l'unicité d'un équilibre de Nash, son calcul effectif, ainsi que des études sur l'efficacité et la sensibilité. De nombreux travaux ont été effectués traitant du cas où tous les joueurs sont affectés de la même manière par la congestion. L'objectif de cette thèse est d'étendre certains de ces résultats au cas où les joueurs peuvent être affectés différemment par la congestion.

Le Chapitre 1 est une présentation détaillée des jeux de congestion, des principaux problèmes rencontrés et des principaux résultats. Il contient toutes les définitions utilisées dans la suite de la thèse ainsi qu'un état de l'art. Nous distinguons les jeux *non-atomiques* (Section 1.2), où chaque joueur est infinitésimal et veut choisir une route de coût minimal, des jeux *atomiques divisibles* (Section 1.3), où chaque joueur a un stock qu'il divise parmi les différentes routes, afin de minimiser son coût total. Les exemples précédents sont des jeux non-atomiques. Les jeux atomiques divisibles peuvent être vus comme des jeux non-atomiques où certains joueurs se regroupent en une *coalition*. Nous présentons dans ces sections les résultats d'existence et d'unicité d'un équilibre de Nash. Sous de faibles hypothèses, un équilibre de Nash existe. Cependant lorsque les fonctions de coût ne sont pas les mêmes pour chaque joueur, un tel équilibre n'est pas nécessairement unique. Nous introduisons également les principaux algorithmes de calcul d'un équilibre.
La Section 1.4 introduit le *prix de l'anarchie*, défini comme le rapport entre le pire coût social à l'équilibre et le coût social optimal. C'est un indicateur de la perte d'efficacité due à l'équilibre de Nash. Nous présentons les principaux résultats connus dans le cas où les fonctions de coût sont les mêmes pour chaque joueur. Finalement la Section 1.5 introduit une extension aux jeux dits *non-séparables*. Il s'agit d'une catégorie de jeux où le coût d'une route n'est plus défini comme la somme des coûts des arcs la constituant. Nous présentons en particulier les jeux avec joueurs sensibles au risque. Dans ces jeux, le coût de chaque arc est défini comme un coût moyen plus une variable aléatoire de moyenne nulle. Chaque joueur veut choisir une route qui minimise le coût moyen plus un facteur, l'aversion au risque, multiplié par la variance du coût total. Certains résultats connus, comme l'existence d'un équilibre, se généralisent à ce jeu. Cependant la question de l'unicité de l'équilibre, même sur des graphes simples et lorsque les joueurs ont les mêmes fonctions de coût, est une question ouverte.

Dans le Chapitre 2, nous nous intéressons à l'unicité des flux à l'équilibre pour les jeux non-atomiques. Une telle étude a un intérêt pratique, par exemple dans le domaine des transports, où un planificateur urbain veut anticiper le comportement des usagers à la suite d'une nouvelle mesure (péage, fermeture ou construction de route par exemple). Dans ce cas il veut non seulement déterminer un équilibre mais également savoir s'il est unique. D'un point de vue pratique, une telle étude permet d'étendre des résultats valables dans le cas où tout les joueurs sont affectés de la même manière par la congestion, au cas où ils peuvent être affectés de manière différente. Milchtaich (2005) a introduit la notion de *propriété d'unicité*. Un graphe possède la propriété d'unicité si, quelles que soient les fonctions de coût strictement croissantes, sur chaque arc, les flux à l'équilibre sont les mêmes dans tous les équilibres. Il montre que lorsque tous les usagers partent d'une même origine et vont vers une même destination, la propriété d'unicité est vérifiée pour une certaine famille de graphes, appelés *presque parallèles*. Il donne également des contre-exemples pour les graphes ne faisant pas partie de cette famille. Dans le Chapitre 2, nous nous intéressons à la généralisation de ce résultat à des graphes avec plusieurs paires origine-destination.

Nous traitons complètement le problème des cercles bi-directionnels. Dans ces graphes, modélisant un périphérique routier par exemple, chaque usager a le choix entre deux routes reliant son origine à sa destination : une dans chaque sens. Nous supposons que le graphe et les différentes paires origine-destination sont fixés. Le Théorème 2.1 stipule alors qu'un graphe circulaire possède la propriété d'unicité si et seulement si chaque arc appartient à au plus deux routes. Autrement dit, s'il existe un arc du graphe et trois paires origine-destination distinctes telles que l'arc appartient à une route de chacune de ces paires origine-destination, alors on peut construire une situation donnant deux équilibres avec des flux distincts. Nous donnons explicitement deux équilibres distincts dans ce cas, dans la Section 2.5.3. En particulier, ces exemples sont construits avec des fonctions de coût affines et peuvent être facilement adaptés avec des fonctions convexes. Ainsi, si l'on veut restreindre l'ensemble des fonctions de coût disponibles afin de retrouver la propriété d'unicité, il faut exclure les fonctions affines et les fonctions convexes.

Dans la Section 2.6, nous donnons d'abord un algorithme permettant de déterminer le nombre maximal de routes contenant un arc d'un graphe circulaire. Cela peut se faire en ne faisant qu'un seul tour du graphe. Nous faisons ensuite une étude topologique permettant de déterminer quels sont les graphes circulaires tels que chaque arc est dans au plus deux routes. La Proposition 2.11 stipule qu'un tel graphe doit être homéomorphe à un mineur d'un certain graphe appartenant à la famille de neuf graphes des Figures 2.2–2.5. Ce résultat permet donc de décrire explicitement tous les graphes circulaires possédant la propriété d'unicité.

Dans la Section 2.7.1, nous utilisons le Théorème 2.1 afin d'obtenir des résultats pour des graphes plus généraux. Nous disons qu'un équilibre est *strict* si chaque usager n'a qu'une seule route de coût minimal. La Proposition 2.13 stipule que lorsqu'on peut construire deux équilibres stricts avec des flux différents pour un certain graphe, alors tout graphe ayant celui-ci comme mineur ne possède pas la propriété d'unicité. En effet, nous pouvons alors adapter les deux équilibres au plus grand graphe. Ce résultat nous permet d'obtenir une condition nécessaire pour posséder la propriété d'unicité : tout graphe ayant comme mineur un des graphes de la Figure 2.10 ne possède pas la propriété d'unicité.

Dans la Section 2.7.3, nous étendons le résultat de Milchtaich (2005) traitant de l'*équivalence* des équilibres au cas des graphes circulaires. Supposons que les usagers sont répartis en *classes*, les usagers d'une même classe ayant tous les mêmes fonctions de coût. Deux équilibres sont dits équivalents si non seulement les flux sont les mêmes sur chaque arc dans tous les équilibres, mais la contribution de chaque classe au flux de chaque arc est la même. Le Théorème 2.16 stipule que pour les graphes circulaires, *génériquement* tous les équilibres sont équivalents si et seulement si le graphe possède la propriété d'unicité.

Dans la Section 2.7.4, nous nous intéressons à la propriété d'unicité *forte*, c'est-à-dire aux graphes possédant la propriété d'unicité pour tout choix de paires origine-destination. Le principal résultat est le Théorème 2.17 indiquant que les graphes possédant la propriété d'unicité forte sont ceux ne possédant pas de cycle de longueur 3 ou plus. Un tel graphe consiste en une forêt dans laquelle on a éventuellement remplacé certaines arêtes par des arêtes parallèles.

Finalement, dans la Section 2.7.5, nous donnons des exemples de graphes possédant plusieurs équilibres avec seulement deux classes. En effet, les exemples de Milchtaich (2005) ou de la Section 2.5.3 utilisent tous trois classes. Ainsi, même en restreignant le nombre de classes, on ne peut espérer obtenir des conditions plus simples d'unicité.

Un des principaux problèmes pratiques est le calcul d'un équilibre de Nash. Les Chapitres 3 et 4 traitent ce sujet pour des jeux non-atomiques multi-classes. Dans un tel jeu, l'ensemble des joueurs est divisé en *classes*, les joueurs d'une même classe ayant les mêmes fonctions de coût. Dans le Chapitre 3, nous construisons un algorithme *efficace* permettant de calculer un équilibre de Nash lorsque les fonctions de coût sont affines (Théorème 3.9). Cet algorithme est un algorithme de pivot, qui s'inspire de l'algorithme de Lemke (1965) pour les problèmes de complémentarité linéaire. Nous ne savons pas si cet algorithme est polynomial, mais nous démontrons qu'il appartient à la classe de complexité PPAD. En pratique il s'avère être rapide sur les instances que nous avons traitées. Le Chapitre 4 est davantage théorique. Nous considérons toujours le cas où les fonctions de coût sont affines. Alors il existe un algorithme polynomial lorsque le nombre de classes et le nombre de sommets du graphe sont fixés (Théorème 4.7).

Plus précisément, le Chapitre 3 définit le problème de la recherche d'un équilibre dans les jeux non-atomiques multi-classes : le *Multiclass Network Equilibrium Problem*. Ce problème peut être reformulé comme le problème de complémentarité linéaire ($MNEP_{gen}$), défini dans la Section 3.2. Plus précisément, lorsque les fonctions de coût sont affines, $c_a^k(x) = \alpha_a^k x + \beta_a^k$, résoudre le *Multiclass Network Equilibrium Problem* revient à résoudre le problème de complémentarité linéaire ($MNEP$) suivant :

$$\begin{aligned}
&\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k && k\in K, v\in V^k\\
&\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k = -\beta_{uv}^k && k\in K, (u,v)\in A^k\\
&x_a^k\mu_a^k = 0 && k\in K, a\in A^k\\
&x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} && k\in K, a\in A^k, v\in V^k\ .
\end{aligned}$$

De façon similaire à l'algorithme de Lemke, nous ajoutons une variable ω et considérons le problème *augmenté* ($AMNEP(\boldsymbol{e})$) suivant, où $\boldsymbol{e}$ est un vecteur choisi ultérieurement.

$$\begin{aligned}
\min\ \ & \omega\\
\text{s.t.}\ \ &\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k && k\in K, v\in V^k\\
&\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k + e_{uv}^k\omega = -\beta_{uv}^k && k\in K, (u,v)\in A^k\\
&x_a^k\mu_a^k = 0 && k\in K, a\in A^k\\
&x_a^k \geq 0, \mu_a^k \geq 0, \omega \geq 0, \pi_v^k \in \mathbb{R} && k\in K, a\in A^k, v\in V^k\ .
\end{aligned}$$

Le point clé de l'algorithme est que la résolution de ($MNEP$) consiste à trouver une solution de ($AMNEP(\boldsymbol{e})$) avec $\omega = 0$. Le vecteur $\boldsymbol{e}$ définissant le problème ($AMNEP(\boldsymbol{e})$) est essentiel : un bon choix permettra de trouver rapidement une solution réalisable.

Dans la Section 3.3, nous redéfinissons les notions classiques de bases et solutions basiques, réalisables et complémentaires. Le but de l'agorithme est de trouver une solution basique réalisable et complémentaire de ($AMNEP(\boldsymbol{e})$), telle que l'indice associé à ω n'appartienne pas à cette base. Pour cela nous construisons un algorithme de pivot, à l'aide des Lemmes 3.3 et 3.4. Ces

deux lemmes stipulent que lorsqu'on a une base réalisable pour le problème ($AMNEP(\boldsymbol{e})$) et une variable hors base, il existe au plus une autre base réalisable contenant toutes les variables de la base initiale sauf une, ainsi que la nouvelle variable. S'il n'existe pas de telle base, alors le polytope des solutions réalisables contient un rayon infini. L'opération consistant à déterminer une nouvelle base connaissant la variable *entrante* s'appelle un *pivot*. Ces deux bases sont alors dites *voisines*. La Section 3.3.4 explique qu'en considérant des bases réalisables *complémentaires*, la détermination de la variable entrante est immédiate, et la nouvelle base est encore complémentaire. L'algorithme peut ainsi pivoter de base réalisable complémentaire en base réalisable complémentaire.
Il reste à déterminer une base réalisable complémentaire initiale afin de donner un point de départ à l'algorithme. Cela est fait dans la Section 3.3.5, en faisant un bon choix du vecteur $\boldsymbol{e}$. Dans la Section 3.3.6, nous prouvons qu'il existe un rayon infini partant de cette base initiale, mais pas de rayon *secondaire*. L'agorithme suit donc un chemin dans le graphe abstrait où les sommets sont les bases réalisables complémentaires et les arêtes connectent deux bases voisines. Comme chaque base a au plus un voisin et qu'il n'y a pas de rayon secondaire, l'algorithme se termine sur une base n'ayant plus de variable entrante. Une telle base ne contient pas la variable ω et est donc une base réalisable complémentaire optimale. Cet algorithme est expliqué en détail dans la Section 3.3.7.

Nous obtenons ainsi un algorithme de pivot permettant de résoudre le problème lorsque les fonctions de coût sont affines, le Théorème 3.9 de la Section 3.4. En particulier, cet algorithme montre que le problème appartient à la classe de complexité PPAD. Dans la Section 3.5, nous effectuons des tests numériques sur des graphes de type Manhattan, i.e. des quadrillages $n \times n$. Les résultats répertoriés dans le Tableau 3.1 montrent que le nombre de pivots reste raisonnable, tout comme le temps d'exécution de l'algorithme. Pour obtenir les résultats finaux, nous devons inverser une matrice, ce qui est coûteux, mais l'algorithme n'a pas été optimisé. Ainsi, ces résultats peuvent être considérés comme très encourageants concernant l'efficacité de l'algorithme.

Le Chapitre 4 présente un autre algorithme permettant de résoudre ce problème. Cet algorithme, plus théorique, vise à améliorer la complexité du problème. Une question naturelle est de savoir si le problème est polynomial. Nous montrons que pour un nombre de sommets et un nombre de classes fixés, il existe un algorithme polynomial en le nombre d'arcs. Cet algorithme repose sur une correspondance entre le *support* de chaque classe pour un équilibre et une *cellule* d'un certain arrangement d'hyperplans.

La Section 4.1 introduit le modèle, et plus précisément le support d'une classe pour un équilibre. Il s'agit de l'ensemble des arcs utilisés par cette classe de joueurs. Nous définissons également pour chaque classe l'ensemble des arcs de *coût minimal*. A l'équilibre, le support est inclus dans l'ensemble des arcs de coût minimal, mais l'inclusion inverse n'est pas forcément vérifiée. Ainsi, comme dans l'Exemple 0.1, il peut exister des arcs de coût minimal mais non utilisés. Pour un équilibre $\vec{\boldsymbol{x}}$, l'ensemble des supports est noté $\mathrm{supp}(\vec{\boldsymbol{x}})$ et l'ensemble des arcs de coût minimal $\mathrm{mincost}(\boldsymbol{x})$.

La Section 4.2 introduit les notions relatives aux arrangements d'hyperplans. Etant donné un ensemble H fini de n hyperplans de $\mathbb{R}^d$, un *arrangement* est une partition de l'espace en ensembles ouverts convexes. Lorsque cette partition est celle déterminée par les hyperplans de H, on parle de l'arrangement d'hyperplans associé à H, noté $\mathcal{A}(H)$. Chaque ensemble de l'arrangement est appelé *cellule*, et le nombre total de cellules de $\mathcal{A}(H)$ est polynomial $O(n^d)$.

L'algorithme est présenté dans la Section 4.3. Il est en deux étapes. La première étape consiste à déterminer un ensemble $\mathcal{S}$ contenant des ensembles d'arcs pour chaque classe et tel que pour chaque équilibre $\vec{\boldsymbol{x}}$, il existe un élément $\boldsymbol{S}(\vec{\boldsymbol{x}})$ de $\mathcal{S}$ tel que $\mathrm{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S}(\vec{\boldsymbol{x}}) \subseteq \mathrm{mincost}(\boldsymbol{x})$. Cet ensemble $\mathcal{S}$ a un nombre polynomial d'éléments. La seconde étape consiste à tester pour chaque élément de $\mathcal{S}$ si effectivement cet élément est un $\boldsymbol{S}(\vec{\boldsymbol{x}})$ pour un certain équilibre $\vec{\boldsymbol{x}}$. Dans ce cas il calcule les flux à l'équilibre. Cette étape peut être effectuée en temps polynomial.

La Section 4.3.1 décrit la première étape. Nous définissons un ensemble de $K^2|A|$ hyperplans de $\mathbb{R}^{K(|V|-1)}$ et considérons l'arrangement d'hyperplans associé. Nous définissons les polyèdres convexes P_a^k permettant de relier pour chaque classe le support à un équilibre au coût à cet équilibre. Plus précisément, le Lemme 4.5 prouve que pour un équilibre $\vec{\boldsymbol{x}}$, une classe k et un arc a, si cet arc appartient au support de la classe k, alors le coût à l'équilibre $\boldsymbol{\pi}(\vec{\boldsymbol{x}})$ appartient au polyèdre P_a^k. Ce lemme nous permet de définir une application de l'ensemble des cellules de l'arrangement vers un ensemble de supports possibles, voir la Proposition 4.4.

La Section 4.3.2 décrit l'outil permettant d'effectuer la deuxième étape. Nous prouvons la Proposition 4.6 qui stipule qu'étant donné $\boldsymbol{S}$, un ensemble d'arcs pour chaque classe, nous pouvons déterminer en temps polynomial s'il existe un équilibre $\vec{\boldsymbol{x}}$ tel que $\mathrm{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S} \subseteq \mathrm{mincost}(\boldsymbol{x})$, et dans ce cas calculer en temps polynomial les flux à l'équilibre. Pour cela nous introduisons le problème ($\mathcal{P}_{\boldsymbol{S}}$) qui a une solution $(\vec{\boldsymbol{x}}, \vec{\boldsymbol{\pi}})$ si et seulement si $\vec{\boldsymbol{x}}$ est un équilibre avec la propriété désirée. Comme ce problème consiste en inégalités linéaires avec $K(|A|+|V|-1)$ variables, nous pouvons le résoudre en temps polynomial en utilisant un algorithme de point intérieur.

La Section 4.3.3 décrit l'agorithme polynomial en le nombre d'arcs, pour un nombre de classes et un nombre de sommets fixés : le Théorème 4.7. Dans la Section 4.4, nous décrivons comment l'algorithme pourrait être amélioré, en réduisant le nombre de cellules de l'arrangement d'hyperplans, ou en considérant un sous-ensemble de graphes.

Un équilibre de Nash est en général non-efficace : le coût pour l'ensemble des joueurs est plus élevé que le coût optimal. Cette situation, illustrée par l'exemple 0.1, peut être quantifiée par le *prix de l'anarchie*, défini par Koutsoupias and Papadimitriou (1999). Comme défini précédemment, il s'agit du rapport entre le pire coût social à l'équilibre et le coût social optimal. Dans le Chapitre 5, nous trouvons une borne pour le prix de l'anarchie dans les jeux atomiques. Nous généralisons le résultat obtenu par Cominetti et al. (2009) pour le cas où les joueurs ont les mêmes fonctions de coût au cas où les fonctions de coût peuvent être spécifiques à chaque joueur.

La Section 5.1 explique plus en détail les motivations du problème et introduit l'état de l'art sur le sujet. La Section 5.2 présente le modèle et les principaux résultats : la borne générale du Théorème 5.1 et sa spécification au cas des fonctions de coût affines dans la Proposition 5.2. Lorsque les fonctions de coût considérées appartiennent à un ensemble $\mathcal{C}$, nous définissons $\lambda(\mathcal{C}) = \sup_{a\in A,\, k,\ell\in[K],\, \vec{c}\in\mathcal{C},\, x\in\mathbb{R}_+} \frac{x(c_a^\ell)'(x)}{c_a^k(x)}$. Alors, le Théorème 5.1 stipule que lorsque $\lambda(\mathcal{C}) < 3$,

$$\text{PoA} \leq \frac{1}{1-\lambda(\mathcal{C})/3}.$$

Les bornes obtenues dépendent en partie de l'ensemble dans lequel nous choisissons les fonctions de coût. Dans le cas où les fonctions sont les mêmes pour chaque joueur et polynomiales, nous savons que le prix de l'anarchie est borné. La Proposition 5.3 prouve que ce résultat n'est plus vrai lorsque chaque joueur a ses propres fonctions de coût, même lorsque celles-ci sont affines.

Dans la Section 5.3, nous prouvons le Théorème 5.1 et la Proposition 5.2. Dans la Section 5.3.2, nous définissons un paramètre $\beta(\mathcal{C})$ similaire à celui défini dans Cominetti et al. (2009) et prouvons dans la Proposition 5.4 que le prix de l'anarchie est borné par $\frac{1}{1-\beta(\mathcal{C})}$. Dans la Section 5.3.3 nous calculons plus précisément cette borne en prouvant la Proposition 5.5 dont le Théorème 5.1 et la Proposition 5.2 sont des corollaires immédiats.

Dans la Section 5.4, nous prouvons la Proposition 5.3. Pour cela nous construisons un jeu sur un graphe avec une seule paire origine-destination et deux arcs a et b. Le premier joueur a une demande totale de M et des fonctions de coût $c_a^1(x) = x$ et $c_b^1(x) = x + 2M$. Le second joueur a une demande totale de 1 et des fonctions de coût $c_a^2(x) = 2M^2x + 1$ et $c_b^2(x) = M^3x$. Nous prouvons que le coût à l'équilibre (unique) est alors $M^2 + M^3$ alors que le coût optimal est majoré par $5M^2 + 1$. Le prix de l'anarchie est alors minoré par une quantité de l'ordre de M. Ainsi, pour tout M nous pouvons construire une instance donc le prix de l'anarchie est plus grand que M. Cela montre qu'il n'est pas borné, même en considérant des fonctions affines.

La Section 5.5 discute ces résultats plus précisément. Nous donnons une généralisation de la Proposition 5.5, dont un corollaire est que le coût social à l'équilibre est borné par le coût optimal du jeu où les demandes sont multipliées par $1 + \beta(\mathcal{C})$.

Le Chapitre 6 traite de la *sensibilité* d'un équilibre de Nash. La sensibilité désigne le comportement d'un équilibre lorsqu'on fait varier les paramètres du jeu : le graphe, les fonctions de coût ou encore la demande de chaque joueur. Nous nous intéressons à un jeu atomique divisible sur un graphe avec une seule paire origine-destination et des arcs parallèles. L'équilibre de Nash est alors unique. Nous étudions la sensibilité de l'équilibre par rapport à la demande de chaque joueur.

La Section 6.1 présente le problème et donne un état de l'art. En particulier il est connu que les flux à l'équilibre sont continus en fonction de la demande mais non différentiables dans les jeux non-atomiques. La Section 6.2 introduit les principaux résultats. Le Théorème 6.1 prouve que le flux à l'équilibre est continu en fonction du vecteur contenant les demandes de chaque joueur. De plus il est différentiable sur tout ensemble où les joueurs gardent le même support à l'équilibre, lorsqu'il y a deux arcs ou deux joueurs. Les Théorèmes 6.2 et 6.3 étudient le cas où un joueur transfère une partie de sa demande à un autre joueur ayant initialement une demande plus importante. Dans ce cas, le Théorème 6.2 stipule que, sur chaque arc, le flux du premier joueur diminue et le flux du deuxième augmente. Si de plus les fonctions de coût sont les mêmes pour chaque joueur, le Théorème 6.3 indique que le coût social à l'équilibre a diminué après le transfert. Nous retrouvons ainsi le résultat indiquant que le coût social à l'équilibre diminue lorsque deux joueurs fusionnent.

La Section 6.3 est dédiée à la preuve du Théorème 6.1. La partie traitant de la continuité est prouvée dans la Proposition 6.5 et utilise la formulation de l'équilibre de Nash comme la solution d'une inégalité variationnelle. L'exemple 6.6 montre que le flux à l'équilibre n'est pas différentiable en général. Dans cet exemple, nous considérons un graphe avec deux arcs parallèles a et b et un seul joueur ayant une demande d. Les fonctions de coût sont $c_a(x) = x + 1$ et $c_b(x) = x$. Alors, à l'équilibre nous avons $x_a = 0$

et $x_b = d$ lorsque $0 \leq d \leq \frac{1}{2}$, et $x_a = \frac{2d-1}{4}$, $x_b = \frac{2d+1}{4}$ lorsque $d \geq \frac{1}{2}$. Le flux à l'équilibre n'est donc pas différentiable au point $d = \frac{1}{2}$. Cependant, nous obtenons la différentiabilité sur tout ensemble ouvert où les joueurs conservent le même support à l'équilibre, lorsqu'il y a deux joueurs ou deux arcs. Cela est prouvé dans la Proposition 6.9.

La Section 6.4 introduit d'autres propriétés concernant les flux à l'équilibre, qui vont être nécessaires pour prouver les Théorèmes 6.2 et 6.3. En particulier, le Lemme 6.13 joue un rôle crucial dans la preuve du Théorème 6.2. L'intuition de ce lemme est la suivante. Supposons qu'un joueur mette davantage de flux sur un arc dans un équilibre $\vec{z}$ que dans un équilibre $\vec{x}$, bien que le coût de cet arc soit plus important. Alors ce joueur met davantage de flux sur tous les arcs dont le coût à l'équilibre $\vec{z}$ est moins important qu'à l'équilibre $\vec{x}$.

La Section 6.5 contient la preuve du Théorème 6.2 ainsi que quelques corollaires. Plus précisément, nous prouvons la Proposition 6.15. Rappelons que nous considérons le cas où un joueur i transfère une partie de sa demande à un autre joueur j ayant initialement une demande plus importante. Nous supposons qu'il y a deux arcs ou deux joueurs. Alors la Proposition 6.15 stipule que sur chaque arc, le flux du joueur i diminue strictement ou reste constant égal à zéro, alors que le flux du joueur j augmente strictement ou reste constant égal à zéro. De plus, sur chaque arc, le flux des autres joueurs a une évolution opposée à l'évolution du flux total. Nous en déduisons en particulier le Corollaire 6.17 qui considère le cas où un arc est de coût marginal égal au coût marginal à l'équilibre pour un certain joueur mais qui n'appartient pas au support de ce joueur. Alors, ce joueur utilisera cet arc si on augmente sa demande.

La Section 6.6 contient la preuve du Théorème 6.3. Pour cela, nous utilisons la différentiabilité des flux à l'équilibre prouvée dans le Théorème 6.1. Nous considérons le cas où tous les joueurs ont les mêmes fonctions de coût. Dans ce cas, le coût social ne dépend pas du flux de chaque joueur, mais seulement du flux total sur chaque arc. Nous prouvons d'abord la Proposition 6.20 valable quel que soit le nombre d'arcs et de joueurs. Cette proposition stipule que lorsque les deux joueurs i et j ont le même support à l'équilibre, et ce support ne change pas après le transfert, alors les flux sur chaque arc sont inchangés. En particulier, le coût social à l'équilibre reste le même après le transfert. Pour prouver ce résultat, nous calculons explicitement le flux de chaque joueur à l'équilibre. Lorsque les joueurs i et j n'ont plus nécessairement le même support, mais que le support de chacun reste le même avant et après le transfert, la Proposition 6.26 indique que le coût social à l'équilibre est décroissant.

Nous pouvons alors prouver le Théorème 6.3. En effet, le Théorème 6.2, et plus précisément le Corollaire 6.16, indique que le support du joueur i est décroissant pour l'inclusion, alors que celui du joueur j est croissant. Alors en augmentant la valeur δ de demande qui est transférée, chaque joueur change de support un nombre fini de fois. Les Propositions 6.20 et 6.26 montrent donc que le coût social à l'équilibre est décroissant sur chaque intervalle d'un ensemble fini d'intervalles de $[0, d^i]$ dont la fermeture vaut $[0, d^i]$. Comme il est également continu, en utilisant le Théorème 6.1, nous obtenons le résultat.

Finalement, dans la Section 6.7, nous étudions la généralisation de ces résultats. Nous introduisons l'exemple de Huang (2011) qui montre que le résultat du Théorème 6.3 n'est plus valable lorsqu'il y a trois arcs et trois joueurs. La validité de celui du Théorème 6.2 dans ce cas n'est pas connue. Un autre exemple montre que le résultat du Théorème 6.3 n'est plus valable lorsque les joueurs ont des fonctions de coût spécifiques. Dans cet exemple, nous considérons un graphe avec deux arcs a et b, et deux joueurs 1 et 2. Nous prenons un coût très élevé sur l'arc b pour le joueur 1 (resp. sur l'arc a pour le joueur 2), de sorte qu'à l'équilibre, quelle que soit la répartition de la demande, le joueur 1 (resp. le joueur 2) met toute sa demande sur l'arc a (resp. l'arc b). La demande totale est fixée égale à 5 et les fonctions de coût sont $c_a^1(x) = 2x$ et $c_b^2(x) = x$. Alors, lorsque la répartition de la demande est $d^1 = 3$, $d^2 = 2$, le coût à l'équilibre est 8, alors qu'après un transfert de 1, i.e. lorsque la répartition de la demande est $d^1 = 4$, $d^2 = 1$, le coût à l'équilibre est 9. Le résultat du Théorème 6.3 ne s'applique pas.

Notations

Nonatomic games

I	Set of users
K	Set of classes
$\boldsymbol{x} = (x_a)_{a \in A}$	Aggregated flow
$\boldsymbol{x}^k = (x_a^k)_{a \in A}$	Flow of all class k users
$\vec{\boldsymbol{x}} = (\boldsymbol{x}^k)_{k \in K}$	All class flows
$c_a^i(x_a)$	Cost of arc a for user i
$c_a^k(x_a)$	Cost of arc a for class k users
$Q_r^i(\boldsymbol{x})$	Cost of route r for user i
$Q_r^k(\boldsymbol{x})$	Cost of route r for class k users

Atomic games

$[K]$	Set of players
$\boldsymbol{x} = (x_a)_{a \in A}$	Aggregated flow
$\boldsymbol{x}^k = (x_a^k)_{a \in A}$	Flow of player k
$\vec{\boldsymbol{x}} = (\boldsymbol{x}^k)_{k \in [K]}$	All player flows
$c_a^k(x_a)$	Per-unit cost of arc a for player k
$\widetilde{c}_a^k(\boldsymbol{x})$	Marginal cost of arc a for player k
$Q^k(\vec{\boldsymbol{x}})$	Total cost for player k
$Q(\vec{\boldsymbol{x}})$	Social cost

Introduction

0.1 Why considering network congestion games?

0.1.1 Practical interest

Transportation planners face a lot of issues: congestion, pollution, accidents... Congestion costs about 5.5 billion euros every year in France and people living in Paris spend 55 hours per year in traffic jams on average (Centre for Economics and Business Research, 2012). Over the last century, a lot of work has been done to improve the traffic management.

To this purpose, a transport model has been developed in the 1960s: this is the classical *four-step model* (Ortuzar and Willumsen, 1994). This model consists in considering *zones* in a network, for example employment spaces, shopping centres, education or recreational facilities. Data gives the frequentation of these zones, and the first step, the *trip generation*, consists in estimating for each zone how many people come inside and how many leave. The second step is the determination of the *distribution* over space. It consists in building the origin-destination matrix, i.e. how many people want to go from a specific zone to another specific one. The third step is the *modal split* and decides for example how many people going from one zone to another choose to take the car, the bike, or the subway. Finally, the last step is the *assignment*. It assigns the set of people associated to an origin-destination pair and a mode over the set of possible routes joining the origin to the destination.

Congestion games are a good way to see the assignment step (Patricksson, 1994). We have a set of selfish users with the information associated to each user: from which zone and to which zone this user wants to go and how he is impacted by the congestion. Each user wants to make his trip at the minimal cost, but the choice of the others impacts his travel cost. After some while, we reach a situation where every user is on a minium-cost route: a Nash (or more precisely Wardrop (1952)) *equilibrium*. Since we want to know what is the total number of users on each part of the network, i.e. what is the *flow at equilibrium*, we need to study such a Nash equilibrium.

These issues arise in any business based on a network structure. In telecommunications for example, the extension of wireless networks and new internet services, such as the video on demand or peer-to-peer based services, appeared in the past few years. This has led to an increase of the bandwidth requirements together with an increase of actors using the networks. These actors include telecom firms, internet service providers, and users. According to the International Telecommunication Union (2013), 39 percent of the worldwide population use the internet in 2013, while only 16 percent did in 2005. In Europe, this ratio raised from 46 percent in 2005 to 75 percent in 2013.

This rapid growth of the number of users has also led to inequalities, often called the *digital divide*. In order to reduce it, a lot of operations have been launched to extend the very high speed networks, such as the fiber-to-the-home (FTTH) network. These operations are expensive, for example in France it has an estimated cost of 20 billion euros for ten years (ARCEP, 2013).

Theoretical models able to deal with these issues are necessary. Network congestion games is one of them and has led to a considerable amount of work, see Altman et al. (2006) for an extended survey.

Among the other main fields of application of congestion games are electrical networks. The development of electrical vehicles, and more generally the increase in energy demand has led to an extensive work in the past years. One of the main idea is to consider a *smart grid* infrastructure. In the smart grid models, facilities needing energy are connected with each other, and can decide when to charge and modulate the power. They are players competing in a congestion game, since the cost they face depends on the number of other facilities charging. This cost is fixed by the suppliers or the regulators.

When a user plugs his electrical vehicle on the evening, he has no preference on *when* the vehicle will start charging, but he only needs it to be fully charged on the next morning. With the present infrastructure, the electrical network will not be able to face the peak demands if the number of electrical vehicles continues to grow. According to CRE (2010), a million of electrical vehicles charging at the same time need between 3000 and 6000 MW, i.e. the power of 2 to 4 EPR reactors. Smart grids should be able to deal with this issue. These topics lead to an intense investigation and investment. GTM Research (2012) forecasts a cumulative global expense on smart-grid-related of 20 billion dollars between 2012 and 2020, with an annual spend of 3.8

billion globally in the year 2020.

In order to design good pricing policies, see Caron and Kesidis (2010) for example, a good understanding of the properties of Nash equilibria is necessary. An extended survey can be found in Saad et al. (2012), and concrete applications to electrical vehicles in Beaude et al. (2014).

The study of the characteristics of a Nash equilibrium is essential before any decision, i.e. before building a new infrastructure for example. Some paradoxes may appear, see Section 0.1.2. Hence, theoretical work on the existence and uniqueness of the equilibrium is necessary. Extensions on the efficiency or stability of the equilibrium can be made. Taking the practical point of view, we need to design algorithms that efficiently compute an equilibrium. A lot of results are known in the specific case where all users are impacted in the same way by the congestion.

The goal of this thesis is to find new results when we allow users to be impacted differently by the congestion.

0.1.2 Examples

We introduce classical examples of congestion games showing some counter-intuitive behaviours.

Example 0.1 (Pigou (1924))**.** Consider the situation where there is a very large number of users having to choose between two ressources. Each user has a negligible weight and the weight of all users is 1. The cost of the first ressource is fixed equal to 1 and the cost of the second one is equal to the proportion of users choosing it. This game can be modelled as a nonatomic congestion game on the graph of Figure 1: each user wants to go from o to d and has to choose between arc a and b. Each arc models a ressource. The total flow is 1, the cost of the arcs are $c_a(x) = 1$ and $c_b(x) = x$.

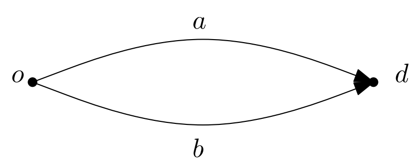

Figure 1: The example of Pigou

The Nash equilibrium of the game is reached when no user has an interest in changing his choice. In this example, at the Nash equilibrium all users choose arc b, and the cost at equilibrium is $Q^{Nash} = 1$. However, the optimal cost is reached when half of the users choose arc a and the other half choose arc b. The optimal cost is

$$Q^{Opt} = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

This example shows that the equilibrium can be inefficient since $Q^{Nash} > Q^{Opt}$. The *price of anarchy* defined as the ratio between the worst equilibrium cost and the optimal cost allows to quantify this loss of efficiency, see Section 1.4.

Example 0.2 (Paradox of Braess (1968))**.** Consider the graph of Figure 2. Nonatomic users want to go from o to d and can choose between three routes: the "north" route *oud*, the "south" route *ovd* or the "middle" route *ouvd*. The total flow is 1.

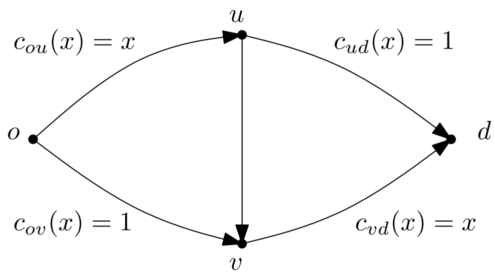

Figure 2: The example of Braess

Suppose first that there is no arc uv, or equivalently that the cost on this arc is prohibitively high. At equilibrium, half of the users take the north route and the other half take the south route. The cost at equilibrium is then $\frac{3}{2}$.

Suppose now that the cost on the arc uv is negligible, say $c_{uv}(x) = 0$. At equilibrium, all users take the middle route, and the cost at equilibrium is 2.

This example shows that deleting an arc can decrease the social cost, or, in other words, that lowering the costs can increase the congestion. This situation is called *Braess' paradox* and is counter-intuitive. For example in transportation, it means that building an highway can actually increase the mean travel time. This paradox happened for example in New York when the 42d Street, usually congested, was closed. According to Kolata (1990),

> To everyone's surprise [...] traffic flow improved when 42d Street was closed.

More recently Youn et al. (2008) proposed roads in Boston, New York, and London that could be closed to reduce predicted travel times. In physics, Pala et al. (2012) showed that Braess' paradox may occur in semiconductor mesoscopic networks: adding a path for electrons in a nanoscopic network paradoxically reduced its conductance.

Example 0.3 (Fisk (1979))**.** We consider nonatomic users on the the graph of Figure 3.

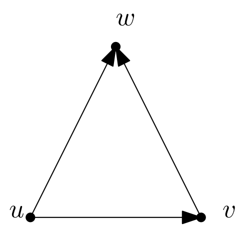

Figure 3: The example of Fisk

There are three *origin-destination pairs*: (u, v), (u, w), and (v, w). Only users with origin-destination pair (u, w) can choose between two routes, all other users have only one choice. Users wishing to go from u to v form a *total flow* of 100. The total flow associated to the origin-destination pair (u, w) is 20 and the one associated to (v, w) is 1.

Suppose that the users have the same cost functions:

$$c_{uv}(x) = c_{vw}(x) = x \quad \text{and} \quad c_{uw}(x) = x + 90.$$

The equilibrium is reached when the total flow of users with origin-destination pair (u, w) is divided in the following way: a flow of 3 chooses the route uvw and a flow of 17 chooses the route uw. The social cost is $Q^{Nash}(100, 20, 1) = 12444$.

Suppose now that the total flow of users with origin-destination pair (v, w) has increased from 1 to 4. The equilibrium is reached when the total flow of users with origin-destination pair (u, w) is divided in the following way: a flow of 2 chooses the route uvw and a flow of 18 chooses the route uw. The social cost is $Q^{Nash}(100, 20, 4) = 12384$.

Then $Q^{Nash}(100, 20, 1) > Q^{Nash}(100, 20, 4)$, while the total flow is greater in the second case. This paradox shows that some intuitive notions may not be correct: increasing the number of users on a graph can actually *improve* the equilibrium cost.

Furthermore, by computing the individual costs, we see that increasing the total flow for the origin-destination pair (v, w) has decreased the equilibrium cost for the origin-destination pair (u, v). This result is also counter-intuitive, since these users have no arc in common in their routes.

0.2 Main concerns and contributions

We distinguish two kinds of games: nonatomic and atomic splittable games, see Section 1.2 and Section 1.3 respectively. In nonatomic games, every user is infinitesimal and has no impact on the congestion. We are interested in the choices of *proportions* of the total number of users. Examples of Section 0.1.2 are nonatomic games. In atomic splittable games, we have a finite set of players, each one controlling a non-negligible amount of flow, and dividing it among the different routes in order to minimize his cost.

We use the terminology *user* for nonatomic games and *player* for atomic splittable games, each one corresponding to the classical term of *player* in the general point of view of game theory. When we consider congestion games in general, the terminology *player* must then be understood as *user* for nonatomic games and *player* for atomic games.

We say that we reach an *equilibrium* when no player has an incentive to change his strategy. The main concerns of congestion games are the **existence**, the **uniqueness**, the **computation**, the **efficiency**, and the **sensitivity** of a Nash equilibrium. A lot of work has been done when all players are impacted in the same way by the congestion, i.e. when they have the same cost functions. The objective of this thesis is to study these concerns when we allow player-specific cost functions.

0.2.1 Existence

The existence of an equilibrium is ensured under mild conditions, see Theorems 1.3 and 1.8 for the nonatomic and atomic cases respectively. We focus on the other issues.

0.2.2 Uniqueness

The equilibrium is not unique when considering the strategies of the players. For example with nonatomic users, when a specific user changes his strategy, he has no impact on the congestion. In particular at a Nash equilibrium, if a negligible subset of users have several strategies giving the same cost, any choice of these strategies gives a different Nash equilibrium. However the flow on each arc remains the same in all these equilibria.

Uniqueness is then considered in a "macroscopic" way: by uniqueness, we mean uniqueness of the *arc flows*. By restricting the set of allowable cost functions or the graph topology, uniqueness can be guaranteed, see Sections 1.2.4 and 1.3.4. However, for general cost functions and general graphs, there can be multiple equilibria. In particular for nonatomic games, Milchtaich (2005) characterized the graphs with one origin-destination pair having the *uniqueness property*, i.e. for which the equilibrium flows are unique for all assignments of increasing cost functions. In Chapter 2, we generalize this result to graphs with several origin-destination pairs.

0.2.3 Computation

When all players have the same nondecreasing cost functions, the game belongs to the class of *potential* games, see Section 1.2.3. In particular, the Nash equilibrium flows coincide with a solution of a convex optimization problem. Then, the computation of such flows can be done with tools of convex optimization. When players are impacted in a different way by the congestion, there are few algorithms that are able to compute an equilibrium, see Sections 1.2.5 and 1.3.5. In Chapters 3 and 4, we give algorithms that compute an equilibrium for nonatomic games with affine cost functions.

0.2.4 Efficiency

As shown by the Example 0.1 of Pigou (1924), the equilibrium can be inefficient. A notion to quantify this loss of efficiency is the *price of anarchy*, introduced by Koutsoupias and Papadimitriou (1999). It is the worst-case ratio between the social cost at equilibrium and the best possible social cost, see Section 1.4. The main issue is to compute bounds on the price of anarchy. Since the seminal paper of Roughgarden and Tardos (2002), tight bounds are known when players have the same cost functions. In Chapter 5, we find bounds on the price of anarchy for atomic games with player-specific cost functions.

0.2.5 Sensitivity

We consider the *sensitivity* of the equilibrium with respect to the demand. The *demand* is the total number of users, or the total flow of a player, associated to an origin-destination pair. We are interested in how the characteristics of an equilibrium change when there is an evolution of the demands. As shown by the Example 0.3 of Fisk (1979), increasing the total demand on a graph can give a better equilibrium cost. In Chapter 6, we study the sensitivity in atomic games in graphs with parallel arcs. We prove regularity results on the flow at equilibrium. We also study the behaviour when a player transfers a part of his demand to another player with initially more demand. When there are two arcs or two players, the flow of the first player increases on no arc while the flow of the second one does not decrease. In particular when players have the same cost functions, such a transfer does not increase the social cost at equilibrium.

CHAPTER 1

Network congestion games

1.1 Preliminaries

1.1.1 Graphs

An *undirected graph* is a pair $G = (V, E)$ where V is a finite set of *vertices* and E is a family of unordered pairs of vertices called *edges*. A *directed graph*, or *digraph* for short, is a pair $D = (V, A)$ where V is a finite set of *vertices* and A is a family of ordered pairs of vertices called *arcs*. The arc (u, v) is an *outgoing arc* of vertex u and an *incoming arc* of vertex v. For a vertex v, we denote by $\delta^+(v)$ the set of outgoing arcs and $\delta^-(v)$ the set of incoming arcs of v.

A *walk* in a directed graph D is a sequence

$$P = (v_0, a_1, v_1, \ldots, a_k, v_k)$$

where $k \geq 0$, $v_0, v_1, \ldots, v_k \in V$, $a_1, \ldots, a_k \in A$, and $a_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, k$. If all v_i are distinct, the walk is called a *path*. If no confusion may arise, we identify sometimes a path P with the set of its vertices or with the set of its arcs, allowing to use the notation $v \in P$ (resp. $a \in P$) if a vertex v (resp. an arc a) occurs in P.

For $(s, t) \in V^2$, a *route* is an s-t path of D and is called an (s, t)-route. The set of all routes (resp. (s, t)-routes) is denoted by $\mathcal{R}$ (resp. $\mathcal{R}_{(s,t)}$).

An undirected graph $G' = (V', E')$ is a *subgraph* of an undirected graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. An undirected graph G' is a *minor* of an undirected graph G if G' is obtained by contracting edges (possibly none) of a subgraph of G. *Contracting* an edge uv means deleting it and identifying both endpoints u and v. Two undirected graphs are *homeomorphic* if they arise from the same undirected graph by subdivision of edges, where a *subdivision* of an edge uv consists in introducing a new vertex w and in replacing the edge uv by two new edges uw and wv.

The same notions hold for directed graphs.

1.1.2 Flows

Given a digraph $D = (V, A)$ and $(s,t) \in V^2$, a (s,t)-*flow* is a vector $\boldsymbol{x} = (x_a)_{a \in A} \in \mathbb{R}^A$ such that $x_a \geq 0$ for every arc $a \in A$ and

$$\sum_{a \in \delta^+(v)} x_a = \sum_{a \in \delta^-(v)} x_a$$

for every vertex $v \in V \setminus \{s,t\}$. This last condition is called the flow conservation law. The *value* of an (s,t)-flow is $\sum_{a \in \delta^+(s)} x_a - \sum_{a \in \delta^-(s)} x_a$, i.e. the amount of flow leaving s. This value is also equal to the amount of flow entering t. Given a value d, called the *demand*, an (s,t)-flow is said to be *feasible* if it is a flow of value d. With a slight abuse of notation, given a flow $\boldsymbol{x}$ and an arc $a \in A$, we say that x_a is the flow on arc a.

Given a graph $G = (V, E)$ and a digraph $H = (T, L)$ with $T \subseteq V$, a *multiflow* is a vector $\vec{\boldsymbol{x}} = (\boldsymbol{x}^{(s,t)})_{(s,t) \in L}$ such that for each $(s,t) \in L$, $\boldsymbol{x}^{(s,t)}$ is an (s,t)-flow. In the literature, the terminology *multicommodity flow* is also used for a multiflow. In this case, each $(s,t) \in L$ is called a *commodity*.

The graph G is called the *supply graph* and H is called the *demand digraph*. The set L is viewed as the set of origin-destination pairs. In the thesis, we may use the set L without defining the demand digraph. In this case, H is naturally defined with T being the set of all endpoints of arcs in L.

1.1.3 Graphs with parallel arcs

A specific class of graphs often considered in congestion games is the *graphs with parallel arcs*. They represent a good modelling of games where players share a set of independent ressources.

A graph with parallel arcs refers to a supply graph $G = (V, E)$ and a demand digraph $D = (V, L)$ where $V = \{s,t\}$ has only two elements, every edge of E is a $\{s,t\}$-edge, and $L = \{(s,t)\}$ has only one element.

1.2 Nonatomic games

1.2.1 Definition

In nonatomic games, the population of *users* is modelled as a bounded real interval I endowed with the Lebesgue measure λ, the *population measure*. Each user is infinitesimal. We are given a digraph $D = (V, A)$ and a set of origin-destination pairs $L \subseteq V^2$ such that for every $(s,t) \in L$, we have $s \neq t$. The set I is partitioned into measurable subsets $I_{(s,t)}$ with $(s,t) \in L$, modelling the users wishing to select an (s,t)-route.

For a given graph and set of origin-destination pairs, and a given partition of users, we define a *strategy profile* as a measurable mapping $\sigma : I \to \mathcal{R}$ such that $\sigma(i) \in \mathcal{R}_{(s,t)}$ for all $(s,t) \in L$ and $i \in I_{(s,t)}$. For each arc $a \in A$, the measure of the set of all users i such that a is in $\sigma(i)$ is the *flow* on a in σ and is denoted x_a:

$$x_a = \lambda\{i \in I : a \in \sigma(i)\}.$$

The vector of flows is then denoted by $\boldsymbol{x} = (x_a)_{a\in A}$.

The cost of each arc $a \in A$ for each user $i \in I$ is given by a *cost function* $c_a^i : \mathbb{R}_+ \to \mathbb{R}_+$. When the flow on a is x_a, the cost for user i of traversing a is $c_a^i(x_a)$. For user i, the cost of a route r is defined as the sum of the costs of the arcs contained in r:

$$Q_r^i(\boldsymbol{x}) = \sum_{a\in r} c_a^i(x_a).$$

The game we are interested in is defined by the digraph D, the set of origin-destination pairs L, the population user set I with its partition, and the cost functions c_a^i for $a \in A$ and $i \in I$.

A strategy profile is a (pure) Nash equilibrium if each route is only chosen by users for whom it is a minimal-cost route. In other words, a strategy profile σ is a Nash equilibrium if for each pair $(s,t) \in L$ and each user $i \in I_{(s,t)}$ we have

$$Q_{\sigma(i)}^i(\boldsymbol{x}) = \min_{r\in\mathcal{R}_{(s,t)}} Q_r^i(\boldsymbol{x}).$$

Remark 1.1. The equilibrium in nonatomic games was first defined by Wardrop (1952). In the literature, the terminology of *Wardrop equilibrium* is often used. However, in order to deal with user-specific cost functions and keep the game theory terminology, we will call it *Nash equilibrium*.

Class of users

In some games, users may be split into classes. A *class* is a set of users having the same cost functions on all arcs, but not necessarily sharing the same origin-destination pair. In this case, the set I is partitioned into a finite number of measurable subsets $(I^k)_{k\in K}$ – the *classes* – modelling the users sharing a same collection of cost functions $(c_a^k : \mathbb{R}_+ \to \mathbb{R}_+)_{a\in A}$. A user in I^k is said to be of *class* k. For each arc $a \in A$, the measure x_a^k of the set of all class k users i such that a is in $\sigma(i)$ is the *class k flow* on a in σ:

$$x_a^k = \lambda\{i \in I^k : a \in \sigma(i)\}.$$

In this case, the class k flow is denoted by $\boldsymbol{x}^k = (x_a^k)_{a\in A}$, while the vector of all class flows is the multiflow $\vec{\boldsymbol{x}} = (\boldsymbol{x}^k)_{k\in K}$. Note that the agregated flow remains denoted by $\boldsymbol{x} = (x_a)_{a\in A}$ with $x_a = \sum_{k\in K} x_a^k$.

Remark 1.2. When considering nonatomic games where the set of users is partitioned into classes, we talk of *multiclass games* or of *the multiclass case*. Note that in the multiclass case, the game remains a noncooperative game. When users of a given class decide to cooperate, we say that they form a *coalition*. In this case, the game belongs to the set of *atomic* games, see Section 1.3. In particular, the terminology *class* will be used only for nonatomic games.

1.2.2 Existence of a Nash equilibrium

Under mild conditions on the cost functions, a Nash equilibrium is always known to exist. The original proof of the existence of an equilibrium was made by Schmeidler (1970) and uses a fixed point theorem. The proof of this result is also made in Milchtaich (2000) or can be deduced from more general results (Rath, 1992).

Theorem 1.3 (Schmeidler (1970)). *Consider a nonatomic congestion game. Suppose that the cost functions satisfy: $c_a^i(\cdot)$ is continuous for every arc a and user i, and $i \mapsto c_a^i(x)$ is measurable for every $x \in [0, \lambda(I)]$. Then there exists a Nash equilibrium.*

The definition of a Nash equilibrium can be reformulated, giving then new characterizations.

1.2.3 Characterizations of a Nash equilibrium

Single-class case: a potential game

The specific case where all users have the same cost functions has been extensively studied. In this context, the game belongs to the class of *potential games*. This class of games was first defined by Monderer and Shapley (1996), extending a result of Rosenthal (1973). Then the Nash equilibrium flows are optimal solutions of a convex optimization problem.

Proposition 1.4 (Beckmann et al. (1956)). *Suppose that the cost functions c_a are continuous and nondecreasing. The multiflow $\vec{\boldsymbol{x}}$ is an equilibrium flow* *if and only if it is a solution of the following problem.*

$$\begin{aligned}
&\min \sum_{a\in A} \int_0^{x_a} c_a(u)\,du && (1.1)\\
s.t.\quad & \sum_{(s,t)\in L} x_a^{(s,t)} = x_a && a\in A,\\
& \sum_{a\in\delta^+(v)} x_a^{(s,t)} = \sum_{a\in\delta^-(v)} x_a^{(s,t)} + b_v^{(s,t)} && (s,t)\in L, v\in V,\\
& x_a^{(s,t)} \geq 0 && (s,t)\in L, a\in A,
\end{aligned}$$

where $b_v^{(s,t)} = \left(1_{\{v=s\}} - 1_{\{v=t\}}\right)\lambda(I_{(s,t)})$.

However, in the multiclass case the game is not in general a potential game.

Multiclass case: a nonlinear complementarity problem

Assume that the users are partitioned into classes. In the single-class case, by writing the Karush-Kuhn-Tucker conditions of the problem (1.1), we get that the equilibrium flows $\boldsymbol{x}$ coincide with the solutions of a nonlinear complementarity problem, see Aashtiani and Magnanti (1981).

In the multiclass case, when the flows $\boldsymbol{x}^{k'}$ for $k' \neq k$ are fixed, finding the equilibrium flows for the class k is again a single-class problem. We get then the following characterization, where for the ease of reading we ask the users of a given class k to have the same origin s^k, and the same destination t^k.

Proposition 1.5. *Suppose that the cost functions* c_a^k *are continuous and nondecreasing. The multiflow* $\vec{\boldsymbol{x}}$ *is an equilibrium flow if and only if there exist* $\boldsymbol{\mu}^k \in \mathbb{R}_+^A$ *and* $\boldsymbol{\pi}^k \in \mathbb{R}^V$ *for all* k *such that* $(\boldsymbol{x}^k, \boldsymbol{\mu}^k, \boldsymbol{\pi}^k)_{k\in K}$ *is a solution of the following complementarity problem:*

$$\begin{aligned}
& \sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k && k\in K, v\in V\\
& c_{uv}^k(x_{uv}) + \pi_u^k - \pi_v^k - \mu_{uv}^k = 0 && k\in K, (u,v)\in A\\
& x_a^k \mu_a^k = 0 && k\in K, a\in A\\
& x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} && k\in K, a\in A, v\in V.
\end{aligned}$$

where $b_v^k = \left(1_{\{v=s^k\}} - 1_{\{v=t^k\}}\right)\lambda(I^k)$ *for every class* k.

Proof. See Proposition 3.1 in Section 3.2. □

Finding solutions for such systems is a complementarity problem, the word "complementarity" coming from the condition $x_a^k \mu_a^k = 0$ for all (a,k) such that $a \in A$.

Multiclass case: a solution of a variational inequality

The equilibrium can also be viewed as a solution of a variational inequality (Smith, 1979, Dafermos, 1980). In our context, we obtain the following characterization.

Proposition 1.6. *Suppose that the cost functions c_a^k are continuous and nondecreasing. The multiflow $\vec{\boldsymbol{x}}$ is an equilibrium flow if and only if, for all k, it satisfies*

$$\sum_{a \in A} c_a^k(x_a)(y_a^k - x_a^k) \geq 0, \quad \textit{for any feasible flow } \boldsymbol{y}^k \textit{ for class } k.$$

1.2.4 Uniqueness of equilibrium

In the single-class case, i.e. when all users are equally affected by the congestion, and when the cost functions are increasing, the equilibrium is unique, since the convex optimization problem (1.1) has a unique solution. However, this is not the case when we allow user-specific cost functions. Two types of restriction can be made to have a unique equilibrium flow: conditions on the cost functions, and conditions on the graph topology.

There are few works giving conditions on the cost functions that ensures uniqueness. Altman and Kameda (2001) proved that when the cost functions are the same for every user, up to an additive constant, the equilibrium flow is unique.

Milchtaich (2005) defined the notion of *uniqueness property* as the uniqueness of the equilibrium flow whatever are the increasing cost functions. He introduced the family of *nearly parallel* graphs. A graph is nearly parallel if it is one of the graphs of Figure 1.1 or a connection in series of those. When all users have the same origin-destination pair, the uniqueness property holds if and only if the graph is nearly parallel.

In Chapter 2, we generalize this result when the users have different origin-destination pairs. We characterize completely bidirectional rings for which the uniqueness property holds: it holds precisely for nine graphs and those obtained from them by elementary operations. We deduce necessary conditions for general graphs to have the uniqueness property. However,

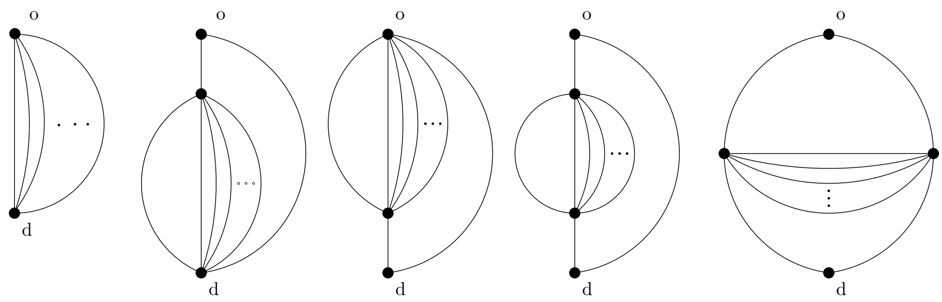

Figure 1.1: Basic graphs defining the class of nearly parallel graphs

there are still graphs for which neither Milchtaich (2005) nor Chapter 2 can be used to prove or disprove the uniqueness property (see Figure 2.8).

1.2.5 Computation of an equilibrium

As noted in Section 1.2.3, when all users have the same cost functions, the computation of a Nash equilibrium amounts to solve a convex optimization problem. The currently most commonly used algorithm for such convex programs is probably the Frank-Wolfe algorithm (Frank and Wolfe, 1956). In the multiclass case, this method can be adapted, since finding the equilibrium flows for a given class when the flows of the other classes is fixed is again a single-class problem. We cite for example two approaches widely used but whose convergence is not always guaranteed.

- In the *Jacobi* approach, we solve at iteration n the K problems

$$\min_{x_a^k} \sum_{a\in A} \int_0^{x_a^k} c_a^k \left(\sum_{i=0}^{k-1} x_a^{(n),i} + u + \sum_{i=k+1}^{K} x_a^{(n),i} \right) du$$

  where the minimum is taken over all feasible flows for class k users.

- In the *Gauss-Seidel* approach, we solve at iteration n the problem

$$\min_{x_a^k} \sum_{a\in A} \int_0^{x_a^k} c_a^k \left(\sum_{i=0}^{k-1} x_a^{(n+1),i} + u + \sum_{i=k+1}^{K} x_a^{(n),i} \right) du$$

  where the minimum is taken over all feasible flows for class k users.

The Jacobi approach consists then in solving a sequence of K problems at each step, which can be done in an efficient way using parallelism. The

Gauss-Siedel approach uses the latest vector update at each step. There are various conditions ensuring the convergence, see Florian (1977), Harker (1988), Mahmassani and Mouskos (1988), Marcotte and Wynter (2004) for example. For a more general survey on methods used to solve variational inequalities, see Marcotte (1997).

However, there is no general algorithm in the literature for solving the problem when the cost of each arc is in an affine dependence with the flow on it. We propose such algorithms in Chapters 3 and 4. The algorithm of Chapter 3 is a pivoting Lemke-like algorithm and relies on the formulation of an equilibrium as a solution of a linear complementarity problem. We prove the convergence and proceed to computational experiments. In Chapter 4 we give an algorithm polynomial in the number of arcs, for a fixed number of vertices and classes. It uses a correspondence between the set of arcs used at equilibrium for each class and cells of a hyperplane arrangement.

1.3 Atomic games

In this thesis, the atomic congestion games we consider are assumed to be *splittable*. In these games, we have a finite set of players, each player having a non-negligible demand to transfer from an origin to a destination. In contrary to *unsplittable* games, players can divide their demand among the different routes. In practice, these games can model companies having a stock to ship for one point to an other, and paradoxes similar as in Section 0.1.2 can occur, see for example Catoni and Pallottino (1991).

Another way to see atomic splittable games is to consider nonatomic games with *coalitions*. In this case, a subset of nonatomic users can decide to act in a centralized way, as a coalition.

1.3.1 Definition

In atomic splittable games, we are given a digraph $D = (V, A)$ and K players. The players are identified with the integers $1, \dots, K$ and the set of all players is denoted by $[K]$. Each player $k \in [K]$ has to send d^k units of flow, his *demand*, from an origin $s^k \in V$ to a destination $t^k \in V$ in the digraph.

For a given digraph and set of players with their demand and origin-destination pair, a *strategy profile* is a multiflow $\vec{\boldsymbol{x}} = (\boldsymbol{x}^1, \dots, \boldsymbol{x}^K) \in \mathbb{R}_+^{A \times K}$ such that for each player k, $\boldsymbol{x}^k \in \mathbb{R}_+^A$ is an (s^k, t^k)-flow of value d^k. Such a flow for player k is an element of

$$\mathcal{F}^k = \left\{ \boldsymbol{y} \in \mathbb{R}_+^A : \sum_{a\in\delta^+(s^k)} y_a - \sum_{a\in\delta^-(s^k)} y_a = d^k \quad \text{and} \right.$$

$$\left. \sum_{a\in\delta^+(v)} y_a = \sum_{a\in\delta^-(v)} y_a, \quad \forall v \in V \setminus \{s^k, t^k\} \right\}$$

and is referred as a *feasible flow for player* k. A strategy profile is then a *feasible* multiflow, i.e. an element of $\mathcal{F}^1 \times \cdots \times \mathcal{F}^K$.

Each player k has his own vector of cost functions $\boldsymbol{c}^k = (c_a^k)_{a\in A}$ where for each arc a the cost function $c_a^k(\cdot)$ is a $\mathbb{R}_+ \to \mathbb{R}_+$ function. We assume given a set of *allowable cost functions* $\mathcal{C}$ in which $\vec{\boldsymbol{c}} = (\boldsymbol{c}^1, \ldots, \boldsymbol{c}^K)$ is taken. Since each player has his own vector of cost functions, the game is said to be with player-specific cost functions.

Remark 1.7. In the literature, this game is sometimes called a *multiclass* game. On the contrary, if $\boldsymbol{c}^1 = \cdots = \boldsymbol{c}^K$ for every $\vec{\boldsymbol{c}} \in \mathcal{C}$, one speaks of a *single-class* game. However, in order to avoid confusion between nonatomic and atomic games, we will use the terminology *class* only for nonatomic games.

We denote the total flow on an arc a by $x_a = \sum_{k\in[K]} x_a^k$. The cost experienced by a player k is

$$Q^k(\vec{\boldsymbol{x}}) = \sum_{a\in A} x_a^k c_a^k(x_a).$$

The goal of this player consists in sending his d^k units of flows while minimizing this cost.

The game we are interested in is defined by the digraph D, the set of players $[K]$ with their demand, origin-destination pair, and set of cost functions.

A strategy profile $\vec{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K)$ is a *Nash equilibrium* if for each player k, we have

$$Q^k(\vec{\boldsymbol{x}}) = \min_{\boldsymbol{y}\in\mathcal{F}^k} Q^k(\boldsymbol{y}, \vec{\boldsymbol{x}}^{-k}), \tag{1.2}$$

where $(\boldsymbol{y}, \vec{\boldsymbol{x}}^{-k}) = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{k-1}, \boldsymbol{y}, \boldsymbol{x}^{k+1}, \ldots, \boldsymbol{x}^K)$.

The *social cost* of a multiflow is defined as

$$Q(\vec{\boldsymbol{x}}) = \sum_{k\in[K]} Q^k(\vec{\boldsymbol{x}}) = \sum_{k\in[K]} \sum_{a\in A} x_a^k c_a^k(x_a).$$

A multiflow of minimal social cost is a *social optimum*.

1.3.2 Existence of a Nash equilibrium

The existence of a Nash equilibrium is ensured when the game is a *convex* game (Rosen, 1965). For an arc a and a player k, the cost function c_a^k is a *per-unit* cost. The contribution of this arc to the total cost of the player is $x_a^k c_a^k(x_a)$. To have a convex game, it is sufficient that the functions $x^k \mapsto x^k c_a^k(x)$ are convex. However we make the stronger assumption that the cost functions c_a^k are nondecreasing.

In this case, the result of Rosen (1965) holds, using the Kakutani fixed point theorem. The formulation of the proof in the framework of congestion games can be found in Orda et al. (1993).

Theorem 1.8 (Rosen (1965), Orda et al. (1993))**.** *Consider an atomic splittable congestion game. Suppose that the cost functions c_a^k are continuous and nondecreasing for every arc a and player k. Then there exists a Nash equilibrium.*

The definition of a Nash equilibrium can be reformulated, giving then new characterizations.

1.3.3 Characterizations of a Nash equilibrium

Throughout the thesis, the components of $\boldsymbol{x} \in \mathbb{R}_+^K$ are denoted x^k. Given a K-tuple of continuously differentiable cost functions $\boldsymbol{c} = (c^1, \ldots, c^K)$, we define for every k the *marginal cost* $\tilde{c}^k : \mathbb{R}_+^K \to \mathbb{R}_+$ by

$$\tilde{c}^k(\boldsymbol{x}) = \frac{\partial}{\partial x^k}\left(x^k c^k(x)\right) \qquad \text{where } x = \sum_{\ell \in [K]} x^\ell.$$

For the ease of reading, we drop the parenthesis over the derivatives and note $c^{k\prime}$ instead of $(c^k)'$. To avoid any confusion, no index with a prime k' will be used in the thesis. We have then

$$\tilde{c}^k(\boldsymbol{x}) = x^k c^{k\prime}(x) + c^k(x).$$

The characterizations of a Nash equilibrium in nonatomic games still hold for atomic splittable games where the cost functions c^k are replaced by the marginal cost $\tilde{c}^k$. In particular, Proposition 1.6 can be reformulated. However, the following proposition is standard in this context and can be directly obtained by writing the optimality conditions of Equation (1.2). This characterization has been used in particular in Haurie and Marcotte (1985).

Proposition 1.9. *Suppose that the cost functions c_a^k are continuously differentiable and nondecreasing. The multiflow $\vec{\boldsymbol{x}}$ is a Nash equilibrium if and only if, for all k, it satisfies*

$$\sum_{a\in A} \tilde{c}_a^k(\boldsymbol{x}_a)(y_a^k - x_a^k) \geq 0, \quad \textit{for any feasible flow } \boldsymbol{y}^k \textit{ for player } k, \tag{1.3}$$

where $\boldsymbol{x}_a = (x_a^1, \ldots, x_a^K) \in \mathbb{R}_+^K$.

Furthermore, Proposition 1.5 can also be transposed for atomic games. We reformulate it in a more convenient way that will be used in the following.

Proposition 1.10. *Suppose that the cost functions c_a^k are continuously differentiable and nondecreasing. The multiflow $\vec{\boldsymbol{x}}$ is a Nash equilibrium flow if and only if, for all k, $\boldsymbol{x}^k$ is a feasible flow for player k such that there exists $\boldsymbol{\pi}^k \in \mathbb{R}^V$ with*

$$\begin{aligned} \tilde{c}_a^k(\boldsymbol{x}_a) &\geq \pi_v^k - \pi_u^k & \textit{for all } a = (u,v) \in A, \\ \tilde{c}_a^k(\boldsymbol{x}_a) &= \pi_v^k - \pi_u^k & \textit{for all } a = (u,v) \in A \textit{ such that } x_a^k > 0. \end{aligned}$$

1.3.4 Uniqueness of equilibrium

Uniqueness of the Nash equilibrium is not guaranteed in general. In their seminal paper, Orda et al. (1993) showed that the equilibrium is unique for *symmetric* players i.e. when all players have the same demand, origin-destination pair and cost functions. They gave a counterexample for more general games. The conditions for uniqueness can be divided into two types: restrictions on the set of allowable cost functions, and restrictions on the graph topology.

Applying the general result of Rosen (1965), uniqueness is guaranteed when we assume conditions on the cost functions, related to the notion of diagonal strict convexity. More recently, Altman et al. (2002) proved that when players have the same monomial cost functions of degree at most three, the equilibrium flow is unique.

Topological restrictions on the graph ensure uniqueness as well. Richman and Shimkin (2007) extended the result of Milchtaich (2005) holding for nonatomic graphs. They proved that when all players have the same origin-destination pair, the uniqueness property holds if and only if the graph is *nearly parallel*, see Figure 1.1. Bhaskar et al. (2009) extended the result to *generalized parallel* graphs, when all players have the same cost functions. The generalization to graphs with several origin-destination pairs is still an open problem. In particular, whether the results of Chapter 2 for nonatomic games can be extended to atomic games deserves future work.

1.3.5 Computation of an equilibrium

Since the equilibria for both nonatomic multiclass and atomic games are solutions of a variational inequality, the methods introduced in Section 1.2.5 hold for atomic games, when replacing the cost functions c_a^k by the marginal costs $\tilde{c}_a^k$. In particular, the algorithms given in Chapters 3 and 4 can be adapted to give an equilibrium flow for atomic games with affine cost functions.

More specific algorithms exist for games with a finite number of players, beginning with Rosen (1965). The algorithm introduced by Rosen (1965) needs restrictive conditions, but the dynamic is shown to converge to an equilibrium. Algorithms have been designed in other specific cases, using supermodularity or monotonicity, see Altman et al. (2006) for a review. In particular, Altman et al. (2001) and Boulogne et al. (2002) proved that some basic algorithms converge to an equilibrium in simple graphs with linear cost functions.

1.4 Efficiency of equilibrium: the price of anarchy

As shown by the Example 0.1, the equilibrium is in general inefficient. To quantify the loss of efficiency, Koutsoupias and Papadimitriou (1999) introduced a ratio that Papadimitriou (2001) named the *price of anarchy*. The price of anarchy is the worst-case ratio between the social cost at equilibrium and the best possible social cost. Given a set of allowable instances $\mathcal{I}$, for any $I \in \mathcal{I}$ let $\mathrm{NE}(I)$ be the set of Nash equilibria of the game and consider $\vec{\boldsymbol{x}}^{OPT}(I)$ a feasible multiflow achieving the minimal social cost. The price of anarchy is defined by

$$\mathrm{PoA}(\mathcal{I}) = \sup_{I \in \mathcal{I}} \sup_{\vec{\boldsymbol{x}} \in \mathrm{NE}(I)} \frac{Q(\vec{\boldsymbol{x}})}{Q(\vec{\boldsymbol{x}}^{OPT}(I))}.$$

1.4.1 Nonatomic games

The interest on the price of anarchy has led to a considerable amount of work for games where the users have the same cost functions. In their seminal paper, Roughgarden and Tardos (2002) proved that the price of anarchy is bounded by $\frac{4}{3}$ for affine cost functions. This bound is reached for simple graphs as in Example 0.1. This has led to the result of Roughgarden (2003) stating that the price of anarchy is independent of the graph topology. These results have been extended, see for example Correa et al. (2004, 2008), Dumrauf and Gairing (2006). In particular, Correa et al. (2004) found a new geometric proof of some previous bounds. The next proposition summarizes these results.

Proposition 1.11. *Let $\mathcal{I}$ be a set of instances of a nonatomic game with users having the same cost functions,*

1. $\text{PoA}(\mathcal{I}) = \frac{4}{3}$ *when the cost functions in $\mathcal{I}$ are all affine cost functions,*
2. $\text{PoA}(\mathcal{I}) = \Theta\left(\frac{d}{\ln d}\right)$ *when the cost functions in $\mathcal{I}$ are all polynomial functions of degree d with nonnegative coefficients.*

Proof. 1. See Roughgarden and Tardos (2002).

2. The exact bound of $\frac{(d+1)^{1+1/d}}{(d+1)^{1+1/d}-d}$ is proved in Roughgarden (2003).

□

The generalization to games with user-specific cost functions is still an open question, considered in very few works, see Gairing et al. (2006) for example.

1.4.2 Atomic games

Most of the works dealing with the price of anarchy for atomic splittable games consider situations where players have the same cost functions. The first general bound on the price of anarchy is probably the one of Cominetti et al. (2009). It yields a bound for polynomial functions of degree at most 3 with nonnegative coefficients. In particular they found a bound of $\frac{3K+1}{2K+2}$ for affine cost functions when the number K of players is fixed. These results have been extended, see for example Harks (2008, 2011), Bhaskar et al. (2010). Finally, Roughgarden and Schoppmann (2011) found a bound holding for polynomial functions with nonnegative coefficients of any degree.

Proposition 1.12 (Roughgarden and Schoppmann (2011))**.** *Let $\mathcal{I}$ be a set of instances of an atomic splittable game with players having the same cost functions,*

1. $\text{PoA}(\mathcal{I}) = \frac{3}{2}$ *when the cost functions in $\mathcal{I}$ are all affine cost functions,*
2. $\text{PoA}(\mathcal{I}) = \left(\frac{1+\sqrt{d+1}}{2}\right)^{d+1}$ *when the cost functions in $\mathcal{I}$ are all polynomial functions of degree $d \geq 2$ with nonnegative coefficients.*

The generalization to games with player-specific cost functions is made in Chapter 5, where we find a general bound, see Theorem 5.1. Unfortunately, this bound uses a parameter depending on the cost functions, and the price of anarchy is generally unbounded, even when considering affine cost functions, see Section 5.4.

1.4.3 Comparison and extensions

When the players have the same cost functions, a way to find a bound for the price of anarchy of an atomic splittable game is to compare it with the one of the corresponding nonatomic game. We can then consider the *price of collusion*, defined by Hayrapetyan et al. (2006) as the ratio between the social cost at the equilibrium of the atomic game and the one at the equilibrium of the corresponding nonatomic game. Since the optimal cost is the same in both games, it can also be viewed as the ratio between the price of anarchy of the atomic game and the one of the nonatomic game.
In some specific situations, Hayrapetyan et al. (2006), Cominetti et al. (2009) and Altman et al. (2011) showed that the price of collusion is at most 1. Unfortunately, this result does not hold in general as shown by the example of Cominetti et al. (2009), adapted from Catoni and Pallottino (1991). This notion has led to several other works, see Bhaskar et al. (2010), Wan (2012b), Blocq and Orda (2013) for example, still in the context where players have the same cost functions. The generalization to games with player-specific cost functions could be done by comparing an atomic splittable game to the corresponding *multiclass* nonatomic game, but it is still an open problem.

The price of anarchy has also been studied in the context where the cost functions are *nonseparable*. In this case, the cost of a route is not defined as the sum of the cost of the arcs in the route, see Section 1.5. Some results on the price of anarchy for games with nonseparable costs are proved in Chau and Sim (2003) and Perakis (2007).

1.5 Extension: games with risk-averse users[1]

A natural extension of the previously defined games is to consider *nonseparable* cost functions. In this case, the cost of a route is not defined as the sum of the cost of the arcs in the route. This kind of cost functions appears for example in congestion games with risk-averse users, see Ordóñez and Stier-Moses (2010).

In these games, we consider nonatomic users. Each user chooses the route that minimizes the *mean travel time* plus a multiple of the *standard deviation* of the travel time. The weight of the standard deviation corresponds to the user *risk aversion*. This problem is a stochastic shortest path problem (Bertsekas and Tsitsiklis, 1991).

[1]This section introduces the work I did in collaboration with Nicolas Stier-Moses, during my visit in 2013 at *Universidad Torcuato di Tella* in Buenos Aires, Argentina.

This has been studied for example in Nie (2011), Cominetti and Torrico (2013), and Nikolova and Stier-Moses (2014)

The model is the same as in Section 1.2, except for the definition of the cost functions. The cost function of an arc $a \in A$ for a user $i \in I$ is defined by

$$c_a^i(x_a) = \ell_a^i(x_a) + \xi_a^i(x_a).$$

The quantity $\ell_a^i(x_a)$ measures the expected travel time for user i on arc a when the flow is x_a, and $\xi_a^i(x_a)$ is a random variable whose expectation is zero and standard deviation is $\sigma_a^i(x_a)$.

The functions ℓ_a^i and σ_a^i are assumed to be $\mathbb{R}^+ \to \mathbb{R}^+$ functions, continuous, increasing, and such that $i \mapsto \ell_a^i(x)$ and $i \mapsto \sigma_a^i(x)$ are measurable for all $a \in A$ and $x \in \mathbb{R}_+$.

The cost of a route r for user i is the mean-standard deviation objective, that is

$$Q_r^i(\boldsymbol{x}) = \sum_{a \in r} \ell_a^i(x_a) + \gamma^i \sqrt{\sum_{a \in r} (\sigma_a^i)^2(x_a)},$$

where $\gamma^i \geq 0$ represents the user's *risk-aversion.*

When the cost functions and the risk-aversion are the same for all users, this is the mean-risk model for nonatomic congestion games defined in Nikolova and Stier-Moses (2014). A Nash equilibrium for this game is called a *mean-stdev equilibrium.*

The generality of the result of Schmeidler (1970) gives that a mean-stdev equilibrium exists. Nikolova and Stier-Moses (2014) gave another proof in the specific case where the cost functions and the risk-aversion are the same for all users. However, some results of Section 1.2 do not hold anymore and further investigation is necessary.

In particular, since the cost of a route is different from the sum of the cost of each arc in this route, there is no *monotonicity* of the cost, as shown in the following example.

Example 1.13 (Nikolova and Stier-Moses (2014))**.** We consider the graph of Figure 1.2.

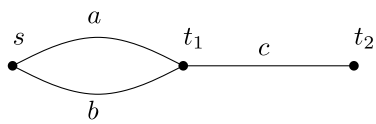

Figure 1.2: An example of non-monotonicity

The cost functions are constant and the same for each user. The means and standard deviations are in the following table.

Arc	a	b	c
Mean	6.9	5	5
Standard deviation	1	3	1

Furthermore, the risk-aversion is equal to 1 for every user. Then, when users want to go from s to t_1, the most attractive arc is the arc a, with a cost of 7.9, while the cost of arc b is 8.

When users want to go from s to t_2, the arc a is not attractive anymore, since the cost of the route bc is 13.16 and the cost of the route ac is 13.31.

Nikolova and Stier-Moses (2014) distinguish two cases of standard deviation: the exogenous and endogenous cases.

In the *exogenous* case, the variability that affects travel times does not depend on the traffic. Hence, the standard deviation is constant : $\sigma_a^i(x) = \sigma_a^i$ for every arc a and user i.

In this case, when all users have the same cost functions, Nikolova and Stier-Moses (2014) proved that the game is a potential game, and the mean-stdev equilibrium is unique when the expected travel time functions ℓ are strictly increasing.

In the more general *endogenous* case, the standard deviation is dependent of the flow. Nikolova and Stier-Moses (2014) showed that, even when players have the same cost functions, the game is not a potential game. There is no general result on uniqueness. The only known results concern extreme cases where users are either risk-neutral ($\gamma^i = 0$) or infinitely risk-averse ($\gamma^i \to \infty$). In these cases, when the expected travel times and standard deviation are strictly increasing, the equilibrium flows are unique.

The question of uniqueness in a more general setting and the design of algorithms for the computation of an equilibrium need further investigation.

CHAPTER 2

The uniqueness property for graphs with several origin-destination pairs

This chapter is based on the paper "The uniqueness property for networks with several origin-destination pairs" (Meunier and Pradeau, 2014). This work was presented at the ISMP 2012 and ROADEF 2013 conferences.

Abstract

In this chapter, we consider congestion games on graphs with nonatomic users and user-specific costs. We are interested in the uniqueness property defined by Milchtaich (2005) as the uniqueness of equilibrium flows for all assignments of strictly increasing cost functions. He settled the case with two-terminal graphs. As a corollary of his result, it is possible to prove that some other graphs have the uniqueness property as well by adding common fictitious origin and destination. In the present work, we find a necessary condition for graphs with several origin-destination pairs to have the uniqueness property in terms of excluded minors or subgraphs. As a key result, we characterize completely bidirectional rings for which the uniqueness property holds: it holds precisely for nine graphs and those obtained from them by elementary operations. For other bidirectional rings, we exhibit affine cost functions yielding to two distinct equilibrium flows. Related results are also proven. For instance, we characterize graphs having the uniqueness property for any choice of origin-destination pairs.

2.1 Introduction

In many areas, different users share a common network to travel or to exchange information or goods. Each user wishes to select a path connecting a certain origin to a certain destination. However, the selection of paths in the network by the users induces congestion on the arcs, leading to an increase of the costs. Taking into account the choices of the other users, each user looks for a path of minimum cost. We expect therefore to reach a Nash equilibrium: each user makes the best reply to the actions chosen by the other users.

This kind of games is studied since the 1950's, with the seminal works by Wardrop (1952) and Beckmann et al. (1956). Their practical interest is high since the phenomena implied by the strategic interactions of users on a network are often counter-intuitive and may lead to an important loss in efficiency. The Braess paradox (Braess, 1968) – adding an arc may deteriorate all travel times – is the classical example illustrating such a counter-intuitive loss and it has been observed in concrete situations, for example in New York (Kolata, 1990). Koutsoupias and Papadimitriou (1999) initiated a precise quantitative study of this loss, which lead soon after to the notion of "Price of Anarchy" that is the cost of the worst equilibrium divided by the optimal cost, see Roughgarden and Tardos (2002) among many other references.

When the users are assumed to be nonatomic – the effect of a single user is negligible – equilibrium is known to exist (Milchtaich, 2000). Moreover, when the users are affected equally by the congestion on the arcs, the costs supported by the users are the same in all equilibria (Aashtiani and Magnanti, 1981). In the present chapter, we are interested in the case when the users may be affected differently by the congestion. In such a case, examples are known for which these costs are not unique. Various conditions have been found that ensure nevertheless uniqueness. For instance, if the user's cost functions attached to the arcs are continuous, strictly increasing, and identical up to additive constants, then we have uniqueness of the equilibrium flows, and thus of the equilibrium costs (Altman and Kameda, 2001). In 2005, continuing a work initiated by Milchtaich (2000) and Konishi (2004) for graphs with parallel routes, Milchtaich (2005) found a topological characterization of two-terminal graphs for which, given any assignment of strictly increasing and continuous cost functions, the flows are the same in all equilibria. Such graphs are said to enjoy the *uniqueness property*. Similar results with atomic users have been obtained by Orda et al. (1993) and Richman and Shimkin (2007).

The purpose of this chapter is to find similar characterizations for graphs with more than two terminals. We are able to characterize completely the ring graphs having the uniqueness property, whatever the number of terminals is. Studying equilibria on rings can be seen as the decentralized counterpart of works on the optimization of multiflows on rings, like the one proposed by Myung et al. (1997). Our main result for ring graphs is that the uniqueness property holds precisely for nine graphs and those obtained from them by elementary operations. For other rings, we exhibit affine cost functions yielding to two distinct equilibrium flows. It allows to describe infinite families of graphs for which the uniqueness property does not hold. For instance, there is a family of ring graphs such that every network with a minor in this family does not have the uniqueness property.

2.2 Preliminaries on graphs

Recall the definitions of Section 1.1.1. We define a *mixed graph* to be a graph having edges and arcs. More formally, it is a triple $M = (V, E, A)$ where V is a finite set of vertices, E is a family of unordered pairs of vertices (edges) and A is a family of ordered pairs of vertices (arcs). Given an undirected graph $G = (V, E)$, we define the *directed version* of G as the digraph $D = (V, A)$ obtained by replacing each (undirected) edge in E by two (directed) arcs, one in each direction. An arc of G is understood as an arc of its directed version. In these graphs, *loops* – edges or arcs having identical endpoints – are not allowed, but pairs of vertices occurring more than once – *parallel edges* or *parallel arcs* – are allowed.

The notions of subgraph, minor and homeomorphism defined in Section 1.1.1 hold for mixed graphs in the same way. Finally, let $G = (V, E)$ be an undirected graph, and $H = (T, L)$ be a directed graph with $T \subseteq V$, then $G + H$ denotes the mixed graph (V, E, L).

2.3 Model

Similarly as in the multiflow theory (see for instance Schrijver (2003) or Korte and Vygen (2000)), we are given a *supply graph* $G = (V, E)$ and a *demand digraph* $H = (T, L)$ with $T \subseteq V$. The graph G models the (transportation) *network*. The arcs of H model the origin-destination pairs, also called in the sequel the *OD-pairs*. H is therefore assumed to be simple, i.e. contains no loops and no multiple edges. A *route* is an (s, t)-path of the directed version of G with $(s, t) \in L$ and is called an (s, t)-route. The set of all routes (resp. (s, t)-routes) is denoted by $\mathcal{R}$ (resp. $\mathcal{R}_{(s,t)}$).

The population of *users* is modelled as in Section 1.2.1. The cost functions $c_a^i : \mathbb{R}_+ \to \mathbb{R}_+$ are assumed to be nonnegative, continuous, strictly increasing, and such that $i \mapsto c_a^i(x)$ is measurable for all $a \in A$ and $x \in \mathbb{R}_+$.

The game we are interested in is defined by the supply graph G, the demand digraph H, the population user set I with its partition, and the cost functions c_a^i for $a \in A$ and $i \in I$. If we forget the graph structure, we get a game for which we use the terminology *nonatomic congestion game with user-specific cost functions*, as in Milchtaich (1996).

Recall the definition of an equilibrium: a strategy profile is a (pure) Nash equilibrium if each route is only chosen by users for whom it is a minimal-cost route. In other words, a strategy profile σ is a Nash equilibrium if for each pair $(s,t) \in L$ and each user $i \in I_{(s,t)}$ we have

$$\sum_{a\in\sigma(i)} c_a^i(x_a) = \min_{r\in\mathcal{R}_{(s,t)}} \sum_{a\in r} c_a^i(x_a).$$

Under the conditions stated above on the cost functions, a Nash equilibrium is always known to exist, see Theorem 1.3. However, such an equilibrium is not necessarily unique, and even the equilibrium flows are not necessarily unique.

2.4 Results

Milchtaich (2005) raised the question whether it is possible to characterize graphs having the *uniqueness property*, i.e. graphs for which flows at equilibrium are unique. A pair (G,H) defined as in Section 2.3 is said to have the *uniqueness property* if, for any partition of I into measurable subsets $I_{(s,t)}$ with $(s,t) \in L$, and for any assignment of (strictly increasing) cost functions, the flow on each arc is the same in all equilibria.

Milchtaich found a positive answer for the two-terminal graphs, i.e. when $|L| = 1$. More precisely, he gave a (polynomial) characterization of a family of two-terminal undirected graphs such that, for the directed versions of this family and for any assignment of (strictly increasing) cost functions, the flow on each arc is the same in all equilibria. For two-terminal undirected graphs outside this family, he gave explicit cost functions for which equilibria with different flows on some arcs exist.

The objective of this chapter is to address the uniqueness property for graphs having more than two terminals. We settle the case of ring graphs and find a necessary condition for general graphs to have the uniqueness property in terms of excluded minors or subgraphs.

In a ring network, each user has exactly two possible strategies. See Figure 2.1 for an illustration of this kind of supply graph G, demand digraph H, and mixed graph $G + H$. We prove the following theorem in Section 2.5.

Theorem 2.1. *Assume that the supply graph G is a cycle. Then, for any demand digraph H, the pair (G,H) has the uniqueness property if and only if each arc of G is contained in at most two routes.*

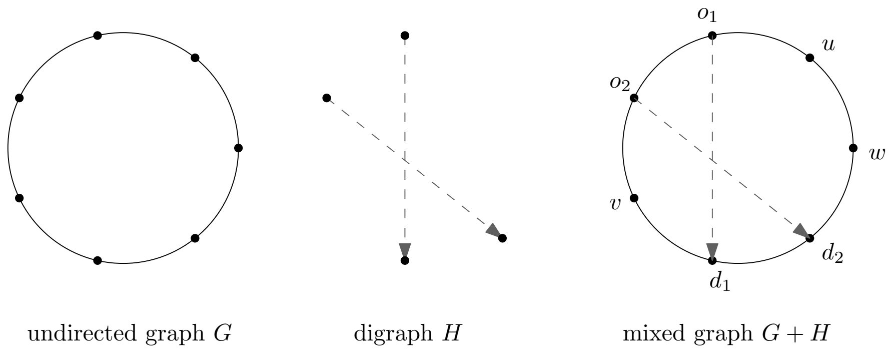

Figure 2.1: Example of a supply graph G, a demand digraph H, and the mixed graph $G + H$. According to Theorem 2.1, (G, H) has the uniqueness property

Whether such a pair (G, H) of supply graph and demand digraph is such that each arc in contained in at most two routes is obviously polynomially checkable, since we can test each arc one after the other. We will show that it can actually be tested by making only one round trip, in any direction, see Section 2.6.2. More generally, Section 2.6 contains a further discussion on the combinatorial structure of such a pair (G, H). Especially, we prove in Section 2.6.3 that such a pair (G, H) has the uniqueness property if and only if $G + H$ is homeomorphic to a minor of one of nine mixed graphs, see Figures 2.2–2.5. Except for the smallest one, none of the uniqueness properties of these graphs can be derived from the results by Milchtaich (2005), even by adding fictitious vertices as suggested p.235 of his article.

Furthermore, we find as a byproduct a sufficient condition for congestion games with nonatomic users to have the uniqueness property when each user has exactly two available strategies (Proposition 2.8).

Section 2.7.1 proves a necessary condition for general graphs to have the uniqueness property in terms of excluded minors (Corollary 2.15). With the help of Theorem 2.1, it allows to describe infinite families of graphs not having the uniqueness property. The remaining of Section 2.7 contains complementary results. For instance, Section 2.7.4 defines and studies a *strong uniqueness property* that may hold for general graphs independently of the demand digraph, i.e. of the OD-pairs.

2.5 Proof of the characterization in case of a ring

2.5.1 Proof strategy and some preliminary results

In this section, we prove Theorem 2.1. The proof works in two steps. The first step, Section 2.5.2, consists in proving Proposition 2.7 below stating that, when each arc is contained in at most two routes, then the uniqueness property holds. The second step, Section 2.5.3, consists in exhibiting cost functions for which flows at equilibrium are non-unique for any pair (G, H) with an arc in at least three routes.

From now on, we assume that the cycle G is embedded in the plane. It allows to use an orientation for G. Each route is now either positive or negative. The same holds for arcs of G: we have positive arcs and negative arcs.

Claim 2.2. *For any $(s, t) \in L$, if a^+ and a^- are the two arcs stemming from an edge $e \in E$, then exactly one of a^+ and a^- is in an (s, t)-route.*

Proof. Indeed, given an $(s, t) \in L$ and an edge $e \in E$, exactly one of the positive and negative (s, t)-routes contains e. □

For any subset $J \subseteq L$, we define A_J^+ (resp. A_J^-) as the set of positive (resp. negative) arcs that are exclusively used by OD-pairs in J. For each OD-pair $\ell \in L$, define r_ℓ^+ (resp. r_ℓ^-) to be the unique positive (resp. negative) route connecting the origin of ℓ to its destination. Then $a \in A_J^+$ if $a \in r_\ell^+$ for all $\ell \in J$ and $a \notin r_\ell^+$ for all $\ell \in L \setminus J$. We proceed similarly for A_J^-. We define moreover $A_J = A_J^+ \cup A_J^-$. In particular, $A_\emptyset$ is the set of arcs contained in no route. The sets A_J form a partition of the set A of arcs of G.

Defining the positive direction as the counterclockwise one on Figure 2.1, we have

$$A^+_{\{(o_1,d_1),(o_2,d_2)\}} = \{(o_2, v), (v, d_1)\}$$
$$A^-_{\{(o_2,d_2)\}} = \{(o_2, o_1)\}$$
$$A_\emptyset = \{(d_1, v), (v, o_2), (d_2, w), (w, u), (u, o_1)\}.$$

The sets A_J^ε enjoy three useful properties.

Claim 2.3. *For any $\varepsilon \in \{-, +\}$ and any $\ell \in L$, there is at least one $J \subseteq L$ containing ℓ such that A_J^ε is nonempty.*

Proof. Let $a \in r_\ell^\varepsilon$. Since the sets A_J form a partition of A, there is a J such that $a \in A_J^\varepsilon$. By definition of A_J^ε, we must have $\ell \in J$. □

Claim 2.4. *For any $J \subseteq L$, we have $A_J^+ \neq \emptyset$ if and only if $A^-_{L \setminus J} \neq \emptyset$.*

Proof. It is a consequence of Claim 2.2: if $a^+ \in A_J^+$, then $a^- \in A_{L\setminus J}^-$. □

Claim 2.5. *For any distinct ℓ and ℓ' in L, there is at least one J such that $|\{\ell,\ell'\} \cap J| = 1$ and $A_J \neq \emptyset$.*

Proof. Indeed, let $\ell = (s,t)$ and $\ell' = (s',t')$ be two distinct OD-pairs of H. Since H is simple, it contains no multiple edges and $s \neq s'$ or $t \neq t'$. It means that there is at least one arc a of G which is in exactly one of the four (s,t)- and (s',t')-routes. We have $a \in A_J$, for some $J \subseteq L$. By definition of A_J, we must have $|\{\ell,\ell'\} \cap J| = 1$. □

2.5.2 If each arc of G is contained in at most two routes, the uniqueness property holds

For each user i, we define r_i^+ (resp. r_i^-) to be the unique positive (resp. negative) route connecting the origin of i to its destination. For a strategy profile σ and a subset $J \subseteq L$, we define x_J^+ and x_J^- to be:

$$x_J^+ = \int_{i\in\bigcup_{\ell\in J} I_\ell} 1_{\{\sigma(i)=r_i^+\}}\,\mathrm{d}\lambda \quad\text{and}\quad x_J^- = \int_{i\in\bigcup_{\ell\in J} I_\ell} 1_{\{\sigma(i)=r_i^-\}}\,\mathrm{d}\lambda.$$

The quantity x_J^+ (resp. x_J^-) is thus the number of users i in a I_ℓ with $\ell \in J$ choosing a positive (resp. negative) route. Note that the quantity $x_J^+ + x_J^- = \sum_{\ell\in J}\lambda(I_\ell)$ does not depend on the strategy profile σ.

Assume that we have two distinct equilibria σ and $\hat\sigma$. The flows induced by $\hat\sigma$ are denoted with a hat: $\hat x$. We define for any subset $J \subseteq L$:

$$\Delta_J = x_J^+ - \hat x_J^+ = \hat x_J^- - x_J^-. \tag{2.1}$$

By a slight abuse of notation, we let $\Delta_\ell := \Delta_{\{\ell\}}$ for $\ell \in L$.

For each user i, we define $\delta(i) = 1_{\{\sigma(i)=r_i^+\}} - 1_{\{\hat\sigma(i)=r_i^+\}} = 1_{\{\hat\sigma(i)=r_i^-\}} - 1_{\{\sigma(i)=r_i^-\}}$. Then, the following lemma holds.

Lemma 2.6. *Let $\ell \in L$ and $i \in I_\ell$ be such that $\delta(i) \neq 0$. Then exactly one of the following alternatives holds.*

- *There is a $J \subseteq L$ with $\ell \in J$, $A_J \neq \emptyset$, and $\delta(i)\Delta_J < 0$.*
- *For all $J \subseteq L$ with $\ell \in J$ and $A_J \neq \emptyset$, we have $\Delta_J = 0$.*

We briefly explain the intuition behind this lemma. Assume that we move from σ to $\hat\sigma$. When a user i changes his chosen route, it is for one of the two following reasons.

The first situation is when the cost of the new route decreases or the cost of the old route increases. If the cost of a route decreases (resp. increases), there is at least one arc of this route whose flow decreases (resp. increases). Since an arc belongs to some set A_J^ε, we get the first point of Lemma 2.6.

The second situation is when the costs remain the same for both routes and both routes have same costs, which implies the second point of Lemma 2.6.

Proof of Lemma 2.6. As σ is an equilibrium, we have for each user i:

$$\sum_{a\in A} c_a^i(x_a)\left(\mathbb{1}_{\{a\in\sigma(i)\}} - \mathbb{1}_{\{a\in\hat{\sigma}(i)\}}\right) \leq 0. \tag{2.2}$$

For $\varepsilon \in \{-,+\}$ and $a \in A_J^\varepsilon$, we have $x_a = x_J^\varepsilon$ and $\mathbb{1}_{\{a\in\sigma(i)\}} = \mathbb{1}_{\{\sigma(i)=r_i^\varepsilon\}}\mathbb{1}_{\{a\in r_i^\varepsilon\}}$. By decomposing the sum (2.2), we obtain that

$$\sum_{J\subseteq L}\left(\sum_{a\in A_J^+\cap r_i^+} c_a^i(x_J^+)\delta(i) - \sum_{a\in A_J^-\cap r_i^-} c_a^i(x_J^-)\delta(i)\right) \leq 0.$$

We can write a similar equation for the equilibrium $\hat{\sigma}$. By summing them, we obtain

$$\delta(i)\sum_{J\subseteq L}\left(\sum_{a\in A_J^+\cap r_i^+}\left(c_a^i(x_J^+) - c_a^i(\hat{x}_J^+)\right) - \sum_{a\in A_J^-\cap r_i^-}\left(c_a^i(x_J^-) - c_a^i(\hat{x}_J^-)\right)\right) \leq 0. \tag{2.3}$$

According to Equation (2.1) and using the fact that the maps c_a^i are strictly increasing, both $\sum_{a\in A_J^+\cap r_i^+}(c_a^i(x_J^+)-c_a^i(\hat{x}_J^+))$ and $-\sum_{a\in A_J^-\cap r_i^-}(c_a^i(x_J^-)-c_a^i(\hat{x}_J^-))$ have the sign of Δ_J. Therefore, if all terms of the sum in Equation (2.3) are equal to 0, the second point of the lemma holds. If at least one term of the sum is < 0, we get the first point. □

With the help of this lemma, we get one direction of Theorem 2.1.

Proposition 2.7. *If each arc of G is contained in at most two routes, the uniqueness property holds.*

Proof. Note that the assumption of the proposition ensures that $A_J = \emptyset$ if $|J| \geq 3$. We want to prove that $\Delta_J = 0$ for all $J \subseteq L$ such that $A_J \neq \emptyset$.

Assume for a contradiction that there is a J_0 such that $\Delta_{J_0} \neq 0$ and $A_{J_0} \neq \emptyset$. Then there is a $\ell_0 \in J_0$ such that $\Delta_{\ell_0} \neq 0$. Since $\Delta_{\ell_0} = \int_{I_{\ell_0}} \delta(i)\,\mathrm{d}\lambda$, at least one user $i_0 \in I_{\ell_0}$ is such that $\delta(i_0)\Delta_{\ell_0} > 0$.

Suppose that the first case of Lemma 2.6 occurs. There exists $\ell_1 \in L$, $\ell_1 \neq \ell_0$ with $A_{\{\ell_0,\ell_1\}} \neq \emptyset$ and $\delta(i_0)\Delta_{\{\ell_0,\ell_1\}} < 0$. Then, $\delta(i_0)\Delta_{\{\ell_0,\ell_1\}} = \delta(i_0)(\Delta_{\ell_0} + \Delta_{\ell_1}) < 0$, which implies that $|\Delta_{\ell_0}| < |\Delta_{\ell_1}|$. It follows that $\Delta_{\ell_1} \neq 0$, and taking $i_1 \in I_{\ell_1}$ with $\delta(i_1)\Delta_{\ell_1} > 0$, only the first case of Lemma 2.6 can occur for $i = i_1$ and $\ell = \ell_1$. Indeed, the second case would imply that $\Delta_{\{\ell_0,\ell_1\}} = 0$ since $A_{\{\ell_0,\ell_1\}} \neq \emptyset$. Repeating the same argument, we build an infinite sequence $(\ell_0, \ell_1, \ldots)$ of elements of L such that, for each $k \geq 0$, $A_{\{\ell_k,\ell_{k+1}\}} \neq \emptyset$ and $|\Delta_{\ell_k}| < |\Delta_{\ell_{k+1}}|$. This last condition implies that the ℓ_k are distinct, which is impossible since $|L|$ is finite.

Thus, the second case of Lemma 2.6 occurs for ℓ_0, and hence $\Delta_{J_0} = 0$, which is in contradiction with the starting assumption. On any arc, we have a total flow that remains the same when changing from σ to $\hat{\sigma}$. □

The only fact we use from the ring structure is that there are two sets A^+ (positive arcs) and A^- (negative arcs) and that each user has exactly two possible strategies, each of them being included in one of these two sets. We can state a result holding for more general nonatomic congestion game with user-specific cost functions. We omit the proof since the one of Proposition 2.7 holds without any change.

Proposition 2.8. *Consider a nonatomic congestion game with user-specific (strictly increasing) cost functions. Let A^+ and A^- be two disjoint finite sets. Assume that every user i has exactly two available strategies r_i^+ and r_i^- with $r_i^+ \subseteq A^+$ and $r_i^- \subseteq A^-$. Then, if all triples of pairwise distinct strategies have an empty intersection, the uniqueness property holds.*

2.5.3 If an arc of G is contained in at least three routes, a counterexample exists

We give an explicit construction of multiple equilibrium flows when an arc is contained in at least three routes.

If $|L| = 3$

In order to ease the notation, we use 1, 2, and 3 to denote the three OD-pairs of H. We denote accordingly by I_1, I_2, and I_3 the three sets of users associated to each of these OD-pairs.

We can assume without loss of generality that $A^+_{\{1,2,3\}} \neq \emptyset$, $A_{\{1,2\}} \neq \emptyset$, and $A_{\{1,3\}} \neq \emptyset$. The first assumption can be done since there is an arc in three routes. For the other ones: with the help of Claim 2.5, and if necessary of Claim 2.4, we get that there is at least a J of cardinality two such that

$A_J \neq \emptyset$. Again, using Claim 2.5, this time with the two elements of J, and if necessary Claim 2.4, we get another J' of cardinality two such that $A_{J'} \neq \emptyset$.

Definition of the cost functions. We define three classes of users. Each of these classes is attached to one of the OD-pairs. For a class $k \in \{1,2,3\}$, we define the cost functions $c_J^{k,\varepsilon}$, for all $J \subseteq \{1,2,3\}$ and $\varepsilon \in \{-,+\}$. The cost function for a class k user i on an arc a of A_J^{ε} is set to $c_a^i := c_J^{k,\varepsilon}$. If the set A_J^{ε} is empty, the definition of $c_J^{k,\varepsilon}$ is simply discarded.

Class 1: We define this class to be the users of the set I_1. We set $\lambda(I_1) = 1.5$ and choose $J_1 \subseteq \{1,2,3\}$ with $1 \in J_1$ such that $A_{J_1}^- \neq \emptyset$ (with the help of Claim 2.3).

$$\begin{cases} c_{\{1,2,3\}}^{1,+}(x) = \dfrac{24x+7}{|A_{\{1,2,3\}}^+|} \\ c_J^{1,+}(x) = \dfrac{x}{|A_J^+|} \quad \text{for any } J \neq \{1,2,3\} \text{ with } 1 \in J \\ c_{J_1}^{1,-}(x) = \dfrac{x+48}{|A_{J_1}^-|} \\ c_J^{1,-}(x) = \dfrac{x}{|A_J^-|} \quad \text{for any } J \neq J_1 \text{ with } 1 \in J. \end{cases}$$

Class 2: We define this class to be the users of the set I_2. We set $\lambda(I_2) = 1$. We have assumed that $A_{\{1,2\}} \neq \emptyset$. We distinguish hereafter the cases $A_{\{1,2\}}^+ \neq \emptyset$ and $A_{\{1,2\}}^- \neq \emptyset$ (which may hold simultaneously, in which case we make an arbitrary choice).

If $A_{\{1,2\}}^+ \neq \emptyset$: We choose $J_2 \subseteq \{1,2,3\}$ with $2 \in J_2$ such that $A_{J_2}^- \neq \emptyset$ (with the help of Claim 2.3).

$$\begin{cases} c_{\{1,2\}}^{2,+}(x) = \dfrac{25x}{|A_{\{1,2\}}^+|} \\ c_J^{2,+}(x) = \dfrac{x}{|A_J^+|} \quad \text{for any } J \neq \{1,2\} \text{ with } 2 \in J \\ c_{J_2}^{2,-}(x) = \dfrac{x+31}{|A_{J_2}^-|} \\ c_J^{2,-}(x) = \dfrac{x}{|A_J^-|} \quad \text{for any } J \neq J_2 \text{ with } 2 \in J. \end{cases}$$

If $A^-_{\{1,2\}} \neq \emptyset$:

$$\begin{cases} c^{2,+}_{\{1,2,3\}}(x) = \dfrac{x+26}{|A^+_{\{1,2,3\}}|} \\ c^{2,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,2,3\} \text{ with } 2 \in J \\ c^{2,-}_{\{1,2\}}(x) = \dfrac{22x}{|A^-_{\{1,2\}}|} \\ c^{2,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq \{1,2\} \text{ with } 2 \in J. \end{cases}$$

Class 3: We define this class to be the users of the set I_3. We set $\lambda(I_3) = 1$. We have assumed that $A_{\{1,3\}} \neq \emptyset$. We distinguish hereafter the cases $A^+_{\{1,3\}} \neq \emptyset$ and $A^-_{\{1,3\}} \neq \emptyset$ (which may hold simultaneously, in which case we make an arbitrary choice).

If $A^+_{\{1,3\}} \neq \emptyset$: We choose $J_3 \subseteq \{1,2,3\}$ with $3 \in J_3$ such that $A^-_{J_3} \neq \emptyset$ (with the help of Claim 2.3).

$$\begin{cases} c^{3,+}_{\{1,3\}}(x) = \dfrac{25x}{|A^+_{\{1,3\}}|} \\ c^{3,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,3\} \text{ with } 3 \in J \\ c^{3,-}_{J_3}(x) = \dfrac{x+31}{|A^-_{J_3}|} \\ c^{3,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq J_3 \text{ with } 3 \in J. \end{cases}$$

If $A^-_{\{1,3\}} \neq \emptyset$:

$$\begin{cases} c^{3,+}_{\{1,2,3\}}(x) = \dfrac{x+26}{|A^+_{\{1,2,3\}}|} \\ c^{3,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,2,3\} \text{ with } 3 \in J \\ c^{3,-}_{\{1,3\}}(x) = \dfrac{22x}{|A^-_{\{1,3\}}|} \\ c^{3,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq \{1,3\} \text{ with } 3 \in J. \end{cases}$$

Definition of two strategy profiles. We define now two strategy profiles σ and $\hat{\sigma}$, inducing distinct flows on some arcs. We check in the next paragraph that each of them is an equilibrium.

Strategy profile σ: For all $i \in I_1$, we set $\sigma(i) = r_i^+$ and for all $i \in I_2 \cup I_3$, we set $\sigma(i) = r_i^-$. Then, the flows are the following:

J	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
x_J^+	1.5	0	0	1.5	1.5	0	1.5
x_J^-	0	1	1	1	1	2	2

Strategy profile $\hat{\sigma}$: For all $i \in I_1$, we set $\hat{\sigma}(i) = r_i^-$ and for all $i \in I_2 \cup I_3$, we set $\hat{\sigma}(i) = r_i^+$. Then, the flows are the following:

J	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\hat{x}_J^+$	0	1	1	1	1	2	2
$\hat{x}_J^-$	1.5	0	0	1.5	1.5	0	1.5

The strategy profiles are equilibria. We check now that σ and $\hat{\sigma}$ are equilibria, by computing the cost of each of the two possible routes for each class.

For a class $k \in \{1,2,3\}$, we denote with a slight abuse of notation the common positive (resp. negative) route of the class k users by r_k^+ (resp. r_k^-).

Class 1: We put in the following tables, the costs experienced by the class 1 users on the various arcs of G for each of σ and $\hat{\sigma}$. For a given $J \subseteq \{1,2,3\}$ with $1 \in J$ and $\varepsilon \in \{-,+\}$, we indicate the cost experienced by any class 1 user on the whole collection of arcs in A_J^ε. For instance in σ, if $J = \{1,2,3\}$, then $x_J^+ = 1.5$, and the cost of all arcs together in A_J^+ is $|A_J^+|c_J^{1,+}(1.5) = 43$.

For the strategy profile σ, we get the following flows and costs on the arcs of G for a class 1 user.

	$\varepsilon = +$		$\varepsilon = -$	
J with $1 \in J$	{1,2,3}	other	J_1	other
x_J^ε	1.5	1.5	0, 1, or 2	0, 1, or 2
Cost on A_J^ε	43	1.5	48, 49, or 50	0, 1, or 2

Using the fact that $A^+_{\{1,2,3\}} \neq \emptyset$, the total cost of r_1^+ in σ for a class 1 user is equal to

$$43 + 1.5 \times \left|\{J \neq \{1,2,3\} \text{ such that } A^+_J \neq \emptyset \text{ and } 1 \in J\}\right| .$$

Since there are at most three sets $J \neq \{1,2,3\}$ such that $A^+_J \neq \emptyset$ and $1 \in J$, we get that the total cost of r_1^+ lies in $[43; 47.5]$. Similarly, using the fact that $A^-_{J_1} \neq \emptyset$, we get that the total cost of r_1^- for a class 1 user lies in $[48; 54]$. Therefore the users of class 1 are not incited to change their choice in σ.

For the strategy profile $\hat{\sigma}$, we get the following flows and costs.

	$\varepsilon = +$		$\varepsilon = -$	
J with $1 \in J$	{1,2,3}	other	J_1	other
$\hat{x}^\varepsilon_J$	2	0 or 1	1.5	1.5
Cost on A^ε_J	55	0 or 1	49.5	1.5

The total cost of r_1^+ for a class 1 user lies in $[55; 58]$ and the total cost of r_1^- for a class 1 user lies in $[49.5; 54]$. Therefore the users of class 1 are not incited to change their choice in $\hat{\sigma}$.

Class 2: If $A^+_{\{1,2\}} \neq \emptyset$: We put in the following tables, the costs experienced by the class 2 users on the various arcs of G for each of σ and $\hat{\sigma}$.

For the strategy profile σ:

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2}	other	J_2	other
x^ε_J	1.5	0 or 1.5	1 or 2	1 or 2
Cost on A^ε_J	37.5	1.5	32 or 33	1 or 2

The total cost of r_2^+ for a class 2 user is precisely 39 (we use the fact that $A^+_{\{1,2,3\}} \neq \emptyset$) and the total cost of r_2^- lies in $[32; 38]$. The users of class 2 are not incited to change their choice in σ.

For the strategy profile $\hat{\sigma}$:

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2}	other	J_2	other
$\hat{x}_J^\varepsilon$	1	1 or 2	0 or 1.5	0 or 1.5
Cost on A_J^ε	25	1 or 2	31 or 32.5	0 or 1.5

The total cost of r_2^+ for a class 2 user lies in $[27; 30]$ and the total cost of r_2^- lies in $[31; 34]$. The users of class 2 are not incited to change their choice in $\hat{\sigma}$.

If $A^-_{\{1,2\}} \neq \emptyset$: We put in the following tables, the costs experienced by the class 2 users on the various arcs of G for each of σ and $\hat{\sigma}$.

For the strategy profile σ:

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2,3}	other	$\{1,2\}$	other
x_J^ε	1.5	0 or 1.5	1	1 or 2
Cost on A_J^ε	27.5	0 or 1.5	22	1 or 2

The total cost of r_2^+ for a class 2 user lies in $[27.5; 29]$ and the total cost of r_2^- lies in $[22; 27]$. The users of class 2 are not incited to change their choice in σ.

For the strategy profile $\hat{\sigma}$:

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2,3}	other	$\{1,2\}$	other
$\hat{x}_J^\varepsilon$	2	1 or 2	1.5	0 or 1.5
Cost on A_J^ε	28	1 or 2	33	0 or 1.5

The total cost of r_2^+ for a class 2 user lies in $[28; 32]$ and the total cost of r_2^- lies in $[33; 34.5]$. The users of class 2 are not incited to change their choice in $\hat{\sigma}$.

Class 3: The symmetry of the cost functions for classes 2 and 3 gives the same tables for class 3 as for class 2, by substituting $\{1,3\}$ to $\{1,2\}$. Therefore, we get the same conclusions: neither in σ, nor in $\hat{\sigma}$, the class 3 users are incited to change their choice.

Therefore, σ and $\hat{\sigma}$ are equilibria and induce distinct flows. It proves that the uniqueness property does not hold. It remains to check the case when $|L| > 3$.

Remark 2.9. A classical question when there are several equilibria is whether one of them dominates the others. An equilibrium is said to *dominate* another one if it is preferable for all users. In this construction, no equilibrium dominates the other, except when $A^+_{\{1,2,3\}} \neq \emptyset$, $A^-_{\{1,2\}} \neq \emptyset$, and $A^-_{\{1,3\}} \neq \emptyset$ where σ dominates $\hat{\sigma}$.

If $|L| > 3$

Denote 1, 2, and 3 three OD-pairs of $H = (T, L)$ giving three routes containing the same arc of G. For these three arcs of H, we make the same construction as above, in the case $|L| = 3$. For the other $\ell \in L$, we set $I_\ell = \emptyset$ to get the desired conclusion.

However, note that we can also get multiple equilibrium flows, while requiring $I_\ell \neq \emptyset$ for all $\ell \in L$. For $\ell \notin \{1, 2, 3\}$, we use a fourth class, whose costs are very small on all positive arcs of G and very large on all negative arcs of G, and whose measure is a small positive quantity δ. Each user of this class chooses always a positive route, whatever the other users do. For δ small enough, the users of this class have no impact on the choices of the users of the classes 1, 2, and 3, as the difference of cost between the routes is always bounded below by 0.5.

2.6 When the supply graph is a ring having each arc in at most two routes

In this section, we provide a further combinatorial analysis of the characterization of the uniqueness property for ring graphs stated in Theorem 2.1.

2.6.1 A corollary

We first state a corollary of Theorem 2.1.

Corollary 2.10. *Suppose that the supply graph is a cycle.*

- *If there are at most two OD-pairs, i.e.* $|L| \leq 2$, *then the uniqueness property holds.*

- *If the uniqueness property holds, then the number of OD-pairs is at most* 4, *i.e.* $|L| \leq 4$.

The first point of Corollary 2.10 is straightforward. The second point is a direct consequence of Claim 2.2: if $|L| \geq 5$, then there is necessarily an arc of G in three routes.

2.6.2 How to compute in one round trip the maximal number of routes containing an arc of G

In an arc (u, v), vertex u is called the *tail* and vertex v is called the *head*. The algorithm starts at an arbitrary vertex of H and makes a round trip in an arbitrary direction, while maintaining a triple `(list, min, max)`. In this triple, `list` is a set of arcs of H whose tail has already been encountered but whose head has not yet been encountered. At the beginning, `list` is empty and `min` and `max` are both zero. When the algorithm encounters a vertex v, it proceeds to three operations.

First operation. It computes the number of arcs in $\delta^-_H(v)$ (arcs of H having v as head) not in `list`. The corresponding routes were "forgotten", since the tail is "before" the starting vertex of the algorithm. To take them into account, this number is added to the values `min` and `max`.

Second operation. All arcs of `list` being also in $\delta^-_H(v)$ are removed from `list`.

Third operation. All arcs in $\delta^+_H(v)$ (arcs of H having v as tail) are added to `list`. The value of `min` is updated to the minimum between the previous value of `min` and the size of `list`, and similarly `max` is updated to the maximum between the previous value of `max` and the size of `list`.

The algorithm stops after one round trip. At the end, the values `min` and `max` are respectively the minimal and maximal number of routes containing an arc in the direction chosen. According to Claim 2.2, $\max(|L| - \texttt{min}, \texttt{max})$ is the maximal number of routes containing an arc of G.

Note that with a second round trip, this algorithm can specify the routes containing a given arc, by scanning the content of `list`.

2.6.3 Explicit description of the graphs having the uniqueness property when the supply graph is a cycle

Proposition 2.11. *Let the supply graph G be a cycle. Then, for any demand digraph H, the pair (G, H) is such that each arc of G is in at most two routes if and only if the mixed graph $G + H$ is homeomorphic to a minor of one of the nine mixed graphs of Figures 2.2–2.5.*

Combined with Theorem 2.1, this proposition allows to describe explicitely all pairs (G, H) having the uniqueness property, when G is a cycle.

Proof. One direction is straightforward. Let us prove the other direction, namely that, if each arc of G is in at most two routes, then $G + H$ is homeomorphic to a minor of one of the nine mixed graphs. We can assume that $V = T$. Moreover, if each arc is in at most two routes, Claim 2.2 implies that $|L| \in \{1, 2, 3, 4\}$, as already noted in the proof of Corollary 2.10.

If $|L| \in \{1, 2\}$, there is nothing to prove: all possible mixed graphs with $|L| = 1$ or $|L| = 2$ are homeomorphic to a minor of the graphs of Figures 2.2 and 2.3.

If $|L| = 3$, we can first assume that L contains two disjoint arcs that are crossing in the plane embedding. By trying all possibilities for the third arc, we get that the only possible configuration is the right one on Figure 2.4 and the ones obtained from it by edge contraction. Second, we assume that there are no "crossing" arcs. The three heads of the arcs cannot be consecutive on the cycle otherwise we would have an arc of G in three routes. Again by enumerating all possibilities, we get that the only possible configuration is the left one on Figure 2.4 and the ones obtained from it by edge contraction.

If $|L| = 4$, Claim 2.4 shows that each arc of G belongs to exactly two routes. It implies that, in H, the indegree of any $\ell \in L$ is equal to its outdegree. There are therefore circuits in H. It is straightforward to check that it is impossible to have a length 3 circuit. It remains to enumerate the possible cases for length 2 and length 4 circuits to get that the only possible configurations are the ones of Figure 2.5 and the common one obtained from them by edge contraction. □

We can also describe the rings having the uniqueness property by minor exclusion, similarly as in Milchtaich (2005).

Proposition 2.12. *Let the supply graph G be a cycle. Then, for any demand digraph H, the pair (G, H) is such that there exists an arc of G belonging to at least three routes if and only if $G + H$ has one of the nine mixed graphs of Figure 2.10 as a minor.*

Proof (sketched). Suppose that one of the nine mixed graphs of Figure 2.10 is a minor of the mixed graph $G+H$. The construction of Section 2.5.3 shows that we can build two distinct equilibria for this minor where all users of a

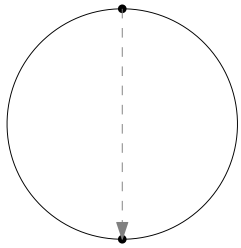

Figure 2.2: All rings with $|L| = 1$ (i.e. one OD-pair) having the uniqueness property are homeomorphic to this graph

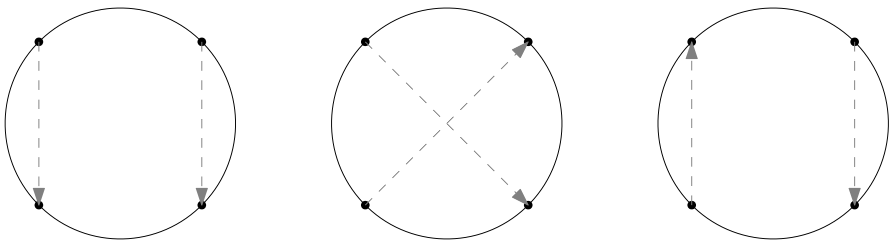

Figure 2.3: All rings with $|L| = 2$ (i.e. two OD-pairs) having the uniqueness property are homeomorphic to one or to minors of these graphs

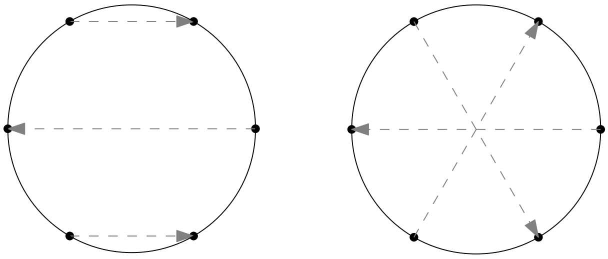

Figure 2.4: All rings with $|L| = 3$ (i.e. three OD-pairs) having the uniqueness property are homeomorphic to one or to minors of these graphs

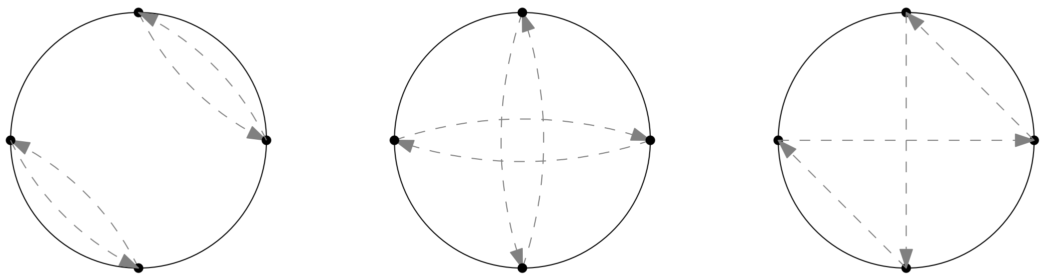

Figure 2.5: All rings with $|L| = 4$ (i.e. four OD-pairs) having the uniqueness property are homeomorphic to one or to minors of these graphs

given class have the same strategy. Then, we can extend this counterexample to the graph $G + H$, see Corollary 2.15 in Section 2.7.1 holding for more general graphs.

To prove that if an arc of G belongs to at least three routes, then one of the nine mixed graphs of Figure 2.10 is a minor of $G + H$, we proceed to an explicit, but tedious, enumeration. We enumerate all possible mixed graphs with $|V| = 6$ and $|L| = 3$ such that each vertex is the tail or the head of exactly one arc in L. Then, we try all possible sequences of edge contractions leading to mixed graphs satisfying two properties: the demand graph is simple and an arc is in three routes. We keep the mixed graphs such that any additional edge contraction leads to a violation of these properties. The details are omitted. □

In particular, the construction in Section 2.5.3 cannot be simplified by exhibiting a counterexample for each mixed graph of Figure 2.10, since the proof of Proposition 2.12 needs this tedious enumeration.

2.7 Discussion

2.7.1 Results for general graphs

For the sake of simplicity, given a supply graph G and a demand digraph H, we say that the mixed graph $G + H$ has the uniqueness property if the pair (G, H) has it.

Using the results of Milchtaich (2005) and Theorem 2.1, we can derive results for more general graphs. Milchtaich suggests to add a fictitious origin, linked to all origins, and similarly for the destinations. If the new graph has the uniqueness property, the original one has it as well. However, this approach cannot be used to prove that a graph does not have the uniqueness property. For instance this method allows us to prove that the graph on the left in Figure 2.6 has the uniqueness property, but fails to settle the status of the graph on the right. Indeed, the new graph does not have the uniqueness property, using the result of Milchtaich (2005), but the original one has it, using Theorem 2.1.

A way for proving that a pair (G, H) does not have the uniqueness property consists in using subgraphs or minors as obstructions to uniqueness property. If $G + H$ has a subgraph without the uniqueness property, then it does not have the property either since we can set prohibitive high costs on the arcs outside the subgraph. However, it is not clear whether having a minor without uniqueness property is an obstruction for having the uniqueness property. Indeed, the cost functions are strictly increasing and we do not see how in general a counterexample to uniqueness at the level of a minor can be extended at the level of the network itself. Yet, we can settle two specific cases.

The first case is when the contractions involve only bridges of G (a *bridge* is an edge whose deletion disconnects the graph). In this case, if the minor does not have the uniqueness property, the pair (G, H) does not have it either. Checking this property is easy.

A second case is formalized in the following proposition. An equilibrium is *strict* if each user has a unique best reply.

Proposition 2.13. *Let G' and H' be respectively a supply and a demand graphs such that $G' + H'$ is a minor of $G + H$. If there are counterexamples of uniqueness property for (G', H') involving strict equilibria, then (G, H) does not have the uniqueness property.*

Proof. We start with a counterexample for (G', H'). We de-contract an edge. We assign to this edge a small enough cost function so that the route followed by any user remains a strict best reply for him, whether the route contains the edge or not. Therefore, we can de-contract all edges and get conterexamples to uniqueness property for subgraphs of $G + H$. As noted above, it allows to conclude that (G, H) does not have the uniqueness property. □

Remark 2.14. Note that Proposition 2.13 remains valid if we restrict the set of cost functions, as soon as this set is a cone. Indeed, we need to choose both functions with sufficiently small costs and functions with prohibitive high costs.

Since the construction of Section 2.5.3 provides strict equilibria, we get the following corollary.

Corollary 2.15. *Any mixed graph containing one of the graphs of Figure 2.10 as a minor does not have the uniqueness property.*

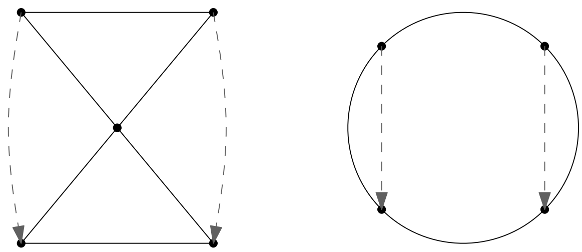

Figure 2.6: Adding a fictitious origin and a fictitious destination settles the case of the left graph but not the case of the right graph

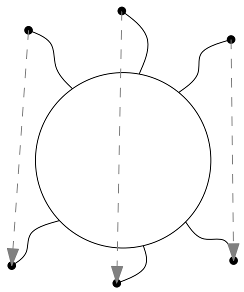

Figure 2.7: Any graph having this one as a subgraph does not have the uniqueness property

Let us now give an example using some of these conditions. According to Corollary 2.15 or to the "bridge-contraction" condition above, the mixed graph of the Figure 2.7 does not have the uniqueness property. Indeed, a ring with an arc in three routes is a minor of it. We can conclude that any network having the mixed graph of the Figure 2.7 as a subgraph (or as a minor) does not have the uniqueness property.

However, there are still graphs for which none of the considerations above allows to conclude, see for example the graph of Figure 2.8.

2.7.2 Possible extensions

We assume in this work that each edge in the graph can be traversed in both directions. When the supply graph is a cycle, this condition is not restrictive, since allowing one-way edges is equivalent to consider a game with fewer OD-pairs and modified cost functions. Indeed, if an arc a is not present in the graph, every user whose OD-pair has a route going through a has no choice and has to take the route in the other direction. Hence we can remove all such users and the associated OD-pairs, and modify the cost functions by

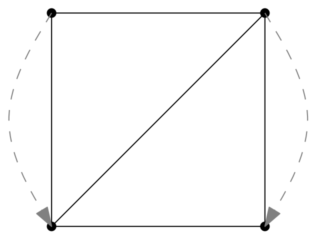

Figure 2.8: A graph for which neither Theorem 2.1 nor Milchtaich (2005) can be used to prove or disprove the uniqueness property

adding the (fixed) congestion due to them.

Another way to extend the model is to add restrictions on the cost functions. As already noted in the Introduction, Altman and Kameda (2001) have shown that if the cost functions are equal for each user up to an additive constant, then the uniqueness property holds for every graph. We can try to extend this set of cost functions. The construction of Section 2.5.3 is made with affine cost functions, and can be extended to strictly convex cost functions, by adding εx^2 at each cost functions, for ε small enough. Using Proposition 2.13 or Remark 2.14, we can build two different equilibria with affine or strictly convex cost functions for any mixed graph having one of the graphs of Figure 2.10 as a minor. Then, to ensure the uniqueness property for a larger class of graphs, we have to exclude both affine and strictly convex functions from the set of possible cost functions.

2.7.3 Equivalence of equilibria

Let us assume that we have a finite set K of classes. We denote by I_ℓ^k the set of class k users in I_ℓ and we assume that all I_ℓ^k are measurable.

Let σ and $\hat{\sigma}$ be two Nash equilibria. We define for an $\ell \in L$, a class k, and an arc a the quantity

$$x_{\ell,a}^k = \lambda\{i \in I_\ell^k : a \in \sigma(i)\},$$

and

$$\hat{x}_{\ell,a}^k = \lambda\{i \in I_\ell^k : a \in \hat{\sigma}(i)\}.$$

Following Milchtaich (2005), we say that the two equilibria are *equivalent* if not only the flow on each arc is the same but the contribution of each pair and each class to the flow on each arc is the same, i.e. $x_{\ell,a}^k = \hat{x}_{\ell,a}^k$ for any arc a, OD-pair ℓ, and class k. Milchtaich proved that a two-terminal network has the uniqueness property if and only if every two Nash equilibria are equivalent for generically all cost functions (Theorem 5.1 in Milchtaich (2005)). A property is considered *generic* if it holds on an open dense set. "Open" and "dense" are understood according to the following metric on the cost functions.

Define the set $\mathcal{G}$ of assignments of continuous and strictly increasing cost functions $(c_a^i)_{a\in A, i\in I}$, with $c_a^i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c_a^i = c_a^{i'}$ whenever i and i' belong to the same class.

Given a particular element of $\mathcal{G}$, the function $i \mapsto c_a^i(x)$ is measurable for all $a \in A$ and $x \in \mathbb{R}_+$. Every element of $\mathcal{G}$ has therefore a nonempty set of Nash equilibria. Note that the set $\mathcal{G}$ depends on the partition of the population in classes. We can define the distance between two elements $(c_a^i)_{a\in A, i\in I}$ and $(\tilde{c}_a^i)_{a\in A, i\in I}$ of $\mathcal{G}$ by $\max|c_a^i(x) - \tilde{c}_a^i(x)|$, where the maximum is taken over all $a \in A$, $i \in I$ and $x \in \mathbb{R}_+$. This defines a metric for $\mathcal{G}$.

Theorem 2.16. *Assume that the supply graph G is a cycle. Then, for any demand digraph H, the following assertions are equivalent:*

(i) *(G, H) has the uniqueness property.*

(ii) *For every partition of the population into classes, there is an open dense set in $\mathcal{G}$ such that for any assignment of cost functions that belongs to this set, every two equilibria are equivalent.*

Proof (sketched). Up to slight adaptations, the proof is the same as the one of Theorem 5.1 in Milchtaich (2005).

If (i) does not hold, we can use the construction of Section 2.5.3 to build two distinct equilibria for an assignment in $\mathcal{G}$. These equilibria are such that the gap between the costs of the two routes available to any user is uniformly bounded from below by a strictly positive number. The equilibria are said to be *strict*. Thus, in a ball centered on this assignment with radius $\rho > 0$ small enough, we still have two equilibria with distinct flows, which cannot be equivalent. Therefore (ii) does not hold either.

If (i) holds, three claims (Claims 1, 2, and 4 of Milchtaich (2005)) lead to the desired conclusion, namely that (ii) holds. These three claims are now sketched. Their original proof does not need to be adapted, except for the second one, which is the only moment where the topology of the network is used. In our case the second claim gets a simpler proof.

For an assignment in $\mathcal{G}$, we denote ϕ_ℓ^k the number of minimal-cost routes for users in I_ℓ^k, which is in our case 1 or 2. Since the uniqueness property is assumed to hold, this number is fully determined by the assignment in $\mathcal{G}$. Define the mean number of minimal-cost routes by

$$\phi = \sum_{k\in K,\ell\in L} \lambda(I_\ell^k)\phi_\ell^k.$$

The first claim states that the map by $\phi : \mathcal{G} \to \mathbb{R}$ is upper semicontinuous and has finite range.

The second claim states that for every assignment of cost functions in $\mathcal{G}$ that is a point of continuity of ϕ, all Nash equilibria are equivalent. To prove this second claim, we consider two Nash equilibria assumed to be nonequivalent σ and $\hat{\sigma}$. Using these two equilibria, a new one is built, $\bar{\sigma}$, such that for some $\ell \in L$, some class k and some ℓ-route r_1 we have $x_{\ell,r_1}^k > 0$ and $\hat{x}_{\ell,r_1}^k > 0$, but $\bar{x}_{\ell,r_1}^k = 0$. As the two ℓ-routes do not share any arc (Claim 2.2 of Section 2.5.1), we have $x_{\ell,a}^k > 0$, $\hat{x}_{\ell,a}^k > 0$, and $\bar{x}_{\ell,a}^k = 0$ for any a in r_1.

The second claim is achieved by choosing any a_1 in r_1 and by adding a small value $\delta > 0$ to the cost function $c_{a_1}^i$ for $i \in I_\ell^k$, while keeping the others unchanged. It can be checked that for δ small enough, the set of minimal-cost routes is the same as for $\delta = 0$, minus the route r_1 for users in I_ℓ^k. The map ϕ has therefore a discontinuity of at least $\lambda(I_\ell^k)$ at the original assignment of cost functions.

Finally, the third claim allows to conclude: in every metric space, the set of all points of continuity of a real-valued upper semicontinuous function with finite range is open and dense. □

2.7.4 The strong uniqueness property

A supply graph is said to have the *strong uniqueness property* if for any choice of the OD-pairs, the uniqueness property holds. In other words, $G = (V, E)$ has the strong uniqueness property if, for any digraph $H = (T, L)$ with $T \subseteq V$, the pair (G, H) has the uniqueness property.

Theorem 2.17. *A graph has the strong uniqueness property if and only if no cycle is of length* 3 *or more.*

Alternatively, this theorem states that a graph has the strong uniqueness property if and only if it is obtained by taking a forest (a graph without cycles) and by replacing some edges by parallel edges.

Before proving this theorem, let us state a preliminary result allowing to extend the strong uniqueness property whenever one "glues" together two supply graphs on a vertex. This latter operation is called a 1-*sum* in the usual terminology of graphs.

Lemma 2.18. *The* 1*-sum operation preserves the strong uniqueness property.*

Proof. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, and let $H = (T, L)$ and $H' = (T', L')$ two directed graphs with $T \subseteq V$ and $T' \subseteq V'$, such that (G, H) and (G', H') have the uniqueness property. Assume that (G, H) and (G', H') have a unique common vertex v, i.e. $V \cap V' = T \cap T' = \{v\}$, and define (G'', H'') as the 1-sum of them: $G'' = (V \cup V', E \cup E')$ and $H'' = (T \cup T', L \cup L' \cup L'')$ with $L'' := \{(u, w) : (u, v) \in L \text{ and } (v, w) \in L'\}$.

Assume that we have an equilibrium on (G'', H'') for some cost functions and some partition $(I_{(s,t)})_{(s,t)\in L\cup L'\cup L''}$ of the population. The restriction of this equilibrium on (G, H) is an equilibrium for (G, H) with the same cost functions and with a partition of the population obtained as follows.

When s and t are both in H, we keep the same $I_{(s,t)}$. Moreover, we complete this collection of subsets. For each vertex s of H, we define $\tilde{I}_{(s,v)}$ to be the union of all $I_{(s,w)}$ with w a vertex of H'. For each vertex t of H, we define $\tilde{I}_{(v,t)}$ to be the union of all $I_{(w,t)}$ with w a vertex of H'. We get the partition of the population I we are looking for. The restriction of the equilibrium on (G, H) is an equilibrium since for each user, the restriction of a minimum cost route of (G'', H'') is a minimum cost route of (G, H).

The same property holds for (G', H'). Therefore, if we had two equilibria inducing two distinct flows on some arc a of the directed version of G'', we would get equilibria inducing two distinct flows on the arc a, which is in the directed version of G or G'. It is in contradiction with the assumption on G and G'. □

Proof of Theorem 2.17. Suppose that there is a cycle C of length 3 in G with vertices u, v, and w. Define H as the digraph with arcs (u, v), (u, w), and (v, w). The mixed graph $C + H$ is then the top left one of Figure 2.10. Corollary 2.15 implies that (G, H) does not have the uniqueness property, and thus that G does not have the strong uniqueness property.

Conversely, suppose that there is no cycle of length 3 or more. The graph G can then be obtained by successive 1-sums of a graph made of two vertices and parallel edges. Since a graph with two vertices and parallel edges has the uniqueness property for any demand digraph (see Konishi (2004) or Milchtaich (2005)), we can conclude with Lemma 2.18 that G has the strong uniqueness property. □

2.7.5 When there are only two classes

When exhibiting multiple equilibrium flows in the proof of Theorem 2.1, we need to define three classes. The same remark holds for the characterization of the two-terminal graphs having the uniqueness property in the article by Milchtaich (2005): all cases of non-uniqueness are built with three classes. We may wonder whether there are also multiple equilibrium flows with only two classes of users. The answer is yes as shown by the following examples. The first example is in the framework of the ring network; according to Theorem 2.1, such an example requires at least three OD-pairs. Since it will contain exactly three OD-pairs, it is in a sense a minimum example for ring network. The second example involves a two-terminal network – K_4, the complete graph on four vertices – as in Bhaskar et al. (2009). They used it in order to answer a question by Cominetti et al. (2009) about the uniqueness of equilibrium in atomic player routing games. However, their cost functions do not suit our framework and we design specific ones.

Multiple equilibrium flows on the ring with only two classes

Consider the graph on top on the left of Figure 2.10. Define the two classes 1 and 2, with the following population measures.

$\ell \in L$	(u,w)	(u,v)	(w,v)
$\lambda(I^1_\ell)$	0	1.5	0
$\lambda(I^2_\ell)$	1	0	1

Cost functions are:

Arc	(u,w)	(w,v)	(v,u)	(w,u)	(u,v)	(v,w)
Class 1	x	$x+48$			$24x+7$	
Class 2	$22x$	$22x$		x	$x+26$	x

For a given class, arcs not used in any route lead to blanks in this table.

We define the strategy profile σ (resp. $\hat{\sigma}$) such that all users of the class 1 select a negative (resp. positive) route and all users of the class 2 select a positive (resp. negative) route. We get the following (distinct) flows.

Arc a	(u,w)	(w,v)	(v,u)	(w,u)	(u,v)	(v,w)
x_a	1	1	0	0	1.5	0
$\hat{x}_a$	1.5	1.5	0	1	2	1

We check that σ is an equilibrium.

For users in $I^1_{(u,v)}$, the cost of the positive route is 50 and of the negative 43. For users in $I^2_{(u,w)}$ and in $I^2_{(w,v)}$, the cost of the positive route is 22 and of the negative 27.5. No user is incited to change his route choice.

We check that $\hat{\sigma}$ is an equilibrium.

For users in $I^1_{(u,v)}$, the cost of the positive route is 51 and of the negative 55. For users in $I^2_{(u,w)}$ and in $I^2_{(w,v)}$, the cost of the positive route is 33 and of the negative 29. No user is incited to change his route choice.

Remark 2.19. Actually, when we specialize the construction of Section 2.5.3 to the graph on top on the left of Figure 2.10, we can merge classes 2 and 3 in a unique class 2 leading to the example above. More generally, using the symmetry of the cost functions for class 2 and class 3 users, we can merge the two classes for any graph such that $A^{\varepsilon}_{\{1,2\}} \neq \emptyset$ and $A^{\varepsilon}_{\{1,3\}} \neq \emptyset$, with $\varepsilon \in \{-,+\}$ in order to get other ring examples with two classes and multiple equilibrium flows.

Multiple equilibrium flows for a two-terminal network with only two classes

Consider the two-terminal network K_4 of Figure 2.9.

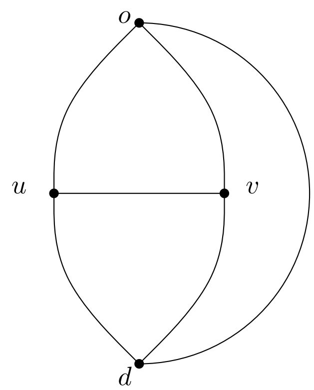

Figure 2.9: A two-terminal network for which multiple equilibrium flows exist with only two classes

Suppose that we have two classes of users I^1 and I^2, with $\lambda(I^1) = 3$ and $\lambda(I^2) = 4$, with the following cost functions on each arc, where "∞" stands for a prohibitively high cost function.

Arc	(o,u)	(o,v)	(u,v)	(v,u)	(u,d)	(v,d)	(o,d)
Class 1	x	"∞"	$x+18$	"∞"	"∞"	x	$7x$
Class 2	$5x$	x	"∞"	"∞"	x	$5x$	$x+10$

Users of class 1 have only the choice between the two routes *ouvd* and *od*, while users of class 2 can choose between the three routes *oud*, *ovd*, and *od*.

The strategy profile σ is defined such that all class 1 users select the route *ouvd* and all class 2 users select the route *od*.

The strategy profile $\hat{\sigma}$ is defined such that all class 1 users select the route *od*, half of class 2 users select the route *oud*, and the other half select the route *ovd*. We get the following (distinct) flows.

Arc a	(o,u)	(o,v)	(u,v)	(v,u)	(u,d)	(v,d)	(o,d)
x_a	3	0	3	0	0	3	4
$\hat{x}_a$	2	2	0	0	2	2	3

We check that σ is an equilibrium.

For users of the class 1, the cost of *ouvd* is 27, and the cost of *od* is 28. For users of the class 2, the cost of *oud* is 15, the cost of *ovd* is 15, and the cost of *od* is 14. No user is incited to change his route choice.

We check that $\hat{\sigma}$ is an equilibrium.

For users of the class 1, the cost of *ouvd* is 22, and the cost of *od* is 21. For users of the class 2, the cost of *oud* is 12, the cost of *ovd* is 12, and the cost of *od* is 13. No user is incited to change his route choice.

Appendix: Minimal ring graphs without the uniqueness property

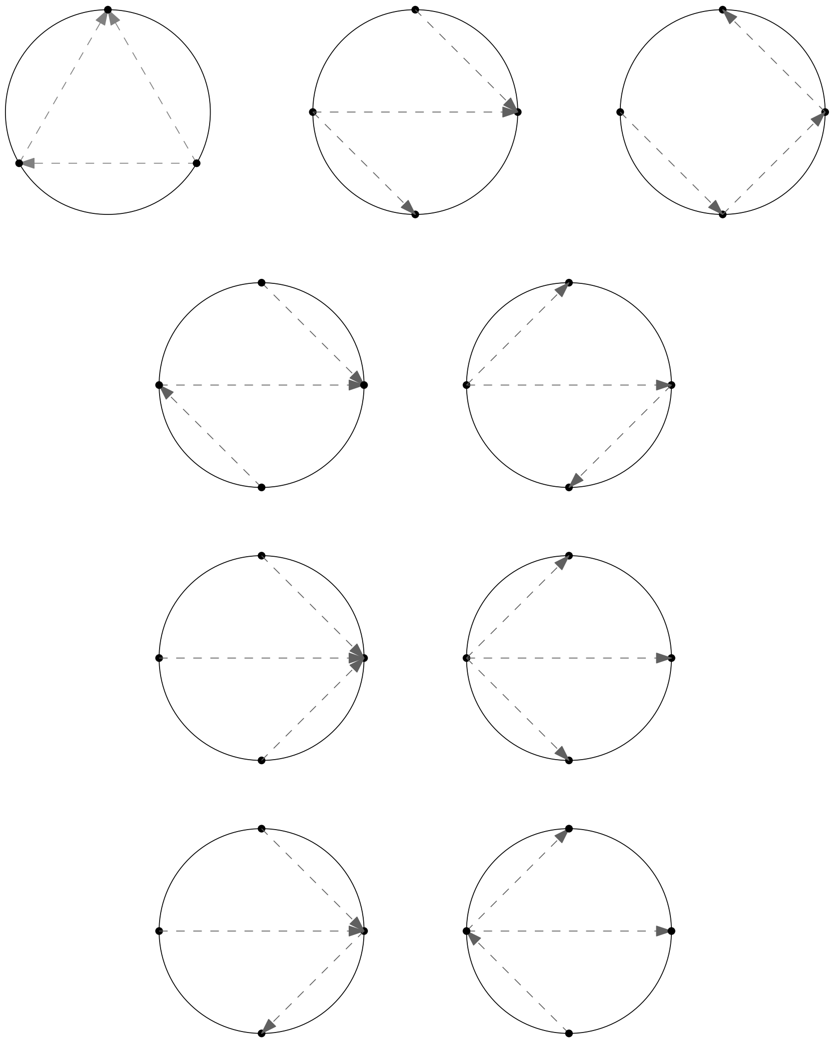

Figure 2.10: Any ring without the uniqueness property has one of these graphs as a minor

CHAPTER 3

A Lemke-like algorithm for the Multiclass Network Equilibrium Problem

This chapter is based on the paper "A Lemke-like algorithm for the Multiclass Network Equilibrium Problem" (Meunier and Pradeau, 2013). This work was presented at the WINE 2013 and ROADEF 2014 conferences.

Abstract

In this chapter, we consider a nonatomic congestion game on a connected graph, with several classes of users. Each user wants to go from his origin vertex to his destination vertex at the minimum cost and all users of a given class share the same characteristics: cost functions on each arc, and origin-destination pair. Under some mild conditions, it is known that a Nash equilibrium exists, but the computation of an equilibrium in the multiclass case is an open problem for general functions. We consider the specific case where the cost functions are affine and propose an extension of Lemke's algorithm able to solve this problem. At the same time, it provides a constructive proof of the existence of an equilibrium in this case.

3.1 Introduction

Context

Being able to predict the impact of a new infrastructure on the traffic in a transportation network is an old but still important objective for transport planners. In 1952, Wardrop (1952) noted that after some while the traffic arranges itself to form an equilibrium and formalized principles characterizing this equilibrium. With the terminology of game theory, the equilibrium is a Nash equilibrium for a congestion game with nonatomic users. In 1956, Beckmann et al. (1956) translated these principles as a mathematical program which turned out to be convex, opening the door to the tools from convex optimization. The currently most commonly used algorithm for such convex programs is probably the Frank-Wolfe algorithm (Frank and Wolfe, 1956), because of its simplicity and its efficiency, but many other algorithms with excellent behaviors have been proposed, designed, and experimented.

One of the main assumptions used by Beckmann to derive his program is the fact that all users are equally impacted by congestion. With the transportation terminology, it means that there is only one *class*. In order to improve the prediction of traffic patterns, researchers started in the 70s to study the *multiclass* situation where each class has its own way of being impacted by the congestion. Each class models a distinct mode of transportation, such as cars, trucks, or motorbikes. Dafermos (1972, 1980) and Smith (1979) are probably the first who proposed a mathematical formulation of the equilibrium problem in the multiclass case. However, even if this problem has been the topic of many research works, an efficient algorithm for solving it remains to be designed, except in some special cases (Florian, 1977, Harker, 1988, Mahmassani and Mouskos, 1988, Marcotte and Wynter, 2004). In particular, there is no general algorithm in the literature for solving the problem when the cost of each arc is in an affine dependence with the flow on it.

Our main purpose is to propose such an algorithm.

Model

We are given a directed graph $D = (V, A)$ modelling the transportation network. We consider the model of multiclass nonatomic games as in Section 1.2.1. We require the users of a given class k to have not only the same cost functions but also the same origin s^k, and the same destination t^k. The set of vertices (resp. arcs) reachable from s^k is denoted V^k (resp. A^k).

Recall the definition of an equilibrium: a strategy profile is a (pure) Nash equilibrium if each route is only chosen by users for whom it is a minimum-cost route. In other words, a strategy profile σ is a Nash equilibrium if for each class $k \in K$ and each user $i \in I^k$ we have

$$\sum_{a \in \sigma(i)} c_a^k(x_a) = \min_{r \in \mathcal{R}_{(s^k,t^k)}} \sum_{a \in r} c_a^k(x_a) \, .$$

This game enters in the category of *nonatomic congestion games with user-specific cost functions*, see Milchtaich (1996). The problem of finding a Nash equilibrium for such a game is called the *Multiclass Network Equilib-*

rium Problem.

Contribution

Our results concern the case when the cost functions are affine and stricly increasing: for all $k \in K$ and $a \in A^k$, there exist $\alpha_a^k > 0$ and $\beta_a^k \geq 0$ such that $c_a^k(x) = \alpha_a^k x + \beta_a^k$ for all $x \in \mathbb{R}_+$. In this case, the Multiclass Network Equilibrium Problem can be written as a linear complementarity problem. In 1965, Lemke (1965) designed a pivoting algorithm for solving a linear complementarity problem under a quite general form. This algorithm has been adapted and extended several times – see for instance Adler and Verma (2011), Asmuth et al. (1979), Cao and Ferris (1996), Cottle et al. (1992), Eaves (1973), Schiro et al. (2012) – to be able to deal with linear complementarity problems that do not directly fit in the required framework of the original Lemke algorithm.

We show that there exists a pivoting Lemke-like algorithm solving the Multiclass Network Equilibrium Problem when the costs are affine. To our knowledge, it is the first algorithm solving this problem. We prove its efficiency through computational experiments. Moreover, the algorithm provides the first constructive proof of the existence of an equilibrium for this problem. The initial proof of the existence from Schmeidler (1970) uses a non-constructive approach with the help of a general fixed point theorem.

On our track, we extend slightly the notion of basis used in linear programming and linear complementarity programming to deal directly with unsigned variables. Even if it is natural, we are not aware of previous use of such an approach. An unsigned variable can be replaced by two variables – one for the nonnegative part and one for the nonpositive part. Such an operation considerably increases the size of the matrices, while, in our approach, we are able to deal directly with the unsigned variables.

Related works

We already gave some references of works related to ours with respect to the linear complementarity. The work by Schiro et al. (2012) is one of them and deals actually with a problem more general than ours. They propose a pivotal algorithm to solve it. However, our problem is not covered by their termination results (the condition of their Proposition 5 is not satisfied by our problem). Another close work is the one by Eaves (1973), which allows additional affine constraints on the variables, but the constraints we need – flow constraints – do not enter in this framework. Note also the work by

De Schutter and De Moor (1995), devoted to the "Extended Linear Complementarity Problem" which contains our problem. They propose a method that exhaustively enumerates all solutions and all extreme rays, without giving *a priori* guarantee for the existence of a solution.

Papers dealing with algorithms for solving the Multiclass Network Equilibrium Problem propose in general a Gauss-Seidel type diagonalization method, which consists in sequentially fixing the flows for all classes but one and solving the resulting single-class problem by methods of convex programming, see Florian (1977), Florian and Spiess (1982), Harker (1988), Mahmassani and Mouskos (1988) for instance. For this method, a condition ensuring the convergence to an equilibrium is not always stated, and, when there is one, it requires that "the interaction between the various users classes be relatively weak compared to the main effects (the latter translates a requirement that a complicated matrix norm be less than unity)" (Mahmassani and Mouskos, 1988). Such a condition does clearly not cover the case with affine cost functions. Another approach is proposed by Marcotte and Wynter (2004). For cost functions satisfying the "nested monotonicity" condition – a notion developed by Cohen and Chaplais (1988) – they design a descent method for which they are able to prove the convergence to a solution of the problem. However, we were not able to find any paper with an algorithm solving the problem when the costs are polynomial functions, or even affine functions.

Structure of the chapter

In Section 3.2, we explain how to write the Multiclass Network Equilibrium Problem as a linear complementarity problem. We get the formulation ($AMNEP(\boldsymbol{e})$) on which the remaining of the chapter focuses. Section 3.3 presents the notions that underly the Lemke-like algorithm. All these notions, likes *basis*, *secondary ray*, *pivot*, and so on, are classical in the context of the Lemke algorithm. They require however to be redefined in order to be able to deal with the features of ($AMNEP(\boldsymbol{e})$). The algorithm is then described in Section 3.4. We also explain why it provides a constructive proof of the existence of an equilibrium. Section 3.5 is devoted to the experiments and shows the efficiency of the proposed approach.

3.2 Formulation as a linear complementarity problem

In this section, we formulate the Multiclass Network Equilibrium Problem as a complementarity problem which turns out to be linear when the cost functions are affine.

From now on, we assume that the cost functions are increasing. As already noted in Section 1.2.3, the equilibrium flows (x_a^k) coincide with the solutions of a system of the following form, where $\boldsymbol{b} = (b_v^k)$ is a given vector with $\sum_{v \in V^k} b_v^k = 0$ for all k.

$$\begin{aligned} &\sum_{a \in \delta^+(v)} x_a^k = \sum_{a \in \delta^-(v)} x_a^k + b_v^k && k \in K, v \in V^k \\ &c_{uv}^k(x_{uv}) + \pi_u^k - \pi_v^k - \mu_{uv}^k = 0 && k \in K, (u,v) \in A^k \\ &x_a^k \mu_a^k = 0 && k \in K, a \in A^k \\ &x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} && k \in K, a \in A^k, v \in V^k \; . \end{aligned} \tag{$MNEP_{gen}$}$$

Actually in our model, we should have moreover $b_v^k = 0$ for $v \notin \{s^k, t^k\}$, and the inequalities $b_{s^k}^k > 0$ and $b_{t^k}^k < 0$, but we relax this condition to deal with a slightly more general problem. Moreover, in this more general form, we can easily require the problem to be non-degenerate, see Section 3.3.2.

Finding solutions for such systems is a *complementarity problem*, the word "complementarity" coming from the condition $x_a^k \mu_a^k = 0$ for all (a, k) such that $a \in A^k$.

We have thus the following proposition.

Proposition 3.1. *$(\boldsymbol{x}^k)_{k \in K}$ is an equilibrium flow if and only if there exist $\boldsymbol{\mu}^k \in \mathbb{R}_+^{A^k}$ and $\boldsymbol{\pi}^k \in \mathbb{R}^{V^k}$ for all k such that $(\boldsymbol{x}^k, \boldsymbol{\mu}^k, \boldsymbol{\pi}^k)_{k \in K}$ is a solution of the complementarity problem ($MNEP_{gen}$).*

Proof. The proof is based on the following fact: $(\boldsymbol{x}^k)_{k \in K}$ is an equilibrium if and only if for each $k \in K$, the vector $\boldsymbol{x}^k$ is an equilibrium flow of the game where all $\boldsymbol{x}^{k'}$, $k' \neq k$ are fixed. Using the result of Beckmann et al. (1956) for single-class problems, we get that $(\boldsymbol{x}^k)_{k \in K}$ is an equilibrium if and only if for each $k \in K$, the vector $\boldsymbol{x}^k$ is a solution of the following problem.

$$\begin{aligned} \min \; & \sum_{a \in A^k} \int_0^{x_a^k} c_a^k(u + x_a^{-k}) \, du \\ \text{s.t.} \; & \sum_{a \in \delta^+(v)} x_a^k = \sum_{a \in \delta^-(v)} x_a^k + b_v^k && v \in V^k, \\ & x_a^k \geq 0 && a \in A^k \; , \end{aligned} \tag{P^k}$$

where $x_a^{-k} = \sum_{k' \neq k} x_a^{k'}$. According to the Karush-Kuhn-Tucker conditions, $\boldsymbol{x}^k$ solves the problem (P^k) if and only if there exist $\boldsymbol{\mu}^k \in \mathbb{R}_+^{A^k}$ and $\boldsymbol{\pi}^k \in \mathbb{R}^{V^k}$

such that $(\boldsymbol{x}^k, \boldsymbol{\mu}^k, \boldsymbol{\pi}^k)_{k\in K}$ satisfies the constraints of the problem ($MNEP_{gen}$). □

When the cost functions are affine $c_a^k(x) = \alpha_a^k x + \beta_a^k$, solving the Multiclass Network Equilibrium Problem amounts thus to solve the following linear complementarity problem

$$
\begin{aligned}
&\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k && k\in K, v\in V^k \\
&\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k = -\beta_{uv}^k && k\in K, (u,v)\in A^k \\
&x_a^k \mu_a^k = 0 && k\in K, a\in A^k \\
&x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} && k\in K, a\in A^k, v\in V^k\,.
\end{aligned}
\tag{MNEP}
$$

Similarly as for the Lemke algorithm, we rewrite the problem as an optimization problem. It will be convenient for the exposition of the algorithm, see Section 3.3. This problem is called the *Augmented Multiclass Network Equilibrium Problem*. It uses a vector $\boldsymbol{e} = (e_a^k)$ defined for all $k \in K$ and $a \in A^k$. The problem ($AMNEP(\boldsymbol{e})$) is

$$
\begin{aligned}
\min\quad & \omega \\
\text{s.t.}\quad & \sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k && k\in K, v\in V^k \\
& \alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k + e_{uv}^k\omega = -\beta_{uv}^k && k\in K, (u,v)\in A^k \\
& x_a^k \mu_a^k = 0 && k\in K, a\in A^k \\
& x_a^k \geq 0, \mu_a^k \geq 0, \omega\geq 0, \pi_v^k \in \mathbb{R} && k\in K, a\in A^k, v\in V^k\,.
\end{aligned}
\tag{$AMNEP(\boldsymbol{e})$}
$$

Some choices of $\boldsymbol{e}$ allow to find easily feasible solutions to this problem. In Section 3.3, $\boldsymbol{e}$ will be chosen in such a way. A key remark is that solving ($MNEP$) amounts to find an optimal solution for ($AMNEP(\boldsymbol{e})$) with $\omega = 0$.

Without loss of generality, we impose that $\pi_{s^k}^k = 0$ for all $k \in K$ and it holds throughout the chapter. It allows to rewrite the problem ($AMNEP(\boldsymbol{e})$)

under the form

$$\begin{aligned} \min \quad & \omega \\ \text{s.t.} \quad & \overline{M}^{\boldsymbol{e}} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\mu} \\ \omega \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ M^T \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ & \boldsymbol{x} \cdot \boldsymbol{\mu} = 0 \\ & \boldsymbol{x} \geq \mathbf{0},\ \boldsymbol{\mu} \geq \mathbf{0},\ \omega \geq 0,\ \boldsymbol{\pi} \in \mathbb{R}^{\sum_k V^k \setminus \{s^k\}}, \end{aligned}$$

where $\overline{M}^{\boldsymbol{e}}$ and C are defined as follows. (The matrix $\overline{M}^{\boldsymbol{e}}$ is denoted with a superscript $\boldsymbol{e}$ in order to emphasize its dependency on $\boldsymbol{e}$).

We define $M = \operatorname{diag}((M^k)_{k\in K})$ where M^k is the incidence matrix of the directed graph (V^k, A^k) from which the s^k-row has been removed:

$$M^k_{v,a} = \begin{cases} 1 & \text{if } a \in \delta^+(v), \\ -1 & \text{if } a \in \delta^-(v), \\ 0 & \text{otherwise .} \end{cases}$$

We also define $C^k = \operatorname{diag}((\alpha^k_a)_{a\in A^k})$ for $k \in K$, and then C the real matrix $C = (\underbrace{(C^k, \cdots, C^k}_{|K| \text{ times}})_{k\in K})$. Then let

$$\overline{M}^{\boldsymbol{e}} = \begin{pmatrix} M & \mathbf{0} & \mathbf{0} \\ C & -I & \boldsymbol{e} \end{pmatrix} .$$

For $k \in K$, the matrix M^k has $|V^k| - 1$ rows and $|A^k|$ columns, while C^k is a square matrix with $|A^k|$ rows and columns. Then the whole matrix $\overline{M}^{\boldsymbol{e}}$ has $\sum_{k\in K}(|A^k| + |V^k| - 1)$ rows and $2\left(\sum_{k\in K} |A^k|\right) + 1$ columns.

3.3 Bases, pivots, and rays

3.3.1 Bases

We define $\mathcal{X}$ and $\mathcal{M}$ to be two disjoint copies of $\{(a,k) : k \in K, a \in A^k\}$. We denote by $\phi^x(a,k)$ (resp. $\phi^\mu(a,k)$) the element of $\mathcal{X}$ (resp. $\mathcal{M}$) corresponding to (a,k). The set $\mathcal{X}$ models the set of all possible indices for the 'x' variables and $\mathcal{M}$ the set of all possible indices for the 'μ' variables for the problem ($AMNEP(\boldsymbol{e})$). We consider moreover a dummy element o as the index for the 'ω' variable.

We define a *basis* for the problem ($AMNEP(\boldsymbol{e})$) to be a subset B of the set $\mathcal{X} \cup \mathcal{M} \cup \{o\}$ such that the square matrix of size $\sum_{k\in K}\left(|A^k| + |V^k| - 1\right)$ defined by

$$\left(\begin{array}{c|c} \overline{M}^{\boldsymbol{e}}_B & \begin{matrix} \mathbf{0} \\ M^T \end{matrix} \end{array} \right)$$

is nonsingular. Note that this definition is not standard. In general, a basis is defined in this way but without the submatrix $\begin{pmatrix} \mathbf{0} \\ M^T \end{pmatrix}$ corresponding to the 'π' columns. We use this definition in order to be able to deal directly with the unsigned variables 'π'. We will see that this approach is natural (and could be used for linear programming as well). However, we are not aware of a previous use of such an approach.

As a consequence of this definition, since M^T has $\sum_{k\in K}(|V^k|-1)$ columns, a basis is always of cardinality $\sum_{k\in K}|A^k|$.

Remark 3.2. In particular, since the matrix is nonsingular and since M^T has $\sum_{k\in K}|A^k|$ rows, the first $\sum_{k\in K}(|V^k|-1)$ rows of $\overline{M}_B^{\boldsymbol{e}}$ have each a nonzero entry. This property is used below, especially in the proof of Lemma 3.8.

The following additional notation is useful: given a subset $Z \subseteq \mathcal{X}\cup\mathcal{M}\cup\{o\}$, we denote by Z^x the set $(\phi^x)^{-1}(Z\cap\mathcal{X})$ and by Z^μ the set $(\phi^\mu)^{-1}(Z\cap\mathcal{M})$. In other words, (a,k) is in Z^x if and only if $\phi^x(a,k)$ is in Z, and similarly for Z^μ.

3.3.2 Basic solutions and non-degeneracy

Let B a basis. If it contains o, the unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ of

$$\begin{cases} \left(\begin{array}{c|c} \overline{M}_B^{\boldsymbol{e}} & \begin{matrix} \mathbf{0} \\ M^T \end{matrix} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{B^x} \\ \boldsymbol{\mu}_{B^\mu} \\ \omega \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ x_a^k = 0 \quad \text{for all } (a,k) \notin B^x \\ \mu_a^k = 0 \quad \text{for all } (a,k) \notin B^\mu \; . \end{cases} \tag{3.1}$$

is called the *basic solution* associated to B.

If B does not contain o, we define similarly its associated *basic solution*. It is the unique solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ of

$$\begin{cases} \left(\begin{array}{c|c} \overline{M}_B^{\boldsymbol{e}} & \begin{matrix} \mathbf{0} \\ M^T \end{matrix} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{B^x} \\ \boldsymbol{\mu}_{B^\mu} \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ x_a^k = 0 \quad \text{for all } (a,k) \notin B^x \\ \mu_a^k = 0 \quad \text{for all } (a,k) \notin B^\mu \\ \omega = 0 \; . \end{cases} \tag{3.2}$$

A basis is said to be *feasible* if the associated basic solution is such that $\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega} \geq 0$.

The problem ($AMNEP(\boldsymbol{e})$) is said to *satisfy the non-degeneracy assumption* if, for any feasible basis B, the associated basic solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ is such that

$$\left((a,k) \in B^x \Rightarrow \bar{x}_a^k > 0\right) \text{ and } \left((a,k) \in B^\mu \Rightarrow \bar{\mu}_a^k > 0\right) .$$

Note that if we had defined the vector $\boldsymbol{b}$ to be 0 on all vertices $v \notin \{s^k, t^k\}$, the problem would not in general satisfy the non-degeneracy assumption. An example of a basis for which the condition fails to be satisfied is the basis B^{ini} defined in Section 3.3.5. Remark 3.7 in that section details the example.

3.3.3 Pivots and polytope

The following lemmas are key results that will eventually lead to the Lemke-like algorithm. They are classical for the usual definition of bases. Since we have extended the definition, we have to prove that they still hold.

Lemma 3.3. *Let B be a feasible basis for the problem ($AMNEP(\boldsymbol{e})$) and assume non-degeneracy. Let i be an index in $\mathcal{X} \cup \mathcal{M} \cup \{o\} \setminus B$. Then there is at most one feasible basis $B' \neq B$ in the set $B \cup \{i\}$.*

Proof. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ be the basic solution associated to B and let $Y = B \cup \{i\}$. The set of solutions

$$\begin{cases} \left(\begin{array}{c|c} \overline{M}_Y^{\boldsymbol{e}} & \begin{array}{c} \boldsymbol{0} \\ M^T \end{array} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{Y^x} \\ \boldsymbol{\mu}_{Y^\mu} \\ \omega \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ x_a^k = 0 \quad \text{for all } (a,k) \notin Y^x \\ \mu_a^k = 0 \quad \text{for all } (a,k) \notin Y^\mu \end{cases}$$

is a one-dimensional line in $\mathbb{R}^{1+\sum_{k\in K}(2|A^k|+|V^k|-1)}$ (the space of all variables) and passing through $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$. The bases in Y correspond to intersections of this line with the boundary of

$$Q = \{(\boldsymbol{x}, \boldsymbol{\mu}, \omega, \boldsymbol{\pi}) : \, x_a^k \geq 0, \mu_a^k \geq 0, \omega \geq 0, \text{ for all } k \in K \text{ and } a \in A^k\} .$$

This latter set being convex (it is a polyhedron), the line intersects at most twice its boundary under the non-degeneracy assumption. $\square$

The operation consisting in computing B' given B and the *entering index* i is called the *pivot operation*.

If we are able to determine an index in $\mathcal{X} \cup \mathcal{M} \cup \{o\} \setminus B$ for any basis B, Lemma 3.3 leads to a "pivoting" algorithm. At each step, we have a current basis B^{curr}, we determine the entering index i, and we compute the new basis in $B^{curr} \cup \{i\}$, if it exists, which becomes the new current basis B^{curr}; and so on. Next lemma allows us to characterize situations where there is no new basis, i.e. situations for which the algorithm gets stuck.

The feasible solutions of ($AMNEP(\boldsymbol{e})$) belong to the polytope

$$\mathcal{P}(\boldsymbol{e}) = \left\{ (\boldsymbol{x}, \boldsymbol{\mu}, \omega, \boldsymbol{\pi}) : \overline{M}^e \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\mu} \\ \omega \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ M^T \end{pmatrix} \boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix}, \right.$$
$$\left. \boldsymbol{x} \geq \mathbf{0},\ \boldsymbol{\mu} \geq \mathbf{0},\ \boldsymbol{\pi} \geq \mathbf{0},\ \omega \in \mathbb{R}_+ \right\}.$$

Lemma 3.4. *Let B be a feasible basis for the problem ($AMNEP(\boldsymbol{e})$) and assume non-degeneracy. Let i be an index in $\mathcal{X} \cup \mathcal{M} \cup \{o\} \setminus B$. If there is no feasible basis $B' \neq B$ in the set $B \cup \{i\}$, then the polytope $\mathcal{P}(\boldsymbol{e})$ contains an infinite ray originating at the basic solution associated to B.*

Proof. The proof is similar as the one of Lemma 3.3, of which we take the same notions and notations. If B is the only feasible basis, then the line intersects the boundary of Q exactly once. Because of the non-degeneracy assumption, it implies that there is an infinite ray originating at $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ and whose points are all feasible. □

3.3.4 Complementarity and twin indices

A basis B is said to be *complementary* if for every (a,k) with $a \in A^k$, we have $(a,k) \notin B^x$ or $(a,k) \notin B^\mu$: for each (a,k), one of the components x_a^k or μ_a^k is not activated in the basic solution. In case of non-degeneracy, it coincides with the condition $\boldsymbol{x} \cdot \boldsymbol{\mu} = 0$. An important point to be noted for a complementary basis B is that if $o \in B$, then there is (a_0, k_0) with $a_0 \in A^{k_0}$ such that

- $(a_0, k_0) \notin B^x$ and $(a_0, k_0) \notin B^\mu$, and
- for all $(a,k) \neq (a_0, k_0)$ with $a \in A^k$, exactly one of the relations $(a,k) \in B^x$ and $(a,k) \in B^\mu$ is satisfied.

This is a direct consequence of the fact that there are exactly $\sum_{k \in K} |A^k|$ elements in a basis and that each (a,k) is not present in at least one of B^x and B^μ. In case of non-degeneracy, this point amounts to say that $x_a^k = 0$ or $\mu_a^k = 0$ for all (a,k) with $a \in A^k$ and that there is exactly one such pair, denoted (a_0, k_0), such that both are equal to 0.

We say that $\phi^x(a_0, k_0)$ and $\phi^\mu(a_0, k_0)$ for such (a_0, k_0) are the *twin indices*.

3.3.5 Initial feasible basis

A good choice of $\boldsymbol{e}$ gives an easily computable initial feasible complementary basis to the problem ($AMNEP(\boldsymbol{e})$).

An *s-arborescence* in a directed graph is a spanning tree rooted at s that has a directed path from s to any vertex of the graph. We arbitrarily define a collection $\mathcal{T} = (T^k)_{k\in K}$ where $T^k \subseteq A^k$ is an s^k-arborescence of (V^k, A^k). Then the vector $\boldsymbol{e} = (e_a^k)_{k\in K, a\in A^k}$ is chosen with the help of $\mathcal{T}$ by

$$e_a^k = \begin{cases} 1 & \text{if } a \notin T^k \\ 0 & \text{otherwise .} \end{cases} \tag{3.3}$$

Lemma 3.5. *Let the set of indices $Y \subseteq \mathcal{X} \cup \mathcal{M} \cup \{o\}$ be defined by*

$$Y = \{\phi^x(a,k) : a \in T^k, k \in K\} \cup \{\phi^\mu(a,k) : a \in A^k \setminus T^k, k \in K\} \cup \{o\} .$$

Then, one of the following situations occurs:

- *$Y \setminus \{o\}$ is a complementary feasible basis providing an optimal solution of the problem ($AMNEP(\boldsymbol{e})$) with $\omega = 0$.*
- *There exists (a_0, k_0) such that $B^{ini} = Y \setminus \{\phi^\mu(a_0, k_0)\}$ is a feasible complementary basis for the problem ($AMNEP(\boldsymbol{e})$).*

Proof. The subset Y has cardinality $\sum_{k\in K} |A^k| + 1$. To show that Y contains a feasible complementary basis, we proceed by studying the solutions of the system

$$\begin{cases} \left(\begin{array}{c|c} \overline{M}_Y^{\boldsymbol{e}} & \begin{array}{c} \boldsymbol{0} \\ M^T \end{array} \end{array} \right) \begin{pmatrix} \boldsymbol{x}_{Y^x} \\ \boldsymbol{\mu}_{Y^\mu} \\ \omega \\ \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b} \\ -\boldsymbol{\beta} \end{pmatrix} \\ x_a^k = 0 \quad \text{for all } (a,k) \notin Y^x \\ \mu_a^k = 0 \quad \text{for all } (a,k) \notin Y^\mu \, . \end{cases} \tag{$S^{\boldsymbol{e}}$}$$

It is convenient to rewrite the problem ($S^{\boldsymbol{e}}$) in the following form.

For all $k \in K$,

$$\begin{cases} M^k_{T^k} x^k_{T^k} = b^k & \\ \alpha^k_{uv} \sum_{k'\in K} x^{k'}_{uv} + \pi^k_u - \pi^k_v - \mu^k_{uv} + e^k_{uv}\omega = -\beta^k_{uv} & \text{for all } (u,v) \in A^k \\ x^k_a = 0 & \text{for all } a \notin T^k \\ \mu^k_a = 0 & \text{for all } a \in T^k \, . \end{cases} \tag{3.4}$$

The matrix $M^k_{T^k}$ is nonsingular (see Poincaré (1900) or for instance the book by Ahuja et al. (1993)). It gives a unique solution $x^k_{T^k}$ of the first equation of (3.4), and since $x^k_a = 0$ for $a \notin T^k$, we get a unique solution $\boldsymbol{x}$ to system ($S^{\boldsymbol{e}}$).

We look now at the second equation of (3.4) for k and (u,v) such that $(u,v) \in T^k$. We get that any solution of system ($S^{\boldsymbol{e}}$) satisfies the equalities

$$\alpha^k_{uv} \sum_{k' \in K} x^{k'}_{uv} + \pi^k_u - \pi^k_v = -\beta^k_{uv}, \qquad \text{for all } k \in K \text{ and } (u,v) \in T^k \,.$$

Indeed, if $(u,v) \in T^k$, we have $e^k_{uv} = 0$ and $\mu^k_{uv} = 0$. Recall that we defined $\pi^k_{s^k} = 0$. Since T^k is a spanning tree of (V^k, A^k) for all k, these equations completely determine $\boldsymbol{\pi}$.

We look then at the second equation of (3.4), this time for k and (u,v) such that $(u,v) \notin T^k$. We get that any solution of system ($S^{\boldsymbol{e}}$) satisfies the equalities

$$\alpha^k_{uv} \sum_{k' \neq k} x^{k'}_{uv} - \mu^k_{uv} + \omega + \pi^k_u - \pi^k_v = -\beta^k_{uv}, \qquad \text{for all } k \in K \text{ and } (u,v) \notin T^k \,. \tag{3.5}$$

Indeed, if $(u,v) \notin T^k$, we have $e^k_{uv} = 1$ and $x^k_{uv} = 0$.

If $\alpha^k_{uv} x_{uv} + \beta^k_{uv} + \pi^k_u - \pi^k_v \geq 0$ for all $k \in K$ and $(u,v) \notin T^k$, then we have an optimal solution of the problem ($AMNEP(\boldsymbol{e})$) with $\omega = 0$, and we get the first point of Lemma 3.5. We can thus assume that $\alpha^k_{uv} x_{uv} + \beta^k_{uv} + \pi^k_u - \pi^k_v < 0$ for at least one triple u, v, k. Let u_0, v_0, k_0 be such a triple minimizing $\alpha^k_{uv} x_{uv} + \beta^k_{uv} + \pi^k_u - \pi^k_v$ and let $a_0 = (u_0, v_0)$. Note that Equation (3.5) implies that

$$\mu^k_{uv} \geq \mu^{k_0}_{u_0 v_0}, \qquad \text{for all } k \in K \text{ and } (u,v) \notin T^k \,. \tag{3.6}$$

We finish the proof by showing that B^{ini}, defined as $Y \setminus \{\phi^\mu(a_0, k_0)\}$, is a feasible complementary basis for the problem ($AMNEP(\boldsymbol{e})$). For B^{ini}, system (3.1) has a unique solution. Indeed, the first part of the proof devoted to the solving of ($S^{\boldsymbol{e}}$) has shown that $\boldsymbol{x}$ and $\boldsymbol{\pi}$ are uniquely determined, without having to compute the values of the μ^k_a's. By definition of (a_0, k_0), since $\phi^\mu(a_0, k_0)$ is not in B^{ini}, we have

$$\mu^{k_0}_{u_0 v_0} = 0 \quad \text{and} \quad \omega = -\alpha^{k_0}_{u_0 v_0} x_{u_0 v_0} - \beta^{k_0}_{u_0 v_0} - \pi^{k_0}_{u_0} + \pi^{k_0}_{v_0} \,.$$

Finally, Equation (3.5) determines the values of the μ^k_{uv} for $k \in K$ and $(u,v) \notin T^k$, and Equation (3.6) ensures that these values are nonnegative. Therefore, B^{ini} is a basis, and it is feasible because all x^k_a and μ^k_a in the solution are nonnegative. Furthermore, for each (a,k) with $a \in A^k$, at least one of $\phi^x(a,k)$ and $\phi^\mu(a,k)$ is not in B^{ini}.

Hence, the subset B^{ini} is a feasible complementary basis. □

We emphasize that B^{ini} depends on the chosen collection $\mathcal{T}$ of arborescences. Note that the basis B^{ini} is polynomially computable.

Remark 3.6. A short examination of the proof makes clear that the following claim is true: *Assuming non-degeneracy, if B is a feasible basis such that $B^x = \{(a,k) : a \in T^k, k \in K\}$, then $B = B^{ini}$.* The fact that the T^k are arborescences fixes completely $\boldsymbol{x}$, and then $\boldsymbol{\pi}$. The fact that B is a feasible basis forces ω to be equal to the maximal value of $-\alpha^k_{uv}x_{uv} - \beta^k_{uv} - \pi^k_u + \pi^k_v$ (except of course if this value is nonpositive, in which case we have already solved our problem), which in turn fixes the values of the μ^k_{uv}.

Remark 3.7. As already announced in Section 3.3.2, if we had defined the vector $\boldsymbol{b}$ to be 0 on all vertices $v \notin \{s^k, t^k\}$, the problem would not satisfy the non-degeneracy assumption as soon as there is $k \in K$ such that T^k has a vertex of degree 3 (which happens when (V^k, A^k) has no Hamiltonian path). In this case, the basis B^{ini} shows that the problem is degenerate. Since the unique solution $\boldsymbol{x}^k_{T^k}$ of $M^k_{T^k}\boldsymbol{x}^k_{T^k} = \boldsymbol{b}^k$ consists in sending the whole demand on the unique route in T^k from s^k to t^k, we have for all arcs $a \in T^k$ not belonging to this route $x^k_a = 0$ while $(a,k) \in B^{ini,x}$.

3.3.6 No secondary ray

Let $(\bar{\boldsymbol{x}}^{ini}, \bar{\boldsymbol{\mu}}^{ini}, \bar{\omega}^{ini}, \bar{\boldsymbol{\pi}}^{ini})$ be the feasible basic solution associated to the initial basis B^{ini}, computed according to Lemma 3.5 and with $\boldsymbol{e}$ given by Equation (3.3). The following inifinite ray

$$\rho^{ini} = \left\{(\bar{\boldsymbol{x}}^{ini}, \bar{\boldsymbol{\mu}}^{ini}, \bar{\omega}^{ini}, \bar{\boldsymbol{\pi}}^{ini}) + t(\mathbf{0}, \boldsymbol{e}, 1, \mathbf{0}) : t \geq 0\right\},$$

has all its points in $\mathcal{P}(\boldsymbol{e})$. This ray with direction $(\mathbf{0}, \boldsymbol{e}, 1, \mathbf{0})$ is called the *primary ray*. In the terminology of the Lemke algorithm, another infinite ray originating at a solution associated to a feasible complementary basis is called a *secondary ray*. Recall that we defined $\pi^k_{s^k} = 0$ for all $k \in K$ in Section 3.2 (otherwise we would have a trivial secondary ray). System ($AMNEP(\boldsymbol{e})$) has no secondary ray for the chosen $\boldsymbol{e}$.

Lemma 3.8. *Let $\boldsymbol{e}$ be defined by Equation (3.3). Under the non-degeneracy assumption, there is no secondary ray in $\mathcal{P}(\boldsymbol{e})$.*

Proof. Suppose that $\mathcal{P}(\boldsymbol{e})$ contains an infinite ray

$$\rho = \left\{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}}) + t(\boldsymbol{x}^{dir}, \boldsymbol{\mu}^{dir}, \omega^{dir}, \boldsymbol{\pi}^{dir}) : t \geq 0\right\},$$

where $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\mu}}, \bar{\omega}, \bar{\boldsymbol{\pi}})$ is a feasible complementary basic solution associated to a basis B.

We first show that $\boldsymbol{x}^{dir} = 0$. For a contradiction, suppose that it is not the case and let k be such that $\boldsymbol{x}^{dir,k}$ is not zero. Since the points of ρ must satisfy the system ($AMNEP(\boldsymbol{e})$) for all $t \geq 0$, we have that $(\boldsymbol{x}^{dir}, \boldsymbol{\mu}^{dir}, \omega^{dir}, \boldsymbol{\pi}^{dir})$ must satisfy for all $v \in V^k$

$$\sum_{a \in \delta^+(v)} x_a^{dir,k} = \sum_{a \in \delta^-(v)} x_a^{dir,k},$$

which shows that $\boldsymbol{x}^k$ is a circulation in the directed graph (V^k, A^k). Moreover, we must have for all $(u,v) \in A^k$

$$\alpha_{uv}^k \sum_{k' \in K} x_{uv}^{dir,k'} + \pi_u^{dir,k} - \pi_v^{dir,k} - \mu_{uv}^{dir,k} + e_{uv}^k \omega^{dir} = 0. \tag{3.7}$$

where we have $\pi_{s^k}^{dir,k} = 0$ since $\pi_{s^k}^k = 0$ for any feasible solution of ($AMNEP(\boldsymbol{e})$), see Section 3.2. The following relations must also be satisfied:

$$\boldsymbol{x}^{dir} \cdot \boldsymbol{\mu}^{dir} = 0, \tag{3.8}$$

and

$$\boldsymbol{x}^{dir} \geq \mathbf{0}, \boldsymbol{\mu}^{dir} \geq \mathbf{0}, \omega^{dir} \geq 0\ . \tag{3.9}$$

Take now any circuit C in $D = (V, A)$ in the support of $\boldsymbol{x}^{dir,k}$. Since we have supposed that $\boldsymbol{x}^{dir,k}$ is not zero and since it is a circulation, such a circuit necessarily exists. According to Equations (3.8) and (3.9), we have $\mu_a^{dir,k} = 0$ for each $a \in C$. The sum $\sum_{a \in C} e_a^k$ is nonzero since no tree T^k can contain all arcs in C. Summing Equation (3.7) for all arcs in C, we get

$$\omega^{dir} = -\frac{\sum_{a \in C} \alpha_a^k \sum_{k' \in K} x_a^{dir,k'}}{\sum_{a \in C} e_a^k} < 0\ .$$

It is in contradiction with Equation (3.9). It implies that $x_a^{dir,k} = 0$ for all $k \in K$ and $a \in A^k$.

We show now that $\boldsymbol{\pi}^{dir} = 0$. We start by noting that Equation (3.7) becomes

$$\pi_u^{dir,k} - \pi_v^{dir,k} - \mu_{uv}^{dir,k} = 0, \qquad \text{for all } k \in K \text{ and } (u,v) \in T^k\ .$$

Since T^k is an s^k-arborescence, we have $0 = \pi_{s^k}^{dir,k} \geq \pi_v^{dir,k}$ for all $v \in V^k$, according to Equation (3.9).

Define now F^k to be the set of arcs $a \in A^k$ such that $(a,k) \in B^x$. Using Remark 3.2 of Section 3.3.1, $\overline{M}_B^{\boldsymbol{e}}$ has a nonzero entry on each of its first

$\sum_{k\in K}(|V^k|-1)$ rows, which implies that the set F^k spans all vertices in $V^k \setminus \{s^k\}$.

According to the non-degeneracy assumption, $\bar{x}_a^k$ is non-zero on all arcs of F^k. The complementarity condition for all points of the ray give that $\bar{\boldsymbol{x}} \cdot \boldsymbol{\mu}^{dir} + \boldsymbol{x}^{dir} \cdot \bar{\boldsymbol{\mu}} = 0$, and since $\boldsymbol{x}^{dir} = \mathbf{0}$, we have $\bar{\boldsymbol{x}} \cdot \boldsymbol{\mu}^{dir} = 0$. Hence $\mu_{uv}^{dir,k} = 0$ for all $(u,v) \in F^k$, and Equation (3.7) becomes

$$\pi_u^{dir,k} - \pi_v^{dir,k} + e_{uv}^k \omega^{dir} = 0 \qquad \text{for all } k \in K \text{ and } (u,v) \in F^k\,. \tag{3.10}$$

Thus, according to Equation (3.9), we have $0 = \pi_{s^k}^{dir,k} \leq \pi_v^{dir,k}$ for all $v \in V^k$. Since we have already shown the reverse inequality, we have $\pi_v^{dir,k} = 0$ for all $v \in V^k$.

Now, if $T^k \neq F^k$ for at least one k, we get the existence of an arc $(u,v) \in F^k$ for which $e_{uv}^k = 1$, while $\pi_u^{dir,k} = \pi_v^{dir,k} = 0$. Equation (3.10) implies then that $\omega^{dir} = 0$. Still using $\boldsymbol{x}^{dir} = \mathbf{0}$, we get then, again with the help of Equation (3.7), that $\boldsymbol{\mu}^{dir} = \mathbf{0}$, which contradicts the fact that ρ is an infinite ray.

Therefore, we have $T^k = F^k$ for all k. Using Remark 3.6 of Section 3.3.5, we are at the initial basic solution: $B = B^{ini}$. According to Equation (3.7), and since $\boldsymbol{x}^{dir} = \mathbf{0}$ and $\boldsymbol{\pi}^{dir} = \mathbf{0}$, we have $\mu_{uv}^{dir,k} = e_{uv}^k \omega^{dir}$ for all $k \in K$ and $(u,v) \in A^k$. Thus $(\boldsymbol{x}^{dir}, \boldsymbol{\mu}^{dir}, \omega^{dir}, \boldsymbol{\pi}^{dir}) = \omega^{dir}(\mathbf{0}, \boldsymbol{e}, 1, \mathbf{0})$ for $\omega^{dir} \geq 0$, and ρ is necessarily the primary ray ρ^{ini}.

Then there is no secondary ray, as required. □

3.3.7 A Lemke-like algorithm

Assuming non-degeneracy, the combination of Lemma 3.3 and the point explicited in Section 3.3.4 give rise to a Lemke-like algorithm. Two feasible complementary bases B and B' are said to be *neighbours* if B' can be obtained from B by a pivot operation using one of the twin indices as an entering index, see Section 3.3.4. Note that is is a symmetrical notion: B can then also be obtained from B' by a similar pivot operation. The abstract graph whose vertices are the feasible complementary bases and whose edges connect neighbour bases is thus a collection of paths and cycles. According to Lemma 3.5, we can find in polynomial time an initial feasible complementary basis for ($AMNEP(\boldsymbol{e})$) with the chosen vector $\boldsymbol{e}$. This initial basis has exactly one neighbour according to Lemma 3.4 since there is a primary ray and no secondary ray (Lemma 3.8).

Algorithm 1 explains how to follow the path starting at this initial feasible complementary basis. Function `EnteringIndex`(B, i') is defined for a feasible complementary basis B and an index $i' \notin B$ being a twin index of B and computes the other twin index $i \neq i'$. Function $\texttt{LeavingIndex}(B, i)$ is defined for a feasible complementary basis B and an index $i \notin B$ and computes the unique index $j \neq i$ such that $B \cup \{i\} \setminus \{j\}$ is a feasible complementary basis (see Lemma 3.3).

Since there is no secondary ray (Lemma 3.8), a pivot operation is possible because of Lemma 3.4 as long as there are twin indices. By finiteness, a component in the abstract graph having an endpoint necessarily has another endpoint. It implies that the algorithm reaches at some moment a basis B without twin indices. Such a basis is such that $o \notin B$ (Section 3.3.4), which implies that we have a solution of the problem ($AMNEP(\boldsymbol{e})$) with $\omega = 0$, i.e. a solution of the problem ($MNEP$), and thus a solution of our initial problem.

input : The matrix $\overline{M}^{\boldsymbol{e}}$, the matrix M, the vectors $\boldsymbol{b}$ and $\boldsymbol{\beta}$, an initial feasible complementary basis B^{ini}
output: A feasible basis B^{end} with $o \notin B^{end}$.

```
φ^μ(a_0, k_0) ← twin index in 𝓜;
i ← EnteringIndex(B^ini, φ^μ(a_0, k_0));
j ← LeavingIndex(B^ini, i);
B^curr ← B^ini ∪ {i} \ {j};
while There are twin indices do
    i ← EnteringIndex(B^curr, j);
    j ← LeavingIndex(B^curr, i);
    B^curr ← B^curr ∪ {i} \ {j};
end
B^end ← B^curr;
return B^end;
```

Algorithm 1: Lemke-like algorithm

3.4 Algorithm and main result

We are now in a position to describe the full algorithm under the non-degeneracy assumption.

1. For each $k \in K$, compute a collection $\mathcal{T} = (T^k)$ where $T^k \subseteq A^k$ is an s^k-arborescence of (V^k, A^k).

2. Define $\boldsymbol{e}$ as in Equation (3.3) (which depends on $\mathcal{T}$).

3. Define $Y = \{\phi^x(a,k) : a \in T^k, k \in K\} \cup \{\phi^\mu(a,k) : a \in A^k \setminus T^k, k \in K\} \cup \{o\}$.

4. If $Y \setminus \{o\}$ is a complementary feasible basis providing an optimal solution of the problem ($AMNEP(\boldsymbol{e})$) with $\omega = 0$, then we have a solution of the problem ($MNEP$), see Lemma 3.5.

5. Otherwise, let B^{ini} be defined as in Lemma 3.5 and apply Algorithm 1, which returns a basis B^{end}.

6. Compute the basic solution associated to B^{end}.

All the elements proved in Section 3.3 lead finally to the following result.

Theorem 3.9. *Under the non-degeneracy assumption, this algorithm solves the problem ($MNEP$), i.e. the Multiclass Network Equilibrium Problem with affine costs.*

This result provides actually a constructive proof of the existence of an equilibrium for the Multiclass Network Equilibrium Problem when the cost are affine and strictly increasing, even if the non-degeneracy assumption is not satisfied. If we compute $\boldsymbol{b} = (b_v^k)$ strictly according to the model, we have

$$b_v^k = \begin{cases} \lambda(I^k) & \text{if } v = s^k \\ -\lambda(I^k) & \text{if } v = t^k \\ 0 & \text{otherwise .} \end{cases} \tag{3.11}$$

In this case, the non-degeneracy assumption is not satisfied as it has been noted at the end of Section 3.3.5 (Remark 3.7). Anyway, we can slightly perturb $\boldsymbol{b}$ and $-\boldsymbol{\beta}$ in such a way that any feasible complementary basis of the perturbated problem is still a feasible complementary basis for the original problem. Such a perturbation exists by standard arguments, see Cottle et al. (1992). Theorem 3.9 ensures then the termination of the algorithm on a feasible complementary basis B whose basic solution is such that $\omega = 0$. It provides thus a solution for the original problem.

It shows also that the problem of finding such an equilibrium belongs to the PPAD complexity class. The PPAD class – defined by Papadimitriou (1994) in 1994 – is the complexity class of functional problems for which we know the existence of the object to be found because of a (oriented) path-following argument. There are PPAD-complete problems, i.e. PPAD problems as hard as any problem in the PPAD class, see Kintali et al. (2009) for examples of such problems. A natural question would be whether the Multiclass Network Equilibrium Problem with affine costs is PPAD-complete. We do not know the answer. Another natural question is whether the problem belongs to other complexity classes often met in the context of congestion games, such as the PLS class (Johnson et al., 1988) or the CLS class (Daskalakis and Papadimitriou, 2011). However, these latter classes require the existence of some potential functions which is not likely to be the case for our problem.

Another consequence of Theorem 3.9 is that if the demands $\lambda(I^k)$ and the cost parameters α_a^k, β_a^k are rational numbers, then there exists an equilibrium inducing rational flows on each arc and for each class k. It is reminiscent of a similar result for two users matrix games: if the matrices involve only rational entries, there is an equilibrium involving only rational numbers (Nash, 1951).

3.5 Computational experiments

3.5.1 Instances

The experiments are made on $n \times n$ grid graphs (Manhattan instances). For each pair of adjacent vertices u and v, both arcs (u, v) and (v, u) are present. We built several instances on these graphs with various sizes n, various numbers of classes, and various cost parameters α_a^k, β_a^k. The cost parameters were chosen uniformly at random such that for all a and all k

$$\alpha_a^k \in [1, 10] \quad \text{and} \quad \beta_a^k \in [0, 100] \, .$$

3.5.2 Results

The algorithm has been coded in C++ and tested on a PC Intel® Core™ i5-2520M clocked at 2.5 GHz, with 4 GB RAM. The experiments are currently in progress. However, some preliminary computational results are given in Table 3.1. Each row of the table contains average figures obtained on five instances on the same graph and with the same number classes, but with various origins, destinations, and costs parameters.

The columns "Classes", "Vertices", and "Arcs" contain respectively the number of classes, the number of vertices, and the number of arcs. The column "Pivots" contains the number of pivots performed by the algorithm. They are done during Step 5 in the description of the algorithm in Section 3.4 (application of Algorithm 1). The column "Algorithm 1" provides the time needed for the whole execution of this pivoting step. The preparation of this pivoting step requires a first matrix inversion, and the final computation of

Classes	Grid	Vertices	Arcs	Pivots	Algorithm 1 (seconds)	Inversion (seconds)
2	2×2	4	8	2	<0.01	<0.01
	4×4	16	48	21	0.01	0.03
	6×6	36	120	54	0.08	0.5
	8×8	64	224	129	0.9	4.0
3	2×2	4	8	4	<0.01	<0.01
	4×4	16	48	33	0.03	0.1
	6×6	36	120	97	0.4	1.9
	8×8	64	224	183	2.6	12
4	2×2	4	8	3	<0.01	<0.01
	4×4	16	48	41	0.06	0.3
	6×6	36	120	126	0.9	4.7
	8×8	64	224	249	5.4	25
10	2×2	4	8	11	<0.01	0.02
	4×4	16	48	107	0.7	4.1
	6×6	36	120	322	15	70
	8×8	64	224	638	87	385
50	2×2	4	8	56	0.3	2.6
	4×4	16	48	636	105	511

Table 3.1: Performances of the complete algorithm for various instance sizes

the solution requires such an inversion as well. The times needed to perform these inversions are given in the column “Inversion”. The total time needed by the complete algorithm to solve the problem is the sum of the “Algorithm 1” time and twice the “Inversion” time, the other steps of the algorithm taking a negligible time.

It seems that the number of pivots remains always reasonable. Even if the time needed to solve large instances is sometimes important with respect to the size of the graph, the essential computation time is spent on the two matrix inversions. The program has not been optimized, since there are several efficient techniques known for inverting matrices. The results can be considered as very positive.

CHAPTER 4

A polynomial algorithm for fixed number of classes and vertices

Abstract

Nonatomic congestion games are a good model of the behaviour of users using a network infrastructure. They are used in particular by transportation planners. Under few conditions, an equilibrium is known to exist. One of the main issue is the computation of a Nash equilibrium.

When all users are impacted in the same way by the congestion, the Nash equilibrium coincides with the solution of an optimization problem, and can be found using the tools of convex optimization. In the multiclass case, i.e. when users can be impacted differently by the congestion, there are few algorithms known. In Meunier and Pradeau (2013), see Chapter 3, an efficient pivoting algorithm is designed, able to find an equilibrium when the cost functions are affine. In particular, the problem belongs to the class of complexity PPAD. Whether the problem is polynomial is an open question.

In this chapter, we give an algorithm polynomial in the number of arcs, when the number of classes and the number of vertices are fixed. This algorithm relies on a correspondence between the set of arcs used at an equilibrium and the cells of some hyperplane arrangement.

4.1 Introduction

4.1.1 Model

We are given a directed graph $D = (V, A)$ modelling the transportation network. We consider the model of multiclass nonatomic games as in Section 1.2.1. As in Chapter 3, we require the users of a given class k to have not only the same cost functions but also the same origin s^k, and the same destination t^k.

Recall the definition of an equilibrium: a strategy profile is a (pure) Nash equilibrium if each route is only chosen by users for whom it is a minimum-cost route. In other words, a strategy profile σ is a Nash equilibrium if for each class $k \in K$ and each user $i \in I^k$ we have

$$\sum_{a \in \sigma(i)} c_a^k(x_a) = \min_{r \in \mathcal{R}_{(s^k,t^k)}} \sum_{a \in r} c_a^k(x_a) \; .$$

This game belongs to the category of *nonatomic congestion games with user-specific cost functions*, see Milchtaich (1996). The problem of finding a Nash equilibrium for such a game is called the *Multiclass Network Equilibrium Problem*.

4.1.2 Supports and arcs of minimal cost

We define the set $\boldsymbol{A}$ as the set of all vectors of subsets of arcs, i.e.

$$\boldsymbol{A} = \left\{ \boldsymbol{X} = (X^k)_{k \in K}, X^k \subseteq A \text{ for all } k \in K \right\} .$$

Given two elements $\boldsymbol{X} = (X^k)_{k \in K}$ and $\boldsymbol{Y} = (Y^k)_{k \in K}$ of $\boldsymbol{A}$, we say that $\boldsymbol{X} \subseteq \boldsymbol{Y}$ if $X^k \subseteq Y^k$ for all k. Note that $\boldsymbol{A} = \mathcal{P}(A)^K$.

Given a flow $\boldsymbol{y} = (y_a)_{a \in A}$, we define its *support* as the set of arcs with a positive flow:

$$\operatorname{supp}(\boldsymbol{y}) = \{a \in A, y_a > 0\}.$$

With a slight abuse of notation, given a multiflow $\vec{\boldsymbol{y}} = (\boldsymbol{y}^k)_{k \in K}$ we define its support as the vector of supports: $\operatorname{supp}(\vec{\boldsymbol{y}}) = (\operatorname{supp}(\boldsymbol{y}^k))_{k \in K}$. It is an element of $\boldsymbol{A}$.

We reformulate Proposition 1.5, characterizing an equilibrium flow as a solution of a complementarity problem, in the following way. For an equilibrium flow $\vec{\boldsymbol{x}}$, recall that $\boldsymbol{x} = (x_a)_{a \in A}$ is the vector of aggregated flow on the arcs. We denote by $\pi_u^k(\boldsymbol{x})$ the minimal cost at equilibrium of the class k users for going from the origin s^k to the vertex u, and $\boldsymbol{\pi}^k(\boldsymbol{x}) = (\pi_u^k(\boldsymbol{x}))_{u \in V \setminus \{s^k\}}$, $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) = (\boldsymbol{\pi}^k(\boldsymbol{x}))_{k \in K}$. Then, we have

$$c_a^k(x_a) \geq \pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x}) \qquad \text{for all } a = (u,v) \in A, \tag{4.1}$$

$$c_a^k(x_a) = \pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x}) \qquad \text{for all } a = (u,v) \in \operatorname{supp}(\boldsymbol{x}^k). \tag{4.2}$$

Remark 4.1. The definition of $\boldsymbol{\pi}$ coincides with the one of Proposition 1.5, where $\pi_{s^k}^k$ is fixed equal to zero for every k.

For an equilibrium flow $\vec{\boldsymbol{x}}$, we define for any k the set of min-cost arcs

$$\mathrm{mincost}^k(\boldsymbol{x}) = \{a = (u, v) \in A, c_a^k(x_a) = \pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x})\}.$$

The vector of these sets is denoted $\mathrm{mincost}(\boldsymbol{x}) = (\mathrm{mincost}^k(\boldsymbol{x}))_{k \in K} \in \boldsymbol{A}$. Note that contrary to the support, it depends only on the aggregated flow $\boldsymbol{x}$. We have $\mathrm{supp}(\vec{\boldsymbol{x}}) \subseteq \mathrm{mincost}(\boldsymbol{x})$. The inclusion is not necessarily an equality as there can exist min-cost arcs with zero flow.

4.1.3 Contribution

We consider the specific case where the cost functions are affine and increasing:

$$c_a^k(x) = \alpha_a^k x + \beta_a^k, \text{ with } \alpha_a^k > 0, \beta_a^k \geq 0,$$

for each class k and arc a.

We prove the existence of a polynomial algorithm solving the problem when the number of classes and vertices is fixed. The main idea of the algorithm relies on properties of hyperplane arrangements.

The remaining of the chapter is organized as follows. Section 4.2 introduces basic definitions on hyperplane arrangements, and in particular on the number of cells. Section 4.3 presents the algorithm. In Section 4.3.1, we prove Proposition 4.4 stating that there is a set of elements of $\boldsymbol{A}$ of polynomial size containing the support of any equilibrium. In Section 4.3.2, we prove Proposition 4.6 stating that given a set of arcs for each class, we can check in polynomial time if it is the support of an equilibrium, and compute the equilibrium when it is the case. These results lead to Theorem 4.7. Finally, Section 4.4 discusses the results, in particular for graphs with parallel arcs.

4.2 Preliminaries on hyperplane arrangements

A *hyperplane* h in $\mathbb{R}^d$ is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. It partitions $\mathbb{R}^d$ into three regions: h itself and the two open half-spaces determined by it. We give an orientation for h and note the two half-spaces $h^{\oplus}$ and $h^{\ominus}$. The closed half-spaces are denoted by $\overline{h^{\oplus}} = h^{\oplus} \cup h$ and $\overline{h^{\ominus}} = h^{\ominus} \cup h$.

Given a finite set H of hyperplanes, an *arrangement* is a partition of $\mathbb{R}^d$ into relatively open convex subsets, called *cells*. A k-cell is a cell of dimension k. A 0-cell is called a point. The terminology *k-face* is also used in the literature to designate a k-cell, in which case the term "cell" refers only to a d-face.

The *hyperplane arrangement* $\mathcal{A}(H)$ associated to the set of hyperplanes H is defined as follows. The d-cells are the connected components of $\mathbb{R}^d \setminus H$. For $0 \leq k \leq d-1$, a *k-flat* is the intersection of exactly $d-k$ hyperplanes of H. Then, the k-faces of the arrangement are the connected components of $L \setminus \{h \in H, L \nsubseteq h\}$ for every k-flat L.

Remark 4.2. Once an orientation is given for each hyperplane, a cell P of the arrangement $\mathcal{A}(H)$ can be described by its *sign vector* $\sigma(P) = (\sigma_h(P))_{h \in H}$. The sign vector is defined as $\sigma_h(P) = 1$ when $P \subseteq h^{\oplus}$, $\sigma_h(P) = -1$ when $P \subseteq h^{\ominus}$ and $\sigma_h(P) = 0$ when $P \subseteq h$.

The sign vector completely determines the cell, but not all possible sign vectors correspond to nonempty cells.

An arrangement of hyperplanes is said to be *simple* if the set of hyperplanes is in general position, i.e. the intersection of any k hyperplanes is $(d-k)$-dimensional for $2 \leq k \leq d+1$. When there is $n \geq d+1$ hyperplanes, to have this condition it suffices that the intersection of any d hyperplanes is a point and the intersection of any $d+1$ hyperplanes is empty.

Given an arrangement of n hyperplanes, the number of k-cells is bounded by

$$\sum_{i=0}^{k} \binom{d-i}{k-i} \binom{n}{d-i}$$

with equality when the arrangement is simple. In particular the total number of cells is a $O(n^d)$.

Further details on hyperplane arrangements can be found in Edelsbrunner (1987) or Matoušek (2002) for example.

Remark 4.3. When considering surfaces or pseudo-lines instead of hyperplanes, a similar result can be obtained under additional conditions such as a bound on the maximal number of intersections between two surfaces. The bound on the number of cells $O(n^d)$ still holds in these cases, see Agarwal (1991), Edelsbrunner et al. (1992) for example.

4.3 The polynomial algorithm

The algorithm consists in two steps:

1. It computes a set $\mathcal{S} \subseteq \boldsymbol{A}$ of polynomial size such that for any equilibrium flow $\vec{\boldsymbol{x}}$ we have

$$\operatorname{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S}(\vec{\boldsymbol{x}}) \subseteq \operatorname{mincost}(\boldsymbol{x})$$

for some $\boldsymbol{S}(\vec{\boldsymbol{x}}) \in \mathcal{S}$, see Section 4.3.1.

2. It tests for every $\boldsymbol{S} \in \mathcal{S}$ whether $\boldsymbol{S} = \boldsymbol{S}(\vec{\boldsymbol{x}})$ for some equilibrium flow $\vec{\boldsymbol{x}}$, and in this case it computes $\vec{\boldsymbol{x}}$, see Section 4.3.2.

4.3.1 Determining a set containing the support of an equilibrium

Proposition 4.4. *For a fixed number of classes and vertices, we can determine in polynomial time a set $\mathcal{S} \subseteq \boldsymbol{A}$ of polynomial size such that for any equilibrium flow $\vec{\boldsymbol{x}}$ there is a $\boldsymbol{S} \in \mathcal{S}$ with $\mathrm{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S} \subseteq \mathrm{mincost}(\boldsymbol{x})$.*

In order to prove Proposition 4.4, we build a hyperplane arrangement. For any classes $k \neq k'$ and arc $a = (u, v)$, we define the following oriented half-spaces of $\mathbb{R}^{K(|V|-1)}$:

$$h_a^{k,k',\ominus} = \left\{\boldsymbol{y} \in \mathbb{R}^{K(|V|-1)}, \alpha_a^{k'} \left(y_v^k - y_u^k - \beta_a^k\right) > \alpha_a^k \left(y_v^{k'} - y_u^{k'} - \beta_a^{k'}\right)\right\},$$

$$h_a^{k,\ominus} = \left\{\boldsymbol{y} \in \mathbb{R}^{K(|V|-1)}, y_v^k - y_u^k > \beta_a^k\right\}.$$

For a class k and an arc $a = (u, v)$ we define moreover the convex polyhedron

$$P_a^k = \bigcap_{k' \neq k} \overline{h_a^{k,k',\ominus}} \cap \overline{h_a^{k,\ominus}}.$$

The P_a^k's have a useful property that links the cost at an equilibrium to the support.

Lemma 4.5. *Let $\vec{\boldsymbol{x}}$ be an equilibrium flow. For any class k and arc a, if $a \in \mathrm{supp}(\boldsymbol{x}^k)$, then $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^k$.*

Proof. Let $a = (u, v) \in \mathrm{supp}(\boldsymbol{x}^k)$. According to Equation (4.2), we have

$$x_a = \frac{\pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x}) - \beta_a^k}{\alpha_a^k}.$$

In particular, since $x_a \geq 0$, we have $\pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x}) \geq \beta_a^k$ and then $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in \overline{h_a^{k,\ominus}}$. In this case, for any other class k', Equation (4.1) gives

$$\alpha_a^{k'} \left(\frac{\pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x}) - \beta_a^k}{\alpha_a^k}\right) + \beta_a^{k'} \geq \pi_v^{k'}(\boldsymbol{x}) - \pi_u^{k'}(\boldsymbol{x}),$$

i.e. $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in \overline{h_a^{k,k',\ominus}}$. Then, $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^k$. $\square$

We consider then the set of hyperplanes

$$H = \left\{h_a^{k,k'}, k \neq k' \in K, a \in A\right\} \cup \left\{h_a^k, k \in K, a \in A\right\}.$$

There are $K^2|A|$ elements in H. We consider then the associated arrangement $\mathcal{A}(H)$.

Proof of Proposition 4.4. Define the map $\varphi : \{\text{cells of } \mathcal{A}(H)\} \to \boldsymbol{A}$ in the following way: for every cell P and class $k \in K$,

$$\varphi(P)_k = \{a \in A, \text{ s.t. } P \cap P_a^k \neq \emptyset\}.$$

Let then $\mathcal{S} = \varphi(\{\text{cells of } \mathcal{A}(H)\})$. According to the number of cells of a hyperplane arrangement – see Section 4.2 – the size of $\mathcal{S}$ is at most $O\left((K^2|A|)^{K(|V|-1)}\right)$.

It remains to show that for any equilibrium flow $\vec{\boldsymbol{x}}$, there exists $\boldsymbol{S} \in \mathcal{S}$ such that $\text{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S} \subseteq \text{mincost}(\boldsymbol{x})$.

Let $\vec{\boldsymbol{x}}$ be an equilibrium flow. Since the cells of $\mathcal{A}(H)$ make a partition of $\mathbb{R}^{K(|V|-1)}$, there is a cell P_0 such that $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_0$. Let $k \in K$ and $a \in \text{supp}(\boldsymbol{x}^k)$. Lemma 4.5 gives that $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^k$, and in particular $P_0 \cap P_a^k \neq \emptyset$, i.e. $a \in \varphi(P_0)_k$. Since this is valid for any $k \in K$, we have $\text{supp}(\vec{\boldsymbol{x}}) \subseteq \varphi(P_0)$. We define $\boldsymbol{S} = \varphi(P_0)$.

Finally, we prove that $\boldsymbol{S} \subseteq \text{mincost}(\boldsymbol{x})$. Consider a class k and an arc a with $P_0 \cap P_a^k \neq \emptyset$. We prove that $a \in \text{mincost}^k(\boldsymbol{x})$. Since P_a^k is an union of cells of $\mathcal{A}(H)$, we have $P_0 \subset P_a^k$. In particular, since $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_0$, we have $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^k$.

Suppose first that $x_a > 0$. If $a \in \text{supp}(\boldsymbol{x}^k)$, we have immediatly that $a \in \text{mincost}^k(\boldsymbol{x})$. Otherwise, there is at least a class $k_0 \neq k$ such that $a \in \text{supp}(\boldsymbol{x}^{k_0})$. Lemma 4.5 gives that $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^{k_0}$, and then $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^{k_0} \cap P_a^k$. In particular, $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in h_a^{k,k_0}$, and by definition

$$\alpha_a^k \left(\frac{\pi_v^{k_0}(\boldsymbol{x}) - \pi_u^{k_0}(\boldsymbol{x}) - \beta_a^{k_0}}{\alpha_a^{k_0}} \right) + \beta_a^k = \alpha_a^k x_a + \beta_a^k = \pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x}),$$

i.e. $a \in \text{mincost}^k(\boldsymbol{x})$.

Suppose then $x_a = 0$. Since $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in P_a^k$, we have in particular $\vec{\boldsymbol{\pi}}(\boldsymbol{x}) \in \overline{h_a^{k,\ominus}}$. It implies that $a \in \text{mincost}^k(\boldsymbol{x})$, since otherwise we would have $\beta_a^k > \pi_v^k(\boldsymbol{x}) - \pi_u^k(\boldsymbol{x})$. We can conclude. $\square$

4.3.2 Determining an equilibrium flow in polynomial time

Proposition 4.6. *Let $\boldsymbol{S} \in \boldsymbol{A}$. We can decide in polynomial time whether there exists an equilibrium flow $\vec{\boldsymbol{x}}$ with* $\text{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S} \subseteq \text{mincost}(\boldsymbol{x})$*, and compute one if there is one.*

Proof. Consider the following problem:

$$\text{Find } (\vec{\boldsymbol{x}}, \vec{\boldsymbol{\pi}}) \text{ such that for every } k \in K \tag{$\mathcal{P}_{\boldsymbol{S}}$}$$

$$\begin{aligned}
&\alpha_a^k \left(\sum_{k' \in K} x_a^{k'} \right) + \beta_a^k = \pi_v^k - \pi_u^k && \text{for } a = (u,v) \in S^k \\
&x_a^k = 0 && \text{for } a \notin S^k \\
&\sum_{a \in \delta^+(s^k)} x_a^k - \sum_{a \in \delta^-(s^k)} x_a^k = \lambda(I^k) && \\
&\sum_{a \in \delta^+(u)} x_a^k - \sum_{a \in \delta^-(u)} x_a^k = 0 && \text{for } u \in V \setminus \{s^k, t^k\} \\
&\alpha_a^k \left(\sum_{k' \in K} x_a^{k'} \right) + \beta_a^k \geq \pi_v^k - \pi_u^k && \text{for } a = (u,v) \notin S^k \\
&x_a^k \geq 0 && \text{for } a \in S^k.
\end{aligned}$$

The problem ($\mathcal{P}_{\boldsymbol{S}}$) consists in linear inequalities with $K(|A|+|V|-1)$ variables and can be solved in polynomial time by an interior point method (see Wright (1997) for example).

The problem ($\mathcal{P}_{\boldsymbol{S}}$) has a solution $(\vec{\boldsymbol{x}}, \vec{\boldsymbol{\pi}})$ if and only if $\vec{\boldsymbol{x}}$ is an equilibrium flow with $\text{supp}(\vec{\boldsymbol{x}}) \subseteq \boldsymbol{S} \subseteq \text{mincost}(\boldsymbol{x})$. In this case, $\vec{\boldsymbol{\pi}} = \vec{\boldsymbol{\pi}}(\boldsymbol{x})$, where with a slight abuse of notation we have denoted by $\vec{\boldsymbol{\pi}}$ an argument of the solution of ($\mathcal{P}_{\boldsymbol{S}}$) and $\vec{\boldsymbol{\pi}}(\cdot)$ the function giving the minimal cost at equilibrium. □

4.3.3 The polynomial algorithm

The algorithm can be described in following way:

1. Build the cells of the arrangement $\mathcal{A}(H)$.

2. For each cell P, test whether there exists an equilibrium flow $\vec{\boldsymbol{x}}$ with $\text{supp}(\vec{\boldsymbol{x}}) \subseteq \varphi(P) \subseteq \text{mincost}(\boldsymbol{x})$, using Proposition 4.6.

3. If it is the case, the algorithm has also computed $\vec{\boldsymbol{x}}$.

There are several techniques for the first step. It can be done in $O(n^d)$ steps where n is the number of hyperplanes and d the dimension of the space. The construction of the cells can also be done step by step. For more details, see Halperin (2004, Chapter 24) or Edelsbrunner (1987).

According to Proposition 4.6, the second and third steps can be done in polynomial time. Moreover, since there is a polynomial number of cells, the second step is made at most a polynomial number of times.

The algorithm stops, since there is at least one cell giving an equilibrium flow, according to Proposition 4.4. This algorithm gives the following Theorem:

Theorem 4.7. *For a fixed number of classes and vertices, there exists an algorithm solving the Multiclass Network Equilibrium Problem with affine costs in polynomial time with respect to the number of arcs.*

4.4 Discussion

We can restrict the number of cells scanned by the algorithm. Indeed, since every class uses at least one arc, for $\boldsymbol{S} \in \mathcal{S}$, we know without any computation that $(\mathcal{P}_{\boldsymbol{S}})$ has no solution as soon as there is a class k with $S^k = \emptyset$. It means that we can consider only the cells P such that for every class k there exists an arc a with $P \cap P_a^k \neq \emptyset$.

Then we can consider only the cells that do not belong to

$$\bigcup_{k \in K} \bigcap_{a=(u,v) \in A} \left(\bigcup_{k' \neq k} h_a^{k,k',\oplus} \cup h_{uv}^{k,\oplus} \right).$$

Computing the improvement of the complexity obtained using this remark deserves future work.

Finally, note that when we consider a graph with parallel arcs, there are only 2 vertices. Furthermore, the set H has size $K^2|A|$, and the number of cells in $\mathcal{A}(H)$ is $O((K^2|A|)^K)$. The question whether there exists an algorithm polynomial in the number of classes, even for graphs with parallel arcs, is an open question.

CHAPTER 5

Bounding the price of anarchy for games with player-specific cost functions

Abstract

We study the efficiency of equilibria in atomic splittable congestion games on graphs. We consider the general case where players are not affected in the same way by the congestion. Extending a result by Cominetti, Correa, and Stier-Moses (2009), we prove a general bound on the price of anarchy for games with player-specific cost functions. This bound generalizes some of their results, especially the bound they obtain for the affine case. However it holds when there is some dependence on the cost functions of the players. In the general case, we prove that the price of anarchy is unbounded, by exhibiting an example with only two players and affine cost functions.

5.1 Introduction

In many economic fields, companies share common resources while being non coordinated. These resources are often owned by agents that are paid for the service they provide to these companies. These resources are for instance machines in a flexible manufacturing environment, means of transportation in a freight context, or arcs in a telecommunication network. An increase of demand for a resource often leads to an increase of its cost, because the fees increase, or because delay is created. This increase of the cost is seen as a congestion on the resource.

Taking a game-theoretical point of view, Cominetti, Correa, and Stier-Moses (2009) studied the efficiency loss implied by the lack of coordination. The companies become *players* in a *congestion game* and the resources become arcs of a graph. If there is no conditional use between the resources, the parallel-link graph often models correctly the interaction between the players, see for instance Orda, Rom, and Shimkin (1993), Hayrapetyan, Tardos, and

Wexler (2006), Wan (2012b). However, in many situations, several resources have to be chosen simultaneously by each company, but not all subsets of resources are possible. Modelling the possible subsets as routes in a graph is a way to cover these situations and makes sense in a freight transportation context for instance. All possible congestion games cannot be modelled with a graph, but this representation is helpful and the results often extend to general congestion games without additional work (this is the case for the game we deal with in the present work). To model the congestion, each arc is endowed with a cost function.

A useful notion to quantify the loss of efficiency is the *price of anarchy*, introduced by Koutsoupias and Papadimitriou (1999). The price of anarchy is the worst-case ratio between the social cost at equilibrium and the best possible social cost. Its interest has lead to a considerable amount of work since the seminal paper of Roughgarden and Tardos (2002). Cominetti, Correa, and Stier-Moses (2009) are able to prove general upper bounds for the price of anarchy of atomic splittable games, which are valid in a large set of situations. These results have been extended by Harks (2011) and Roughgarden and Schoppmann (2011).

One of the main assumptions in these papers is that, for each arc of the graph, the players all have the same cost function. However, depending on their size, the nature of the goods they carry, or other features, the companies are not equally affected by the congestion. Thus, allowing the companies to have their own cost functions makes sense. To the best of our knowledge, this assumption has not been relaxed in the context of the computation of the price of anarchy, except in Gairing, Monien, and Tiemann (2006) for different models (atomic unsplittable and nonatomic games). In this chapter, we allow player-specific cost functions and extend some of the results of Cominetti, Correa, and Stier-Moses (2009) in this more general setting.

5.2 Model and main results

5.2.1 Model

The description of the game we deal with in this chapter goes as follows. We are given a directed graph $D = (V, A)$ and K players identified with the integers $1, \ldots, K$. We consider the model of atomic splittable games as in Section 1.3.1.

Recall the definition of an equilibrium and a social optimum: a feasible multiflow $\vec{\boldsymbol{x}}^{NE} = (\boldsymbol{x}^{NE,1}, \ldots, \boldsymbol{x}^{NE,K})$ is a *Nash equilibrium* if for each player

k, we have

$$Q^k(\vec{\boldsymbol{x}}^{NE}) = \min_{\boldsymbol{y}\in\mathcal{F}^k} Q^k(\boldsymbol{y}, \vec{\boldsymbol{x}}^{NE,-k}), \tag{5.1}$$

where $(\boldsymbol{y}, \vec{\boldsymbol{x}}^{NE,-k}) = (\boldsymbol{x}^{NE,1}, \ldots, \boldsymbol{x}^{NE,k-1}, \boldsymbol{y}, \boldsymbol{x}^{NE,k+1}, \ldots, \boldsymbol{x}^{NE,K})$.
The *social cost* of a multiflow is defined as

$$Q(\vec{\boldsymbol{x}}) = \sum_{k\in[K]} Q^k(\vec{\boldsymbol{x}}) = \sum_{k\in[K]}\sum_{a\in A} x_a^k c_a^k(x_a).$$

A multiflow of minimal social cost is a *social optimum.*

This kind of games belongs to the class of *atomic splittable network congestion games*. Atomic, because each player has non-negligible impact. Splittable, because the goods are seen as a flow whose support is not necessarily a unique route. For the game we consider in this chapter, a Nash equilibrium always exists, see Section 1.3.2. However, it may not be unique and even the total flows on the arcs may vary among the multiple Nash equilibria (Richman and Shimkin, 2007, Bhaskar et al., 2009).

5.2.2 Main results

An instance of the game is defined by the graph, the set of players with their origin-destination pairs, their demands, and their cost functions. We denote by $\mathrm{NE}(I)$ the set of Nash equilibria for an instance I and by $\vec{\boldsymbol{x}}^{OPT}(I)$ a feasible multiflow achieving the minimal social cost.

This chapter is focused on the *price of anarchy* (PoA) of the game, see Section 1.4. Given a set of allowable instances $\mathcal{I}$, the price of anarchy is

$$\mathrm{PoA} = \sup_{I\in\mathcal{I}} \sup_{\vec{\boldsymbol{x}}\in\mathrm{NE}(I)} \frac{C(\vec{\boldsymbol{x}})}{C(\vec{\boldsymbol{x}}^{OPT}(I))}.$$

Theorem 5.1. *Consider an atomic splittable network congestion game with player-specific cost functions. Suppose that the available cost functions in $\mathcal{C}$ are differentiable, nonnegative, increasing, and convex, and define*

$$\lambda(\mathcal{C}) = \sup_{a\in A,\, k,\ell\in[K],\, \vec{c}\in\mathcal{C},\, x\in\mathbb{R}_+} \frac{x(c_a^\ell)'(x)}{c_a^k(x)}.$$

If $\lambda(\mathcal{C}) < 3$, we have

$$\mathrm{PoA} \leq \frac{1}{1-\lambda(\mathcal{C})/3}.$$

This theorem is also valid when all players have the same cost functions, in which case $\lambda(\mathcal{C})$ coincides with $\gamma(\mathcal{C})-1$ in Cominetti et al. (2009).

The upper bound on the price of anarchy given in Theorem 5.1 can be improved with a more complicated formula, see Proposition 5.5 below. A corollary is then the following proposition, which is the generalization of Proposition 3.5 in Cominetti et al. (2009) to the case with player-specific cost functions.

Proposition 5.2. *Consider an atomic splittable network congestion game with player-specific cost functions. Suppose that the allowable cost functions are affine of the form* $c_a^k(x) = p_a^k x + q_a^k$ *with* $p_a^k > 0, q_a^k \geq 0$, *and define* $\Delta = \sup_{a\in A} \frac{\sup_{\ell\in[K]} p_a^\ell}{\inf_{k\in[K]} p_a^k}$.
If $\Delta < 3$, *we have*

$$\text{PoA} \leq \frac{3\Delta(K-1)+4}{\Delta(3-\Delta)(K-1)+4}.$$

Proposition 5.2 uses the parameter Δ with the same definition and with a similar purpose as in Gairing et al. (2006). This parameter is independent of the constant terms of the functions. This bound is valid for $\Delta < 3$, i.e. when the marginal cost caused by the congestion does not vary too much among the players. In particular, it handles cases where the marginal impact of the congestion for any player is at most twice the one of the others. Note that when players have the same cost functions, we have $\Delta = 1$, and our proposition coincides with the aforementioned Proposition 3.5 of Cominetti et al. (2009) giving a bound of $\frac{3K+1}{2K+2}$.

If the cost functions are affine, Proposition 5.2 shows that the price of anarchy is bounded when $\Delta < 3$. However, according to the next proposition, it is unbounded in general, even when the cost functions are affine.

Proposition 5.3. *For any* $M > 0$, *there is an instance of an atomic splittable network congestion game with player-specific affine cost functions, with two players, and with*

$$\text{PoA} > M.$$

This result contrasts with the case where players have the same cost functions. Indeed, in this case it has been showed that the price of anarchy is bounded for polynomial cost functions of degree d: Harks (2011), followed by Roughgarden and Schoppmann (2011) found the closed-form upper bounds of $\left(\frac{1+\sqrt{d+1}}{2}\right)^{d+1}$ for $d \geq 2$ and $\frac{3}{2}$ for affine costs. Bounds also exist for different games that are nonatomic games (Roughgarden and Tardos, 2004) and atomic unsplittable games (Aland et al., 2011).

The remaining of the chapter is organized as follows. Section 5.3 gives a general but not explicit bound on the price of anarchy, from which we deduce Theorem 5.1 and Proposition 5.2. Section 5.4 presents the proof of Proposition 5.3. In Section 5.5 we discuss the results and some open questions.

5.3 The price of anarchy for general cost functions

5.3.1 Preliminary remarks

Throughout the chapter, the components of an $\boldsymbol{x} \in \mathbb{R}_+^K$ are denoted x^k. Recall the notations of Section 1.3.3: given a cost function c^k, we define $\widetilde{c}^k : \mathbb{R}_+^K \to \mathbb{R}_+$ by $\widetilde{c}^k(\boldsymbol{x}) = \frac{\partial}{\partial x^k}\left(x^k c^k(x)\right)$, where $x = \sum_{\ell \in [K]} x^\ell$.

We have then

$$\widetilde{c}^k(\boldsymbol{x}) = x^k (c^k)'(x) + c^k(x).$$

Note that $\widetilde{c}^k$ coincides with the notation c^k in Cominetti et al. (2009).

According to Proposition 1.9, the multiflow $\vec{\boldsymbol{x}}^{NE}$ is a Nash equilibrium if and only if, for all $k \in [K]$, it satisfies

$$\sum_{a \in A} \widetilde{c}_a^k(\boldsymbol{x}_a^{NE})(y_a^k - x_a^{NE,k}) \geq 0, \quad \text{for any feasible flow } \boldsymbol{y}^k \text{ for player } k, \tag{5.2}$$

where $\boldsymbol{x}_a = (x_a^1, \ldots, x_a^K) \in \mathbb{R}_+^K$.

5.3.2 A general bound

Following Cominetti et al. (2009), for a K-tuple of cost functions $\boldsymbol{c} = (c^1, \ldots, c^K)$, we define

$$\beta(\boldsymbol{c}) = \sup_{\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^K} \frac{\sum_{k \in [K]} \left[\left(\widetilde{c}^k(\boldsymbol{x}) - c^k(y)\right) y^k + \left(c^k(x) - \widetilde{c}^k(\boldsymbol{x})\right) x^k \right]}{\sum_{k \in [K]} x^k c^k(x)},$$

where x stands for $\sum_{k \in [K]} x^k$ and y for $\sum_{k \in [K]} y^k$. We assume $0/0 = 0$. Note that $\beta(\boldsymbol{c}) \geq 0$, since the function we want to maximize is zero when $\boldsymbol{x} = \boldsymbol{y}$. We also define $\beta(\mathcal{C}) = \sup_{\vec{\boldsymbol{c}} \in \mathcal{C}, a \in A} \beta(\boldsymbol{c}_a)$, where $\boldsymbol{c}_a = (c_a^1, \ldots, c_a^K)$.

The following proposition gives a general bound, yet nonexplicit, on the price of anarchy. For sake of simplicity, we assume that $(1-\beta(\mathcal{C}))^{-1}=+\infty$ for $\beta(\mathcal{C})\geq 1$.

Proposition 5.4. *Let $\vec{\boldsymbol{x}}^{NE}$ be a Nash equilibrium and $\vec{\boldsymbol{x}}^{OPT}$ be a social optimum. Then*

$$C(\vec{\boldsymbol{x}}^{NE})\leq \frac{1}{1-\beta(\mathcal{C})}C(\vec{\boldsymbol{x}}^{OPT}).$$

This proposition coincides with Proposition 3.2 of Cominetti et al. (2009) in the case where players have the same cost functions. They refer to Roughgarden (2005) and use ideas of Correa et al. (2004). The proof is routine.

Proof of Proposition 5.4. We have

$$\begin{aligned}
C(\vec{\boldsymbol{x}}^{NE}) &= \sum_{k\in[K]}\sum_{a\in A}\left(c_a^k(x_a^{NE})-\tilde{c}_a^k(\boldsymbol{x}_a^{NE})\right)x_a^{NE,k}+\tilde{c}_a^k(\boldsymbol{x}_a^{NE})x_a^{NE,k}\\
&\leq \sum_{k\in[K]}\sum_{a\in A}\left(c_a^k(x_a^{NE})-\tilde{c}_a^k(\boldsymbol{x}_a^{NE})\right)x_a^{NE,k}+\tilde{c}_a^k(\boldsymbol{x}_a^{NE})y_a^k\\
&\leq \sum_{a\in A}\left[\beta(\boldsymbol{c}_a)\sum_{k\in[K]}x_a^{NE,k}c_a^k(x_a^{NE})\right]+C(\vec{\boldsymbol{y}})\\
&\leq \beta(\mathcal{C})C(\vec{\boldsymbol{x}}^{NE})+C(\vec{\boldsymbol{y}})
\end{aligned}$$

where we use Equation (5.2) to get the first inequality and the definition of $\beta(\cdot)$ to get the second inequality.

We finish by taking $\vec{\boldsymbol{y}}=\vec{\boldsymbol{x}}^{OPT}$. □

5.3.3 Computation of the bound

We give now an explicit upper bound on $\beta(\boldsymbol{c})$.

Proposition 5.5. *Consider a K-tuple of cost functions $\boldsymbol{c}=(c^1,\dots,c^K)$ and define*

$$\delta(\boldsymbol{c})=\sup_{k,\ell\in[K],\,x\in\mathbb{R}_+}\frac{(c^\ell)'(x)}{(c^k)'(x)}\quad\text{and}\quad \lambda(\boldsymbol{c})=\sup_{k,\ell\in[K],\,x\in\mathbb{R}_+}\frac{x(c^\ell)'(x)}{c^k(x)}.$$

If each c^k is differentiable, nonnegative, increasing, and convex, then the following inequality holds

$$\beta(\boldsymbol{c})\leq\frac{\lambda(\boldsymbol{c})}{3}\frac{1}{1+\frac{4}{3}\frac{1}{\delta(\boldsymbol{c})(K-1)}}.$$

Before proving this proposition, let us note that the special case $K = 1$ gives $\beta(\boldsymbol{c}) = 0$ and a price of anarchy of 1 as expected: if there is only one player, Nash equilibrium and social optimum coincide.

Proof of Proposition 5.5. Here and throughout the proof, ${c^k}'$ stands for the derivative of c^k. We have thus

$$\beta(\boldsymbol{c}) = \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}_+^K} \frac{f(\boldsymbol{x},\boldsymbol{y})}{\sum_{k\in[K]} x^k c^k(x)},$$

where

$$f(\boldsymbol{x},\boldsymbol{y}) = \sum_{k\in[K]} \left[\left(x^k {c^k}'(x) + c^k(x) - c^k(y) \right) y^k - \left(x^k\right)^2 {c^k}'(x) \right].$$

The proof goes as follows. We will first find an upper bound on $f(\boldsymbol{x},\boldsymbol{y})$ depending only on $\boldsymbol{x}$, Equation (5.5) below. Then, using the quantities $\delta(\boldsymbol{c})$ and $\lambda(\boldsymbol{c})$ defined in the statement of the proposition, we will get an upper bound on $\beta(\boldsymbol{c})$ expressed as a supremum of a concave function on a convex domain, Equation (5.6), which will lead by straightforward computations to the desired expression.

We compute now an upper bound on $f(\boldsymbol{x},\boldsymbol{y})$ depending only on $\boldsymbol{x}$. We have

$$f(\boldsymbol{x},\boldsymbol{y}) \leq \sup_{y\in\mathbb{R}_+,\ell\in[K]} \left(x^\ell {c^\ell}'(x) + c^\ell(x) - c^\ell(y) \right) y - \sum_{k\in[K]} \left(x^k\right)^2 {c^k}'(x), \tag{5.3}$$

since for a $\boldsymbol{y} \in \mathbb{R}_+^K$ with fixed sum $y = \sum_{k\in[K]} y^k$, the first sum in the definition of f can be made maximum by putting all the weight on a single term.

The map $g : y \mapsto \left(x^\ell {c^\ell}'(x) + c^\ell(x) - c^\ell(y) \right) y$ is concave on $\mathbb{R}_+$. Its derivative is

$$g'(y) = x^\ell {c^\ell}'(x) + c^\ell(x) - c^\ell(y) - y{c^\ell}'(y).$$

Let $y^\star \in \mathbb{R}_+$ such that $g'(y^\star) = 0$. We have

$$y^\star {c^\ell}'(y^\star) + c^\ell(y^\star) = x^\ell {c^\ell}'(x) + c^\ell(x)$$

and thus, since $x^\ell \leq x$,

$$x^\ell {c^\ell}'(x^\ell) + c^\ell(x^\ell) \leq y^\star {c^\ell}'(y^\star) + c^\ell(y^\star) \leq x{c^\ell}'(x) + c^\ell(x). \tag{5.4}$$

Since the map $u \mapsto u{c^\ell}'(u) + c^\ell(u)$ is nondecreasing, these inequalities imply

$$x^\ell \leq y^\star \leq x.$$

Hence

$$g(y^\star) = \left(x^\ell c^{\ell\prime}(x) + c^\ell(x) - c^\ell(y^\star)\right) y^\star \leq (x^\ell + x - y^\star) c^{\ell\prime}(x) y^\star \leq \left(\frac{x + x^\ell}{2}\right)^2 c^{\ell\prime}(x),$$

where the first inequality is a consequence of the convexity of $c^\ell(\cdot)$ and where the second is obtained via direct calculations. Using this bound in Equation (5.3), we get an upper bound that does not depend on $\boldsymbol{y}$ and which is valid for all $\boldsymbol{x} \in \mathbb{R}_+^K$:

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) &\leq \sup_{\ell \in [K]} \left(\frac{x + x^\ell}{2}\right)^2 c^{\ell\prime}(x) - \sum_{k \in [K]} \left(x^k\right)^2 c^{k\prime}(x) \\ &\leq \sup_{\ell \in [K]} \frac{x^2 c^{\ell\prime}(x)}{4} \left[\left(1 + 2\frac{x^\ell}{x} - 3\left(\frac{x^\ell}{x}\right)^2\right) - \frac{4}{\delta(\boldsymbol{c})} \sum_{k \neq \ell} \left(\frac{x^k}{x}\right)^2\right]. \end{aligned} \tag{5.5}$$

Using Equation (5.5) in the definition of $\beta(\boldsymbol{c})$ and with the help of the parameter $\lambda(\boldsymbol{c})$ defined in the statement of the proposition, we get

$$\beta(\boldsymbol{c}) \leq \frac{\lambda(\boldsymbol{c})}{4} \sup_{\boldsymbol{x} \in \mathbb{R}_+^K, \ell \in [K]} \left[\left(1 + 2\frac{x^\ell}{x} - 3\left(\frac{x^\ell}{x}\right)^2\right) - \frac{4}{\delta(\boldsymbol{c})} \sum_{k \neq \ell} \left(\frac{x^k}{x}\right)^2\right].$$

Without loss of generality we can assume that the maximum is attained with $\ell = 1$. This inequality can thus be rewritten as

$$\beta(\boldsymbol{c}) \leq \frac{\lambda(\boldsymbol{c})}{4} \sup_{\boldsymbol{z} \in \triangle} \left[\left(1 + 2z^1 - 3\left(z^1\right)^2\right) - \frac{4}{\delta(\boldsymbol{c})} \sum_{k=2}^{K} \left(z^k\right)^2\right], \tag{5.6}$$

where $\triangle$ is the $(K-1)$-dimensional simplex $\{\boldsymbol{z} \in \mathbb{R}_+^K : \sum_{k \in [K]} z^k = 1\}$. The value of right-hand side can be obtained by maximizing a concave function on a convex domain. We compute $\boldsymbol{z}^\star \in \triangle$ maximizing its value:

$$z^{\star,1} = \frac{1 + \theta(\boldsymbol{c})}{3 + \theta(\boldsymbol{c})} \quad \text{and} \quad z^{\star,2} = \cdots = z^{\star,K} = \frac{1 - z^{\star,1}}{K - 1},$$

where $\theta(\boldsymbol{c}) = \frac{4}{\delta(\boldsymbol{c})(K-1)}$. Inequality (5.6) becomes then

$$\beta(\boldsymbol{c}) \leq \frac{\lambda(\boldsymbol{c})}{4} \left[1 + 2z^{\star,1} - 3(z^{\star,1})^2 - \theta(\boldsymbol{c})(1 - z^{\star,1})^2\right],$$

which leads to the upper bound given in the statement of the proposition. □

Note that when we adapt the proof of Proposition 5.5 with a parameter ω analogue to the one of Harks (2011), we are not able to retrieve an equation similar to (5.4).

We explain now how to deduce Theorem 5.1 and Proposition 5.2 from Proposition 5.5.

Proof of Theorem 5.1. We have $\lambda(\mathcal{C})$ larger that any $\lambda(\boldsymbol{c})$, and we use the inequality $\frac{1}{1+\frac{4}{3}\frac{1}{\delta(\boldsymbol{c})(K-1)}} \leq 1$. □

Proof of Proposition 5.2. In this special case, we have Δ larger than any $\delta(\boldsymbol{c})$, and we have moreover $\delta(\boldsymbol{c}) = \lambda(\boldsymbol{c})$ for all $\boldsymbol{c}$. A straightforward calculation leads to the desired formula. □

5.4 The price of anarchy with affine cost functions is unbounded

In this section, we prove Proposition 5.3 by exhibiting an instance of the game with affine cost functions giving a price of anarchy which can be made arbitrarily large.

Consider the graph with one origin-destination pair and two parallel arcs a and b. There are two players. The first player has a total demand of M and his cost functions are $c_a^1(x) = x$ and $c_b^1(x) = x + 2M$. The second player has a total demand of 1 and his cost functions are $c_a^2(x) = 2M^2x + 1$ and $c_b^2(x) = M^3x$. Denote by $I(M)$ this instance.

Since the graph has parallel arcs, the Nash equilibrium is unique (Richman and Shimkin, 2007). We prove that it is reached with the multiflow $\vec{\boldsymbol{x}}$ where player 1 puts all his demand on the arc a and player 2 puts all his demand on the arc b. Indeed, we have in this case, for any $\boldsymbol{y}^1 \in \mathcal{F}^1$ and $\boldsymbol{y}^2 \in \mathcal{F}^2$,

$$\begin{aligned}
&\widetilde{c}_a^1(\boldsymbol{x}_a)(y_a^1 - x_a^1) + \widetilde{c}_b^1(\boldsymbol{x}_b)(y_b^1 - x_b^1) \\
&\qquad = 2M(y_a^1 - M) + (2M+1)(M - y_a^1) = M - y_a^1 \geq 0, \\
&\widetilde{c}_a^2(\boldsymbol{x}_a)(y_a^2 - x_a^2) + \widetilde{c}_b^2(\boldsymbol{x}_b)(y_b^2 - x_b^2) \\
&\qquad = (2M^3+1)(1 - y_b^2) + 2M^3(y_b^2 - 1) = 1 - y_b^2 \geq 0.
\end{aligned}$$

Proposition 1.9 gives then that $\vec{\boldsymbol{x}}$ is a Nash equilibrium. The social cost at equilibrium is $C(\vec{\boldsymbol{x}}) = M^2 + M^3$.

Consider now the multiflow $\vec{\boldsymbol{z}}$ where player 1 puts all his demand on the arc b and player 2 puts all his demand on the arc a. These flows are feasible and give a social cost $C(\vec{\boldsymbol{z}}) = 5M^2 + 1$. We have then

$$\mathrm{PoA}(I(M)) \geq \frac{C(\vec{\boldsymbol{x}})}{C(\vec{\boldsymbol{z}})} = \frac{M^3 + M^2}{5M^2 + 1}.$$

By $\lim_{M \to +\infty} \mathrm{PoA}(I(M)) = +\infty$, we get the result.

Proof of Proposition 5.3. The instance $I(6M)$ gives a price of anarchy greater than M. □

5.5 Discussion and open questions

The bound of Proposition 5.5 makes sense only when $\lambda(\boldsymbol{c}) - 3 < \frac{4}{\delta(\boldsymbol{c})(K-1)}$, since otherwise it is larger than 1. When $\lambda(\boldsymbol{c}) > 3$, this condition is met only when

$$K < 1 + \frac{4}{\delta(\boldsymbol{c})(\lambda(\boldsymbol{c}) - 3)}.$$

In other words, there is a whole range of cost functions and numbers of players for which we are unable to provide any concrete bounds. It would be interesting to extend the bound to a larger set of instances. Proposition 5.3 shows that the price of anarchy is unbounded as soon as the set of affine cost functions is included in the set of allowable cost functions. Its proof needs that the demand of one player becomes infinitely larger than the demands of the others. The question whether the price of anarchy is bounded with affine cost functions, when for instance the quantity $\frac{d^j}{\sum_{k \in [K]} d^k}$ remains bounded for each player j, remains an open question.

Another result from Cominetti et al. (2009) that can be extended to the case with player-specific cost functions is their Proposition 3.7 stating that the social cost at equilibrium is bounded by the optimal cost of the game where the demands are multiplied by $1 + \beta(\mathcal{C})$. More precisely, we can adapt their proof to show the following proposition.

Proposition 5.6. *Consider an atomic splittable network congestion game with player-specific cost functions. Suppose that the available cost functions in $\mathcal{C}$ are differentiable, nonnegative, increasing, and convex. Consider an instance I with an equilibrium multiflow $\vec{\boldsymbol{x}}^{NE}(I)$ and the instance αI where all demands are multiplied by $\alpha \geq 1$, with an optimal multiflow $\vec{\boldsymbol{x}}^{OPT}(\alpha I)$. Suppose that $\beta(\mathcal{C}) < 1$, we have*

$$C(\vec{\boldsymbol{x}}^{NE}(I)) \leq \frac{1}{\alpha - \beta(\mathcal{C})} C(\vec{\boldsymbol{x}}^{OPT}(\alpha I)).$$

Proof. The proof is almost the same as in Cominetti et al. (2009). For the ease of reading, we denote $\vec{\boldsymbol{x}}^{NE} = \vec{\boldsymbol{x}}^{NE}(I)$. Let $\vec{\boldsymbol{y}}$ be a flow feasible for the instance αI, then

$$\begin{aligned}
\alpha C(\vec{\boldsymbol{x}}^{NE}) &= \alpha \left[\sum_{k\in[K]} \sum_{a\in A} \left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \right) x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) x_a^{NE,k} \right] \\
&\leq \alpha \left[\sum_{k\in[K]} \sum_{a\in A} \left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \right) x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \frac{y_a^k}{\alpha} \right] \\
&\leq \sum_{k\in[K]} \sum_{a\in A} \left(c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \right) x_a^{NE,k} + \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) y_a^k \\
&\leq \beta(\mathcal{C}) C(\vec{\boldsymbol{x}}^{NE}) + C(\vec{\boldsymbol{y}})
\end{aligned}$$

where we use Equation (5.2) with $\frac{y_a^k}{\alpha}$ to get the first inequality, the fact that $\alpha \geq 1$ and $c_a^k(x_a^{NE}) - \tilde{c}_a^k(\boldsymbol{x}_a^{NE}) \leq 0$ to get the second inequality, and the definition of $\beta(\cdot)$ to get the last inequality.

We finish by taking $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{OPT}(\alpha I)$. □

This proposition extends Proposition 5.4 which deals with the case $\alpha = 1$. In particular we have $C(\vec{\boldsymbol{x}}^{NE}(I)) \leq C(\vec{\boldsymbol{x}}^{OPT}((1 + \beta(\mathcal{C}))I))$.

Another possible further development would be to compare the game studied in this chapter with the nonatomic case. Cominetti et al. (2009) proved that when all players have same cost functions, same demand, and same origin-destination pair, the social cost at equilibrium of the atomic game is bounded by the one of the corresponding nonatomic game. In particular, the price of anarchy of the atomic game is bounded by the one of the nonatomic game. The key point in their context is that the atomic game is potential, which is unlikely to be the case here. Whether this bound holds with player-specific cost functions is an open question.

CHAPTER 6

Sensitivity of the equilibrium to the demand

Abstract

We consider an atomic congestion game on a graph with parallel arcs. In this game, each player has a non-negligible demand that he splits over the different arcs, in order to minimize his total cost. We consider games with player-specific cost functions: players are impacted in their own way by the congestion. In this context, a Nash equilibrium is known to exist and to be unique. We are interested in the sensitivity of the equilibrium to the repartition of the demand.

We prove regularity results on the Nash equilibrium flows. Moreover, we study the impact of a transfer of a part of the demand of one player to another player with initally more demand. When there are two arcs or two players, on each arc, the flow of the player giving a part of his demand decreases or remains constant equal to zero. Symmetrically, on each arc, the flow of the other player increases or remains constant equal to zero. In the specific case where the cost functions are equal for each player, the social cost at equilibrium does not increase. As a corollary of this result, we found the already known result stating that the social cost at equilibrium decreases when players form coalitions.

6.1 Introduction

The sensitivity analysis of a congestion game is the evaluation of the impact of the input (graph, cost functions and demands) on the equilibrium flow. In practice, these analyses are used for example for designing networks, estimating origin-destination matrices or fixing pricing rules. In this case, the model is a nonatomic congestion game and formulas have been designed in order to perform the sensitivity analysis, see Tobin and Friesz (1988), Qiu and Magnanti (1989), Bell and Iida (1997). However the conditions of application are not always satisfied, even for simple graphs as shown by Josefsson and Patriksson (2007).

Taking a more theoretical point of view, Hall (1978) proved that the equilibrium flow of nonatomic games is continuous with respect to the demand when all users have the same cost functions. This result have been extended for more general cost functions by Dafermos and Nagurney (1984). A more general study, concerning in particular the differentiability, has been made by Patriksson and Rockafellar (2003) and Patriksson (2004). The latter gave a characterization for the existence of a directional derivative of the equilibrium flow with respect to the demand. Josefsson and Patriksson (2007) showed that while equilibrium costs are directionally differentiable, this does not hold for the flows.

A natural intuition would be that an increase of the demand gives an increase of the equilibrium cost. Hall (1978) proved that this result is true for two-terminal graphs when players have the same cost functions, and Lin et al. (2011) gave another combinatorial proof. However this intuition is false in general, as noted for example by Fisk (1979), see Example 0.3. Dafermos and Nagurney (1984) proved that an "average" total cost will necessarily increase.

More recently, Englert et al. (2008) proved that there are networks for which a slight increase of the total demand changes the strategies of all users. This "microscopic" instability, in the sense that the total flow on each arc does not change too much, happens in particular for the class of *generalized Braess graphs* introduced by Roughgarden (2006).

Our purpose is to make a sensitivity analysis for *atomic splittable* games. We consider a set of players having a stock, the *demand*, to divide among different resources. For example, a freight company may have to choose between several means of transportation. This situation can be modelled by an atomic game on a network with parallel arcs, each arc representing a resource. This kind of games were extensively studied, see for example Orda et al. (1993), Altman et al. (2002), Richman and Shimkin (2007), Bhaskar et al. (2009, 2010). We say that we reach a Nash equilibrium when no player can improve his cost by changing unilaterally his repartition. More results and extensions on Nash equilibria can be found in Gairing et al. (2006), Harks (2008), Cominetti et al. (2009). These works consider mostly the case when every player is impacted in the same way by the congestion.

In this chapter, we study the sensitivity of the equilibrium in atomic splittable games on parallel networks. We show that the equilibrium flow is continuous with respect to the demand, but not necessarily differentiable.

We find conditions ensuring the differentiability. Furthermore, we consider the case where a player i transfers a part of his demand to an another player j with initially more demand. In this case, we show that if there are two arcs (i.e. two choices) or two players, on each arc, the flow of player i (resp. player j) decreases (resp. increases) or remains constant equal to zero. Moreover, when all players have same cost functions, the social cost at equilibrium decreases of remains constant.

As a corollary of this result, when several players make a coalition, the social cost at equilibrium decreases or remains constant. However, this last result does not hold when players have specific cost functions.

6.2 Model and main results

We are given a two-terminal graph with a set A of parallel arcs and $K \geq 2$ players identified with the integers $1, \dots, K$. We consider the model of atomic splittable games as in Section 1.3.1. We assume that the cost functions c_a^k are increasing and *strictly* convex. The topology of the graph allows to identify arcs and routes. The strategies for the player k are then the repartitions of his demand d^k among the arcs.

This kind of games belongs to the class of *atomic splittable network congestion games*, for which a Nash equilibrium always exists, see Section 1.3.2. Furthermore, the graph belongs to the class of nearly-parallel graphs, and the equilibrium flow is unique (Richman and Shimkin, 2007).

We are interested by the *sensitivity* of the equilibrium flow with respect to the demand. We define the application $\vec{\boldsymbol{x}}^{NE}$ giving for a vector of demands $\boldsymbol{d} \in \mathbb{R}_+^K$ the (unique) equilibrium multiflow $\vec{\boldsymbol{x}}^{NE}(\boldsymbol{d})$.

Recall that the total cost faced by all players is called the *social cost* and is defined by

$$Q(\vec{\boldsymbol{x}}) = \sum_{k \in [K]} \sum_{a \in A} x_a^k c_a^k(x_a).$$

The *support* of player k is the set of arcs used by this player

$$\operatorname{supp}(\boldsymbol{x}^k) = \{a \in A,\, x_a^k > 0\}.$$

The first result deals with the regularity of the equilibrium flow.

Theorem 6.1. *The equilibrium multiflow $\vec{\boldsymbol{x}}^{NE}$ is continuous with respect to the vector of demands.*

Furthermore, when there are two arcs or two players, and the cost functions are twice continuously differentiable, the equilibrium multiflow is differentiable on every open set of demands where the support at equilibrium is constant.

Our two other main results are the following Theorems.

Theorem 6.2. *Suppose that there are two arcs or two players and that a player i tranfers a part of his demand to a player j with $d^i \leq d^j$. Then, on each arc, the flow of player i (resp. player j) decreases (resp. increases) or remains constant equal to zero.*

Theorem 6.3. *Suppose that there are two arcs or two players and that a player i tranfers a part of his demand to a player j with $d^i \leq d^j$. Assume that all players have the same twice continuously differentiable cost functions. Then the social cost at equilibrium decreases or remains constant.*

Remark 6.4. The assumption of twice differentiability is not necessary when there are two arcs.

Theorem 6.3 is not valid when there are three arcs or three players, or when players have specific cost functions, see Section 6.7. Whether the differentiability part of Theorem 6.1 and Theorem 6.2 are valid when there are three arcs or three players is an open question.

The remaining of the chapter is organized as follows. We prove Theorem 6.1 in Section 6.3. Section 6.4 introduces general properties of the Nash equilibrium that will be used to prove Theorems 6.2 and 6.3. Section 6.5 is devoted to the proof of Theorem 6.2, and the proof of Theorem 6.3 is in Section 6.6. Finally, Section 6.7 gives examples disproving the results for more general games.

6.3 Regularity of the equilibrium flow

This section is devoted to the proof of Theorem 6.1.
Theorem 6.1 is a direct corollary of Proposition 6.5, where we prove the continuity, and Proposition 6.9, where we prove the differentiability of the equilibrium flows.

6.3.1 Continuity

A natural question is the continuity of the equilibrium with respect to the vector of demands.

Proposition 6.5. *The application $\vec{\boldsymbol{x}}^{NE}$ is continuous on $\mathbb{R}_+^K$.*

Proof. Let $\triangle = \{\boldsymbol{p} \in \mathbb{R}_+^{|A|} : \sum_{a \in A} p_a = 1\}$ be the $(|A|-1)$-dimensional simplex, and $F : \triangle^K \times \mathbb{R}_+^K \to \mathbb{R}^{|A|K}$ defined by

$$F_a^k(\vec{\boldsymbol{p}}, \boldsymbol{d}) = \left[c_a^k \left(\sum_{\ell \in [K]} p_a^\ell d^\ell \right) + p_a^k d^k c_a^{k\prime} \left(\sum_{\ell \in [K]} p_a^\ell d^\ell \right) \right] d^k,$$

for every a and k. The application F is continuous in both variables.

Then, according to Proposition 1.9, the multiflow $\vec{\boldsymbol{x}}^{NE}(\boldsymbol{d})$ is the Nash equilibrium flow if and only if $x_a^{NE,k}(\boldsymbol{d}) = p_a^k d^k$, for all a and k, where $\vec{\boldsymbol{p}}$ satisfies the variational inequality

$$F(\vec{\boldsymbol{p}}, \boldsymbol{d}) \cdot (\vec{\boldsymbol{q}} - \vec{\boldsymbol{p}}) \geq 0, \quad \text{for any } \vec{\boldsymbol{q}} \in \triangle^K. \tag{$VI(\boldsymbol{d})$}$$

By a sequential argument, using the continuity of F and the compactness of $\triangle^K$, elementary calculations give that the solution of ($VI(\boldsymbol{d})$) is continuous with respect to $\boldsymbol{d}$. We can conclude. □

6.3.2 Differentiability

The application $\vec{\boldsymbol{x}}^{NE}$ is not differentiable in general as shown by the following example, inspired from Hall (1978).

Example 6.6. Consider a graph with two arcs a and b and only one player with demand d. Suppose that the cost functions are

$$c_a(x) = x + 1 \quad \text{and} \quad c_b(x) = x.$$

A direct calculation gives that at equilibrium $x_a = 0$ and $x_b = d$ when $0 \leq d \leq \frac{1}{2}$, and $x_a = \frac{2d-1}{4}$, $x_b = \frac{2d+1}{4}$ when $d \geq \frac{1}{2}$. Hence $\vec{\boldsymbol{x}}^{NE}$ is not differentiable at the point $d = \frac{1}{2}$.

However we can prove that when the players have the same cost functions, $\vec{\boldsymbol{x}}^{NE}$ is differentiable on the sets where the support remains constant.

We begin with technical lemmas.

Lemma 6.7. *Let $I = \{i_1, \ldots, i_p\}$ be a subset of indices of $[K]$. For any $\boldsymbol{g}$ and $\boldsymbol{h} \in \mathbb{R}^I$ such that $g_{i_k} > 0$ and $h_{i_k} > 0$ for every $k \in [p]$, we define the $p \times p$ matrix $M_I(\boldsymbol{g}, \boldsymbol{h})$ by*

$$m_{k,\ell}(\boldsymbol{g}, \boldsymbol{h}) = \begin{cases} g_{i_k} & \text{for } k \neq \ell \\ g_{i_k} + h_{i_k} & \text{for } k = \ell. \end{cases}$$

Then, $M_I(\boldsymbol{g}, \boldsymbol{h})$ is nonsingular.

Let then $A_I(\boldsymbol{g},\boldsymbol{h}) = M_I(\boldsymbol{g},\boldsymbol{h})^{-1}$, and define the $K \times K$ matrix $\widetilde{A}_I(\boldsymbol{g},\boldsymbol{h})$ by

$$\tilde{a}_{k,\ell}(\boldsymbol{g},\boldsymbol{h}) = \begin{cases} a_{k,\ell} & \text{for } k \text{ and } \ell \in I, \\ 0 & \text{for } k \text{ or } \ell \notin I. \end{cases}$$

Consider a set $(I_i, \boldsymbol{g}_i, \boldsymbol{h}_i)_{i\in[n]}$ with $I_i \subseteq [K]$, $\boldsymbol{g}_i$ and $\boldsymbol{h}_i \in \mathbb{R}_+^{|I_i|}$ for every i. If every $k \in [K]$ belongs to at least a I_i, the $K \times K$ matrix

$$\sum_{i\in[n]} \widetilde{A}_I(\boldsymbol{g}_i,\boldsymbol{h}_i)$$

is nonsingular.

Proof. Without loss of generality, assume that $I = [p]$. A straightforward calculation using classical techniques gives that

$$\det M_{[p]}(\boldsymbol{g},\boldsymbol{h}) = \left(\sum_{k\in[p]} \frac{g_k}{h_k} + 1\right) \prod_{k\in[p]} h_k.$$

Since the h_k and g_k are positive, we get that $\det M_{[p]}(\boldsymbol{g},\boldsymbol{h}) > 0$ and hence $M_{[p]}(\boldsymbol{g},\boldsymbol{h})$ is nonsingular.

More precisely, $M_{[p]}(\boldsymbol{g},\boldsymbol{h})^{-1} = A_{[p]}(\boldsymbol{g},\boldsymbol{h})$ is defined by

$$a_{k,\ell}(\boldsymbol{g},\boldsymbol{h}) = \begin{cases} -\dfrac{g_k}{\left(\sum_{i\in[p]} \frac{g_i}{h_i} + 1\right) h_k h_\ell} & \text{for } k \neq \ell \\[2ex] \dfrac{1}{h_k} - \dfrac{g_k}{\left(\sum_{i\in[p]} \frac{g_i}{h_i} + 1\right) (h_k)^2} & \text{for } k = \ell. \end{cases} \tag{6.1}$$

In particular, $a_{k,\ell}(\boldsymbol{g},\boldsymbol{h}) < 0$ for $k \neq \ell$, and $a_{k,k}(\boldsymbol{g},\boldsymbol{h}) > 0$. since the g_i and h_i are positive. Moreover, the sum of the coefficients of column ℓ is $\frac{1}{\left(\sum_{i\in[p]} \frac{g_i}{h_i}+1\right)h_\ell} > 0$.

Fiedler and Pták (1962) defined a Z-matrix as a square matrix A with $a_{k,\ell} \leq 0$ for every $k \neq \ell$. Minkowski (1900) proved that a Z-matrix whose column sums are positive has a positive determinant, and in particular is nonsingular. For more details, see (Berman and Plemmons, 1979, Chapter 6).

We saw that $A_{I_i}(\boldsymbol{g}_i,\boldsymbol{h}_i)$ is a Z matrix with positive column sums for every i, hence $\widetilde{A}_{I_i}(\boldsymbol{g}_i,\boldsymbol{h}_i)$ is a Z-matrix with nonnegative column sums, since we only add lines and columns of zeros. Furthermore, for every $k \in [K]$, since k is in at least a set I_i, the column k of $\sum_{i\in[n]} \widetilde{A}_{I_i}(\boldsymbol{g}_i, \boldsymbol{h}_i)$ has a positive sum.

Then $\sum_{i\in[n]} \widetilde{A}_{I_i}(\boldsymbol{g}_i, \boldsymbol{h}_i)$ is a Z matrix with positive column sums and is nonsingular. □

Fix a set of arcs for each player, i.e. $\boldsymbol{S} = (S^k)_{k\in[K]} \in \mathcal{P}(A)^K$. We denote any vector of $\mathbb{R}^{\sum_{k\in[K]}|S^k|}$ by $\vec{\boldsymbol{x}}_{\boldsymbol{S}} = (x_a^k)_{k\in[K],a\in S^k}$ and define $x_{\boldsymbol{S},a} = \sum_{k\in[K],a\in S^k} x_a^k$.

Lemma 6.8. *Suppose that the cost functions are twice continuously differentiable. Fix a nonempty set of arcs for each player, $\boldsymbol{S} = (S^k)_{k\in[K]} \in \mathcal{P}(A)^K$.*

Define the application $H_{\boldsymbol{S}} : \mathbb{R}^{\sum_{k\in[K]}|S^k|} \times \mathbb{R}^K \to \mathbb{R}^{\sum_{k\in[K]}|S^k|} \times \mathbb{R}^K$ by $H_{\boldsymbol{S}}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) = (G(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}), D(\vec{\boldsymbol{x}}_{\boldsymbol{S}}))$ where

$$G_a^k(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) = c_a^k(x_{\boldsymbol{S},a}) + x_a^k c_a^{k\prime}(x_{\boldsymbol{S},a}) - \pi^k \qquad k \in [K], a \in S^k$$
$$D^k(\vec{\boldsymbol{x}}_{\boldsymbol{S}}) = \sum_{a\in S^k} x_a^k \qquad k \in [K].$$

Then $H_{\boldsymbol{S}}$ is continuously differentiable, invertible on any subset $U \times \mathbb{R}^K$ such that $c_a^{k\prime}(x_{\boldsymbol{S},a}) > 0$ for every a and $\vec{\boldsymbol{x}}_{\boldsymbol{S}} \in U$. Moreover, $H_{\boldsymbol{S}}^{-1}$ is continuously differentiable on $H_{\boldsymbol{S}}(U \times \mathbb{R}^K)$.

Note that for the ease of notation, we drop the dependency on $\boldsymbol{S}$ of the functions G and D.

Proof of Lemma 6.8. Since c_a is twice continuously differentiable for every a, $H_{\boldsymbol{S}}$ is continuously differentiable. To prove that $H_{\boldsymbol{S}}$ is invertible with an inverse continuously differentiable, we use the inverse function theorem. We only need to show that for every $(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) \in U \times \mathbb{R}^K$, the Jacobian matrix of $H_{\boldsymbol{S}}$ at this point is nonsingular.

Let then $(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) \in U \times \mathbb{R}^K$ and denote by J the Jacobian matrix of $H_{\boldsymbol{S}}$ at this point. For every k, let $g_a^k = c_a^{k\prime}(x_{\boldsymbol{S},a}) + x_a^k c_a^{k\prime\prime}(x_{\boldsymbol{S},a})$ for $a \in S^k$ and $h_a^k = c_a^{k\prime}(x_{\boldsymbol{S},a})$ for every $a \in A$.

A direct calculation gives that for any $k \in [K]$ and $a \in S^k$,

$$\frac{\partial G_a^k}{\partial x_b^\ell}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) = \begin{cases} g_a^k & \text{for } a = b,\ k \neq \ell \\ g_a^k + h_a^k & \text{for } a = b,\ k = \ell \\ 0 & \text{for } a \neq b. \end{cases}$$

$$\frac{\partial G_a^k}{\partial \pi^\ell}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) = \begin{cases} 0 & \text{for } k \neq \ell \\ -1 & \text{for } k = \ell \end{cases}$$

$$\frac{\partial D^k}{\partial x_b^\ell}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) = \begin{cases} 0 & \text{for } k \neq \ell \\ 1 & \text{for } k = \ell. \end{cases}$$

$$\frac{\partial D^k}{\partial \pi^\ell}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}) = 0.$$

For every a, we denote by K_a the set of players effectively using the arc:

$$K_a = \{k \in [K],\ a \in S^k\},$$

and consider the vectors $\boldsymbol{g}_a = (g_a^k)_{k \in K_a}$ and $\boldsymbol{h}_a = (h_a^k)_{k \in K_a}$. Let then the $|K_a| \times |K_a|$ matrix $M_{K_a}(\boldsymbol{g}_a, \boldsymbol{h}_a)$ be as in Lemma 6.7. Since we assumed that $c_a^{k\prime}(x_{\boldsymbol{S},a}) > 0$, we can apply the lemma, and this matrix is nonsingular.

Let $(\vec{\boldsymbol{\lambda}}, \boldsymbol{\mu}) \in \mathbb{R}^{\sum_{k \in [K]} |S^k|} \times \mathbb{R}^K$ be in the kernel of J. $J(\vec{\boldsymbol{\lambda}}, \boldsymbol{\mu}) = 0$ if and only if

$$\begin{cases} M_{K_a}(\boldsymbol{g}_a, \boldsymbol{h}_a)\boldsymbol{\lambda}_a = \boldsymbol{\mu}_{K_a} & a \in A, \\ \displaystyle\sum_{a \in S^k} \lambda_a^k = 0 & k \in [K]. \end{cases} \tag{6.2}$$

where $\boldsymbol{\lambda}_a = (\lambda_a^k)_{k \in K_a}$ and $\boldsymbol{\mu}_{K_a} = (\mu^k)_{k \in K_a}$.

In order to consider matrices with same dimension, we consider for every a the $K \times K$ matrix $\widetilde{A}_{K_a}(\boldsymbol{g}_a, \boldsymbol{h}_a)$ defined in Lemma 6.7. We also define $\vec{\boldsymbol{\Lambda}} \in \mathbb{R}^{K|A|}$ by $\Lambda_a^k = \lambda_a^k$ for $a \in K_a$ and 0 elsewhere. Then, the system (6.2) is equivalent to the following.

$$\begin{cases} \boldsymbol{\Lambda}_a = \widetilde{A}_{K_a}(\boldsymbol{g}_a, \boldsymbol{h}_a)\boldsymbol{\mu} & a \in A, \\ \displaystyle\sum_{a \in A} \boldsymbol{\Lambda}_a = 0. \end{cases}$$

These conditions imply that

$$\sum_{a \in A} \boldsymbol{\Lambda}_a = \left(\sum_{a \in A} \widetilde{A}_{K_a}(\boldsymbol{g}_a, \boldsymbol{h}_a) \right) \boldsymbol{\mu} = \boldsymbol{0}.$$

Since the sets S^k are nonempty, for every $k \in [K]$, there is at least an arc a such that $k \in K_a$. We can apply the second part of Lemma 6.7, and $\sum_{a\in A} \widetilde{A}_{K_a}(\boldsymbol{g}_a, \boldsymbol{h}_a)$ is nonsingular. It gives $\boldsymbol{\mu} = \mathbf{0}$ and then $\vec{\boldsymbol{\Lambda}} = \vec{\mathbf{0}}$. In particular, $\vec{\boldsymbol{\lambda}} = 0$, and J is nonsingular. We can conclude. □

To prove that the equilibrium flow is differentiable when the support of the players remains constant, we have to define notions relative to this support. Fix a set of arcs for each player i.e. $\boldsymbol{S} = (S^k)_{k\in[K]} \in \mathcal{P}(A)^K$.

We can decompose any multiflow $\vec{\boldsymbol{x}} \in \mathbb{R}^{K|A|}$ in the following way

$$\vec{\boldsymbol{x}} = (\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \vec{\boldsymbol{x}}_{\bar{\boldsymbol{S}}}),$$

with $\vec{\boldsymbol{x}}_{\boldsymbol{S}} = (x_a^k)_{k\in[K],a\in S^k}$ and $\vec{\boldsymbol{x}}_{\bar{\boldsymbol{S}}} = (x_a^k)_{k\in[K],a\notin S^k}$.

Proposition 6.9. *Suppose that there are two arcs or two players and that the cost functions are twice continuously differentiable. The application $\vec{\boldsymbol{x}}^{NE}$ is differentiable on every open subset D of $\mathbb{R}_+^K$ such that $\boldsymbol{d} \mapsto \operatorname{supp}(\vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}))$ is constant on D.*

Proof. Denote by $\boldsymbol{S} = (S^k)_{k\in[K]}$ the constant support: for every $\boldsymbol{d} \in D$, $\operatorname{supp}(\vec{\boldsymbol{x}}^{NE,k}(\boldsymbol{d})) = S^k$.

We extend the definition of G_a^k made in Lemma 6.8. For a multiflow $\vec{\boldsymbol{x}} \in \mathbb{R}^{K|A|}$ and $\boldsymbol{\pi} \in \mathbb{R}^K$,

$$G_a^k(\vec{\boldsymbol{x}}, \boldsymbol{\pi}) = c_a^k(x_a) + x_a^k {c_a^k}'(x_a) - \pi^k, \qquad k \in [K], a \in A.$$

Let $\boldsymbol{d}_0 \in D$, we prove that there is a neighbourhood $V(\boldsymbol{d}_0)$ of $\boldsymbol{d}_0$ such that $\vec{\boldsymbol{x}}^{NE}$ is differentiable on $V(\boldsymbol{d}_0)$. Let $\vec{\boldsymbol{x}}$ be a multiflow, then $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_0)$ if and only if

$$H_{\boldsymbol{S}}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}, \boldsymbol{\pi}(\vec{\boldsymbol{x}}_{\boldsymbol{S}})) = (\mathbf{0}, \boldsymbol{d}_0) \tag{6.3}$$

$$\vec{\boldsymbol{x}}_{\bar{\boldsymbol{S}}} = \mathbf{0} \tag{6.4}$$

$$G_a^k(\vec{\boldsymbol{x}}, \boldsymbol{\pi}(\vec{\boldsymbol{x}})) > 0 \qquad k \in [K], a \notin S^k. \tag{6.5}$$

The condition (6.5) is valid since D is an open subset and G_a^k and $\vec{\boldsymbol{x}}^{NE}$ are continuous. Indeed, if $G_a^k(\vec{\boldsymbol{x}}, \boldsymbol{\pi}(\vec{\boldsymbol{x}})) = 0$ for a given k and $a \notin S^k$, the player k will use the arc a when we increase his demand, see Corollary 6.17 valid when there are two arcs or two players and proved in Section 6.5.

Furthermore, there exists a neighbourhood $V(\boldsymbol{d}_0)$ such that for any $\boldsymbol{d} \in V(\boldsymbol{d}_0)$, $\vec{\boldsymbol{x}}_{\bar{\boldsymbol{S}}}^{NE}(\boldsymbol{d})$ still satisfies the condition (6.5).

Since $x_a^{NE}(\boldsymbol{d}) > 0$ for any $a \in \cup_{k\in[K]} S^k$ and $\boldsymbol{d} \in D$, we have $c_a'(x_a^{NE}(\boldsymbol{d})) > 0$. We can apply Lemma 6.8. Since $(\vec{\boldsymbol{x}}_{\boldsymbol{S}}^{NE}(\cdot), \boldsymbol{\pi}(\vec{\boldsymbol{x}}_{\boldsymbol{S}}^{NE}(\cdot)) = H_{\boldsymbol{S}}^{-1}(\boldsymbol{0}, \cdot)$ is differentiable, we get that $\vec{\boldsymbol{x}}_{\boldsymbol{S}}^{NE}$ is differentiable on D.

Finally, condition (6.4) is satisfied on D by definition. In particular $\vec{\boldsymbol{x}}_{\bar{\boldsymbol{S}}}^{NE}$ is differentiable.

We proved that $\vec{\boldsymbol{x}}^{NE}$ is differentiable on a neighbourhood of $\boldsymbol{d}_0$ for every $\boldsymbol{d}_0 \in D$. We can conclude. □

6.4 Further properties of the Nash equilibrium

6.4.1 Characterization of the Nash equilibrium

We recall the definition of an equilibrium as in Proposition 1.10, adapted to the context of parallel arcs.

Proposition 6.10. *The multiflow $\vec{\boldsymbol{x}}$ is a Nash equilibrium flow if and only if, for all k, $\boldsymbol{x}^k$ is a feasible flow for player k such that there exists $\pi^k \in \mathbb{R}$ with*

$$c_a^k(x_a) + x_a^k {c_a^k}'(x_a) \geq \pi^k \text{ for all } a \in A, \tag{6.6}$$

$$c_a^k(x_a) + x_a^k {c_a^k}'(x_a) = \pi^k \text{ for all } a \in A \text{ such that } x_a^k > 0. \tag{6.7}$$

Then $\pi^k = \pi^k(\vec{\boldsymbol{x}})$ is the marginal cost at equilibrium for the player k.

Remark 6.11. Let u and v be the two vertices, such that all arcs are (u, v). Then π^k coincides with π_v^k in Proposition 1.10, where π_u^k is fixed equal to zero.

This characterization shows that the cost of each arc in the support of a player is smaller than the marginal cost of this player. More precisely, we have the following corollary.

Corollary 6.12. *Let $\vec{\boldsymbol{x}}$ be an equilibrium flow. Then*

$$\max_{a\in \operatorname{supp}(\boldsymbol{x}^k)} c_a^k(x_a) < \pi^k(\vec{\boldsymbol{x}}) \leq \min_{b\notin \operatorname{supp}(\boldsymbol{x}^k)} c_b^k(x_b).$$

Proof. Let $a \in \operatorname{supp}(\boldsymbol{x}^k)$ and $b \notin \operatorname{supp}(\boldsymbol{x}^k)$. Proposition 6.10 gives $c_a^k(x_a) + x_a^k {c_a^k}'(x_a) = \pi^k(\vec{\boldsymbol{x}}) \leq c_b^k(x_b)$ and then $c_a^k(x_a) < c_b^k(x_b)$ since $x_a^k > 0$. □

6.4.2 Comparison of two equilibria

In order to prove Theorem 6.2 we use the following useful lemma, holding for any number of arcs and players. The intuition behind this lemma is the following. Suppose that a player puts more flow on an arc in an equilibrium $\vec{z}$ than in an equilibrium $\vec{x}$, although the cost of this arc is greater. Then this player puts more flow on any arc whose cost is lower in $\vec{z}$ than in $\vec{x}$.

Lemma 6.13. *Let* $\mathbf{d}$ *and* $\tilde{\mathbf{d}}$ *be two vectors of demands and* $\vec{x}$ *and* $\vec{z}$ *the associated equilibrium flows . For any arcs* a *and* b *and player* k*,*

$$\begin{pmatrix} z_a \geq x_a \\ z_a^k > x_a^k \\ z_b \leq x_b \end{pmatrix} \Rightarrow \left(z_b^k > x_b^k \text{ or } z_b^k = x_b^k = 0 \right).$$

Proof. Suppose then that $z_a \geq x_a$, $z_a^k > x_a^k$ and $z_b \leq x_b$.

Suppose first that $x_b^k = 0$. Then $z_b^k \geq x_b^k$, with equality iff $z_b^k = x_b^k = 0$. Suppose then that $x_b^k > 0$. Proposition 6.10 at the equilibrium $\vec{x}$ gives

$$c_b^k(x_b) + x_b^k c_b^{k\prime}(x_b) \leq c_a^k(x_a) + x_a^k c_a^{k\prime}(x_a).$$

Furthermore, $z_a^k > x_a^k \geq 0$, and Proposition 6.10 at the equilibrium $\vec{z}$ gives

$$c_a^k(z_a) + z_a^k c_a^{k\prime}(z_a) \leq c_b^k(z_b) + z_b^k c_b^{k\prime}(z_b).$$

These two equations together with the facts that c_a^k and $c_a^{k\prime}$ are increasing, $z_a \geq x_a$ and $z_a^k > x_a^k$ give that

$$c_b^k(x_b) + x_b^k c_b^{k\prime}(x_b) < c_b^k(z_b) + z_b^k c_b^{k\prime}(z_b).$$

Finally, since $z_b \leq x_b$ and c_b^k, $c_b^{k\prime}$ are increasing, we must have $z_b^k > x_b^k$ in order to satisfy this last equation. □

Lemma 6.13 gives the following corollary, which precises the situation when the flow on each arc is the same for two vectors of demands, and whose proof is straightforward.

Corollary 6.14. *Let* $\mathbf{d}$ *and* $\tilde{\mathbf{d}}$ *be two vectors of demands and* $\vec{x}$ *and* $\vec{z}$ *the associated equilibrium flows. Suppose that the aggregated flows are the same in both equilibria:* $\boldsymbol{x} = \boldsymbol{z}$. *Then for every player* k*, there is an arc* $a \in A$ *with* $z_a^k > x_a^k$ *if and only if* $\tilde{d}^k > d^k$.

Furthermore, when $\tilde{d}^k > d^k$*, we have* $z_a^k > x_a^k$ *or* $z_a^k = x_a^k = 0$ *for every arc* $a \in A$.

6.5 Evolution of the equilibrium flows after the transfer

We fix $\boldsymbol{d}$ a vector of demands and two players i and j, with $d^i \leq d^j$. We consider the equilibrium after player i transfers a part $\delta > 0$ of his demand to player j. We define $\boldsymbol{d}_\delta$ the vector of demands after player i has transfered a part δ of his demand to player j:

$$d^i_\delta = d^i - \delta, \qquad d^j_\delta = d^j + \delta, \quad \text{and} \quad d^k_\delta = d^k \quad \text{for } k \neq i, j.$$

6.5.1 Proof of Theorem 6.2

This section is devoted to the proof of Theorem 6.2. More precisely, we prove the following Proposition:

Proposition 6.15. *Suppose that there are two arcs or two players and let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_\delta)$. Then for every arc $a \in A$,*

$$\begin{aligned} y^i_a < x^i_a \quad &\text{or} \quad \left(y^i_a = x^i_a = 0 \ \text{ and } \ y_a \geq x_a\right), \\ y^j_a > x^j_a \quad &\text{or} \quad \left(y^j_a = x^j_a = 0 \ \text{ and } \ y_a \leq x_a\right). \end{aligned}$$

Moreover, when there are two arcs,

$$(y^k_a - x^k_a)(y_a - x_a) \leq 0 \quad \text{for } k \neq i, j.$$

Theorem 6.2 is a direct corollary of Proposition 6.15.

Proof of Proposition 6.15. When $\boldsymbol{y} = \boldsymbol{x}$, the result holds for any number of arcs and classes according to Corollary 6.14. Suppose then that $\boldsymbol{y} \neq \boldsymbol{x}$.

We first consider the case where there are two arcs, a and b. Suppose without loss of generality that $y_a > x_a$ and $y_b < x_b$. We first prove the evolution for the arc a.

Consider a player $k \neq j$, then $d^k_\delta \leq d^k$. Suppose that $y^k_a > x^k_a$, Lemma 6.13 applied with $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}$ gives that $y^k_b \geq x^k_b$. Thus $d^k_\delta = y^k_a + y^k_b > x^k_a + x^k_b = d^k$ which is impossible. Then $y^k_a \leq x^k_a$. In particular, $y^i_a \leq x^i_a$.
Summing over all these players, we find that in order to satisfy $y_a > x_a$, we have $y^j_a > x^j_a$.

We prove now the evolution for the arc b. Consider a player $k \neq i, j$. Since $d^k_\delta = d^k$ and $y^k_a \leq x^k_a$, we must have $y^k_b \geq x^k_b$. For player j, since $y^j_a > x^j_a$, Lemma 6.13 applied with $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}$ and $k = j$ ensures that $y^j_b > x^j_b$ or $y^j_b = x^j_b = 0$.
Again, summing over all players except i, we find that in order to satisfy $y_b < x_b$, we have $y^i_b < x^i_b$.

To conclude, apply Lemma 6.13 with $\vec{\boldsymbol{z}} = \vec{\boldsymbol{x}}$, $\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$, $k = i$ and a and b exchanged. It gives that whenever $y_a^i = x_a^i$, they are both equal to zero.

Consider now the case where there are two players. Since $\boldsymbol{y} \neq \boldsymbol{x}$, there is an arc $a_0 \in A$ such that $y_{a_0} \neq x_{a_0}$. Without loss of generality $y_{a_0} > x_{a_0}$, and there is an arc $b_0 \in A$ such that $y_{b_0} < x_{b_0}$. We have $y_{a_0}^k > x_{a_0}^k$ for a player $k \in \{i, j\}$ and $y_{b_0}^\ell < x_{b_0}^\ell$ for a player $\ell \in \{i, j\}$.

For every arc b such that $y_b \leq x_b$, Lemma 6.13 applied with $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}$ and $a = a_0$ gives that $y_b^k > x_b^k$ or $y_b^k = x_b^k = 0$. In particular when applied with b_0, we get $k \neq \ell$. Then, in order to satisfy $y_b \leq x_b$, we must have $y_b^\ell \leq x_b^\ell$.

For every arc a such that $y_a \geq x_a$, apply Lemma 6.13 with $\vec{\boldsymbol{z}} = \vec{\boldsymbol{x}}$, $\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$, $k = \ell$, $a = b_0$ and $b = a$. It gives that $y_a^\ell < x_a^\ell$ or $y_a^\ell = x_a^\ell = 0$. Again, in order to satisfy $y_a \geq x_a$, we must have $y_a^k \geq x_a^k$.

Summing these inequalities over all arcs, we get that $d_\delta^k \geq d^k$ and $d_\delta^\ell \leq d^\ell$, and we conclude that $k = j$ and $\ell = i$.

We have proved that, for player i, $y_a^i < x_a^i$ or $y_a^i = x_a^i = 0$ for every arc a such that $y_a \geq x_a$, and $y_b^i \leq x_b^i$ for every arc b such that $y_b \leq x_b$. It remains to prove that the inequality is strict for arcs b such that $y_b < x_b$. Consider then such an arc b. Since $y_b^j \geq x_b^j$ and there are only two players, we must have a strict inequality for player i too: $y_b^i < x_b^i$. The same argument for player j with reverse inequalities allows to conclude. □

6.5.2 Corollaries

In addition to Theorem 6.2, Proposition 6.15 gives other corollaries that will be useful for the remaining of the chapter. The first one is straightforward.

Corollary 6.16. *Suppose that there are two arcs or two players, and let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_\delta)$. Then,*

1. $\operatorname{supp}(\boldsymbol{y}^i) \subseteq \operatorname{supp}(\boldsymbol{x}^i)$ *and* $\operatorname{supp}(\boldsymbol{x}^j) \subseteq \operatorname{supp}(\boldsymbol{y}^j)$.
2. $\{a \in A, y_a < x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^i)$ *and* $\{a \in A, y_a > x_a\} \subseteq \operatorname{supp}(\boldsymbol{y}^j)$.

We consider now the "limit" cases when there are two arcs or two players. These cases appear when the cost of an arc with no flow is equal to the marginal cost.

Corollary 6.17. *Suppose that there are two arcs or two players, and let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_\delta)$.*

1. *Let $b \notin \operatorname{supp}(\boldsymbol{x}^j)$ be such that $c_b^j(x_b) = \pi^j(\vec{\boldsymbol{x}})$. Then $b \in \operatorname{supp}(\boldsymbol{y}^j)$.*
2. *Let $b \notin \operatorname{supp}(\boldsymbol{y}^i)$ be such that $c_b^i(y_b) = \pi^i(\vec{\boldsymbol{y}})$. Then $b \in \operatorname{supp}(\boldsymbol{x}^i)$.*

Proof. We prove only the first point since the proof of the second one is exactly the same, where we replace j by i and switch $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$.

Suppose first that $y_b > x_b$, then Proposition 6.15 gives $y_b^j > x_b^j$, and in particular $b \in \operatorname{supp}(\boldsymbol{y}^j)$. Suppose then that $y_b \leq x_b$ and for a contradiction that $b \notin \operatorname{supp}(\boldsymbol{y}^j)$. For every arc $a \neq b$, we have since c_b^j is increasing

$$c_a^j(x_a) + x_a^j {c_a^j}'(x_a) \geq \pi^j(\vec{\boldsymbol{x}}) = c_b^j(x_b) \geq c_b^j(y_b) \geq \pi^j(\vec{\boldsymbol{y}}). \tag{6.8}$$

Consider then an arc $a \in \operatorname{supp}(\boldsymbol{y}^j)$. Equation (6.8) gives

$$c_a^j(y_a) + y_a^j {c_a^j}'(y_a) = \pi^j(\vec{\boldsymbol{y}}) \leq c_a^j(x_a) + x_a^j {c_a^j}'(x_a).$$

Moreover, Proposition 6.15 gives that $y_a^j > x_a^j$, and since c_a^j and ${c_a^j}'$ are increasing we get that $y_a < x_a$.

Since the total demand is the same before and after the transfer, we get a contradiction in the case where there are two arcs. In the case where there are two players, there must be an arc a_0 such that $y_{a_0} > x_{a_0}$. According to Corollary 6.16, we have $a_0 \in \operatorname{supp}(\boldsymbol{y}^j)$. Since we proved that $y_a < x_a$ for every $a \in \operatorname{supp}(\boldsymbol{y}^j)$, we get a contradiction. □

6.6 Evolution of the social cost at equilibrium when players have same cost functions

In this section, we suppose that every player has the same cost functions: $c_a^k = c_a$ is independent of k for every arc a. In this context, we redefine the function Q giving the social cost, which does not depend on the multiflow $\vec{\boldsymbol{x}}$ but only on the aggregated flow $\boldsymbol{x} = (x_a)_{a \in A}$:

$$Q(\boldsymbol{x}) = \sum_{a \in A} x_a c_a(x_a).$$

Recall that we consider the equilibrium after player i transfers a part $\delta > 0$ of his demand to player j, and that the vector of demands after the transfer is denoted by $\boldsymbol{d}_\delta$. We define $\bar{\boldsymbol{x}}$ and $\bar{Q}$ on $[0, d^i]$ by $\bar{\boldsymbol{x}}(\delta) = \boldsymbol{x}^{NE}(\boldsymbol{d}_\delta)$ and $\bar{Q}(\delta) = Q(\bar{\boldsymbol{x}}(\delta))$.

We prove Theorem 6.3, i.e. that $\bar{Q}$ is nonincreasing when there are two arcs or two players. This result is a generalization of Theorem 3.23 of Wan (2012a) where there is no nonatomic players. Wan (2012a) proved that when there are two arcs the social cost at equilibrium cannot increase when two players merge, i.e. in our context when player i tranfers all his demand to player j. The question whether our result remains valid with nonatomic players deserves future work.

Note that this result is not valid with three arcs, as shown by Huang (2011), or when we allow player-specific cost functions, see Section 6.7.

In order to prove Theorem 6.3, we consider the situation when the support of the players remains the same after the transfer. We treat separately the case where players i and j have the same support in Section 6.6.2, and the case where they have a different one in Section 6.6.3. Hence the proof comes form the monotonicity of the supports and the continuity of the equilibrium cost, see Section 6.6.4.

We use extensively the following well-known result.

Lemma 6.18 (Orda et al. (1993))**.** *Let $\boldsymbol{d}$ be a vector of demands and two players k_1 and k_2 be such that $d^{k_1} \leq d^{k_2}$. Let $\vec{\boldsymbol{x}}$ the equilibrium flow, then $x_a^{k_1} \leq x_a^{k_2}$ for every arc $a \in A$. In particular,* $\operatorname{supp}(\boldsymbol{x}^{k_1}) \subseteq \operatorname{supp}(\boldsymbol{x}^{k_2})$ *and $\pi^{k_1}(\vec{\boldsymbol{x}}) \leq \pi^{k_2}(\vec{\boldsymbol{x}})$.*

6.6.1 Preliminary results when the support of each player remains constant

Given a flow $\boldsymbol{z} = (z_a)_{a\in A}$ and an arc $a \in A$, we define the vector $\boldsymbol{z}^{-a} \in \mathbb{R}_+^{|A|-1}$ by $\boldsymbol{z}^{-a} = (z_{a'})_{a'\neq a}$. Then for every arc $a \in A$, we consider the function $Q_a : \mathbb{R}_+^{|A|-1} \to \mathbb{R}_+$ defined by

$$Q_a(\boldsymbol{z}^{-a}) = \sum_{a'\neq a} z_{a'} c_{a'}(z_{a'}) + \left(d - \sum_{a'\neq a} z_{a'}\right) c_a \left(d - \sum_{a'\neq a} z_{a'}\right),$$

where $d = \sum_{k\in[K]} d^k$ is the total demand.

For a feasible flow $\boldsymbol{z}$, we have $Q_a(\boldsymbol{z}^{-a}) = Q(\boldsymbol{z})$. Furthermore, Q_a is differentiable since the cost functions are differentiable. For a feasible flow $\boldsymbol{x}$, we have for any $b \neq a$

$$\frac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a}) = c_b(x_b) + x_b c_b'(x_b) - (c_a(x_a) + x_a c_a'(x_a))\,.$$

Lemma 6.19. *Suppose that the cost functions are twice continuously differentiable. Let $\delta_0 > 0$ and suppose that for some $a \in A$ and any δ small enough, the support of each player keeps remains constant after a transfer of δ, and*

$$\sum_{b\neq a} \frac{\partial Q_a}{\partial z_b}\left(\bar{\boldsymbol{x}}^{-a}(\delta_0)\right)\left(\bar{x}_b(\delta_0+\delta) - \bar{x}_b(\delta_0)\right) \leq 0. \tag{6.9}$$

Then $\bar{Q}'(\delta_0) \leq 0$.

Proof. According to Proposition 6.9, the equilibrium flow $\vec{\boldsymbol{x}}^{NE}$ is differentiable, since the support of the players remains constant. In particular, $\bar{\boldsymbol{x}}$ and then $\bar{Q}$ are differentiable. Moreover, $\bar{Q}(\delta) = Q(\bar{\boldsymbol{x}}(\delta)) = Q_a(\bar{\boldsymbol{x}}^{-a}(\delta))$.

Then,

$$\begin{aligned}\bar{Q}'(\delta_0) &= \sum_{b \neq a} \frac{\partial Q_a}{\partial z_b}(\bar{\boldsymbol{x}}^{-a}(\delta_0))\bar{x}_b'(\delta_0) \\ &= \lim_{\delta \to 0} \sum_{b \neq a} \frac{\partial Q_a}{\partial z_b}(\bar{\boldsymbol{x}}^{-a}(\delta_0)) \frac{\bar{x}_b(\delta_0 + \delta) - \bar{x}_b(\delta_0)}{\delta} \\ &\leq 0.\end{aligned}$$

□

6.6.2 When the two players keep the same common support

We look first at the situation when players i and j have the same support, and this support is the same before and after the tranfer. In this case, we can describe explicitely the flows.

Proposition 6.20. *Let $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d})$ and $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_\delta)$ and suppose that* $\mathrm{supp}(\boldsymbol{x}^i) = \mathrm{supp}(\boldsymbol{x}^j) = \mathrm{supp}(\boldsymbol{y}^i) = \mathrm{supp}(\boldsymbol{y}^j)$. *Then $Q(\boldsymbol{x}) = Q(\boldsymbol{y})$.*

Proof. Denote by S_0 be the common support. Let $\delta^{min} = \min(\delta_0, \delta_1)$, where

$$\delta_0 = \min_{a \in S_0} \left(x_a^i c_a'(x_a)\right) \sum_{a \in S_0} \frac{1}{c_a'(x_a)},$$

$$\delta_1 = \left(\min_{b \notin S_0} c_b(x_b) - \pi^j(\vec{\boldsymbol{x}})\right) \sum_{a \in S_0} \frac{1}{c_a'(x_a)}.$$

Note that δ^{min} is well defined since $c_a'(x) > 0$ for every $x > 0$, and is nonnegative, according to Corollary 6.12. We show that when $\delta \leq \delta^{min}$, the players keep the same support after the transfer.

Let then $\delta \leq \delta^{min}$. We consider the flow $\vec{\boldsymbol{z}}$ defined by

$$z_a^k = x_a^k + \frac{d_\delta^k - d^k}{\beta c_a'(x_a)} 1_{\{a \in S_0\}} \qquad \text{for every player } k,$$

where

$$\beta = \sum_{a \in S_0} \frac{1}{c_a'(x_a)}.$$

Then we show that $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}$. Since $z_a = x_a$ for every arc a, it will give the result.

We first check that $\vec{\boldsymbol{z}}$ is an admissible flow. We have more precisely

$$\begin{aligned} z_a^i &= x_a^i - \frac{\delta}{\beta c_a'(x_a)} 1_{\{a \in S_0\}} \\ z_a^j &= x_a^j + \frac{\delta}{\beta c_a'(x_a)} 1_{\{a \in S_0\}} \\ z_a^k &= x_a^k \qquad \text{for } k \neq i, j. \end{aligned}$$

For each player k and arc a, we have $z_a^k \geq 0$ since $\delta \leq \delta_0$. Furthermore, we can check easily that for every player k, $\sum_{a \in A} z_a^k = d_\delta^k$. Hence, the flow $\vec{\boldsymbol{z}}$ is admissible.

We check now that $\vec{\boldsymbol{z}}$ is an equilibrium flow, by checking that it satisfies the conditions of Proposition 6.10 with $\pi^k(\vec{\boldsymbol{z}}) = \pi^k(\vec{\boldsymbol{x}}) + \frac{d_\delta^k - d^k}{\beta}$ for every player k. Consider first a player $k \in \{i, j\}$. For every arc $a \in S_0$.

$$\begin{aligned} c_a(z_a) + z_a^k c_a'(z_a) &= c_a(x_a) + x_a^k c_a'(x_a) + \frac{d_\delta^k - d^k}{\beta} \\ &= \pi^k(\vec{\boldsymbol{x}}) + \frac{d_\delta^k - d^k}{\beta} \qquad \text{with (6.7).} \end{aligned}$$

Consider now an arc $a \notin S_0$. In particular, $z_a^k = x_a^k = 0$. We have

$$c_a(z_a) = c_a(x_a) \geq \pi^j(\vec{\boldsymbol{x}}) + \frac{\delta}{\beta} \geq \pi^k(\vec{\boldsymbol{x}}) + \frac{d_\delta^k - d^k}{\beta},$$

where the first equality holds since $z_a = x_a$, the first inequality since $\delta \leq \delta_1$, and the last inequality for player i since $\pi^j(\vec{\boldsymbol{x}}) \geq \pi^i(\vec{\boldsymbol{x}})$, according to Lemma 6.18. The conditions of Proposition 6.10 are satisfied for any player $k \in \{i, j\}$.

Consider then a player $k \neq \{i, j\}$, we have $d_\delta^k = d^k$ and we retrieve

$$\begin{aligned} c_a(z_a) + z_a^k c_a'(z_a) &= \pi^k(\vec{\boldsymbol{x}}) \qquad && \text{for } a \in \text{supp}(\boldsymbol{x}^k) \\ c_a(z_a) &\geq \pi^k(\vec{\boldsymbol{x}}) && \text{for } a \notin \text{supp}(\boldsymbol{x}^k). \end{aligned}$$

Then, using Proposition 6.10, $\vec{\boldsymbol{z}}$ is an equilibrium, with $\pi^k(\vec{\boldsymbol{z}}) = \pi^k(\vec{\boldsymbol{x}}) + \frac{d_\delta^k - d^k}{\beta}$ for every player k. By uniqueness of the equilibrium, $\vec{\boldsymbol{z}} = \vec{\boldsymbol{y}}$. □

Remark 6.21. Proposition 6.20 is valid for any number of arcs and players.

Remark 6.22. We can extend Proposition 6.20 in a more general case. It remains valid for any demand $\tilde{\boldsymbol{d}}$ in the neighbourhood of $\boldsymbol{d}$ defined by

$$\begin{cases} \text{there exists } S_0 \text{ s.t. } \operatorname{supp}(\boldsymbol{x}^k) = S_0 \text{ for all } k \text{ s.t. } \tilde{d}^k \neq d^k, \\ d^k - \tilde{d}^k \leq \min_{a \in S_0} \left(x_a^k c_a'(x_a) \right) \beta, \\ \tilde{d}^k - d^k \leq \left(\min_{a \notin S_0} c_a(x_a) - \pi^k(x_a) \right) \beta, \\ \tilde{d} = \sum_{k \in [K]} \tilde{d}^k = \sum_{k \in [K]} d^k = d. \end{cases}$$

We can also extend this result in the case where players have specific cost functions. It remains valid as soon as for all players k such that $\tilde{d}^k \neq d^k$, $\sum_{a \in S_0} \frac{1}{c_a^{k\prime}(x_a)}$ does not depend on k. For example when the cost functions are the same up to an additive constant.

6.6.3 When the two players keep the same different support

We suppose now that the players keep the same support, but there is at least an arc in the support of one player which is not in the support of the other one.

Since the players have the same cost functions, we can precise the Corollary 6.16 coming from Proposition 6.15.

Corollary 6.23. *Suppose that there are two arcs or two players. Let δ_0 and $\delta > 0$ and note $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_{\delta_0})$, $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_{\delta_0+\delta})$. Suppose that* $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i) \neq \operatorname{supp}(\boldsymbol{x}^j) = \operatorname{supp}(\boldsymbol{y}^j)$. *Suppose that $\boldsymbol{x} \neq \boldsymbol{y}$, then*

$$\{a \in A, y_a < x_a\} = \operatorname{supp}(\boldsymbol{x}^i),$$

$$\{a \in A, y_a > x_a\} = \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i).$$

Proof. When there are two arcs, Corollary 6.16 gives that $\{a \in A, y_a < x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^i)$, and $\{a \in A, y_a > x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i)$. A cardinality argument gives the equalities.

When there are two players, let $a \in \operatorname{supp}(\boldsymbol{x}^i)$. Lemma 6.18 gives that $a \in \operatorname{supp}(\boldsymbol{x}^j)$, and we have then

$$2c_a(x_a) + x_a c_a'(x_a) = \pi^i(\vec{\boldsymbol{x}}) + \pi^j(\vec{\boldsymbol{x}}). \tag{6.10}$$

Since $\operatorname{supp}(\vec{\boldsymbol{x}}) = \operatorname{supp}(\vec{\boldsymbol{y}})$, we have also

$$2c_a(y_a) + y_a c_a'(y_a) = \pi^i(\vec{\boldsymbol{y}}) + \pi^j(\vec{\boldsymbol{y}}). \tag{6.11}$$

Since $\boldsymbol{x} \neq \boldsymbol{y}$, there is an arc a such that $y_a < x_a$. According to Corollary 6.16, we have $a \in \operatorname{supp}(\boldsymbol{x}^i)$. Then, since the function $u \mapsto 2c_a(u)+uc_a'(u)$ is increasing, we get with Equations (6.10) and (6.11)

$$\pi^i(\vec{\boldsymbol{y}}) + \pi^j(\vec{\boldsymbol{y}}) < \pi^i(\vec{\boldsymbol{x}}) + \pi^j(\vec{\boldsymbol{x}}).$$

In particular, if there is an arc $a \in \operatorname{supp}(\boldsymbol{x}^i)$ with $y_a \geq x_a$, Equations (6.10) and (6.11) give $\pi^i(\vec{\boldsymbol{y}}) + \pi^j(\vec{\boldsymbol{y}}) \geq \pi^i(\vec{\boldsymbol{x}}) + \pi^j(\vec{\boldsymbol{x}})$, which is not possible. Then $y_a < x_a$ for all arcs of $\operatorname{supp}(\boldsymbol{x}^i)$. Since Corollary 6.16 gives that $\{a \in A, y_a < x_a\} \subseteq \operatorname{supp}(\boldsymbol{x}^i)$, we can conclude that $\{a \in A, y_a < x_a\} = \operatorname{supp}(\boldsymbol{x}^i)$.

Finally, let $a \in \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i)$. Since there are two players and $\operatorname{supp}(\vec{\boldsymbol{x}}) = \operatorname{supp}(\vec{\boldsymbol{y}})$, we have $x_a = x_a^j$ and $y_a = y_a^j$, and Proposition 6.15 gives that $y_a > x_a$. Conversely every arc a such that $y_a > x_a$ belongs to $\operatorname{supp}(\boldsymbol{x}^j)$ according to Corollary 6.16, but not to $\operatorname{supp}(\boldsymbol{x}^i)$ as we just proved. Then $\{a \in A, y_a > x_a\} = \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i)$. □

In order to prove Theorem 6.3 we use the following lemmas.

Lemma 6.24. *Let δ_0 and $\delta > 0$ and note $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_{\delta_0})$, $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_{\delta_0+\delta})$. Suppose that $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i) \neq \operatorname{supp}(\boldsymbol{x}^j) = \operatorname{supp}(\boldsymbol{y}^j)$. For every $a \in \operatorname{supp}(\boldsymbol{x}^i)$ and $b \in \operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i)$,*

$$\frac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a})(y_b - x_b) \leq 0.$$

Proof. Since $b \notin \operatorname{supp}(\boldsymbol{x}^i)$, Corollary 6.16 gives that $y_b \geq x_b$ and since $a \in \operatorname{supp}(\boldsymbol{x}^i)$, Corollary 6.12 gives that $c_b(x_b) > c_a(x_a)$.

Consider the set I_b of players k such that $b \in \operatorname{supp}(\boldsymbol{x}^k)$. Then I_b is nonempty and for every player $k \in I_b$ we have, according to Proposition 6.10,

$$c_b(x_b) + x_b^k c_b'(x_b) \leq c_a(x_a) + x_a^k c_a'(x_a).$$

By summing over these players, we get

$$\begin{aligned}
&|I_b| c_b(x_b) + x_b c_b'(x_b) \\
&\quad \leq |I_b| c_a(x_a) + \left(x_a - \sum_{k \notin I_b} x_a^k \right) c_a'(x_a) \\
&\quad < |I_b| c_a(x_a) + x_a c_a'(x_a).
\end{aligned}$$

Since $|I_b| \geq 1$, and $c_b(x_b) > c_a(x_a)$ we must have

$$c_b(x_b) + x_b c_b'(x_b) < c_a(x_a) + x_a c_a'(x_a),$$

i.e. $\frac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a}) < 0$. □

Lemma 6.25. *Suppose that there are only two players. Let* δ_0 *and* $\delta > 0$ *and note* $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_{\delta_0})$, $\vec{\boldsymbol{y}} = \vec{\boldsymbol{x}}^{NE}(\boldsymbol{d}_{\delta_0+\delta})$. *Suppose that* $\operatorname{supp}(\boldsymbol{x}^i) = \operatorname{supp}(\boldsymbol{y}^i) \neq \operatorname{supp}(\boldsymbol{x}^j) = \operatorname{supp}(\boldsymbol{y}^j)$. *For* δ *small enough, there exists* $a \in \operatorname{supp}(\boldsymbol{x}^i)$ *such that for every* $b \in \operatorname{supp}(\boldsymbol{x}^i) \setminus \{a\}$,

$$\frac{\partial Q_a}{\partial z_b}(\boldsymbol{x}^{-a})(y_b - x_b) \leq 0.$$

Proof. Consider an arc $a \in \arg\max\{c_{a'}(x_{a'}), a' \in \operatorname{supp}(\boldsymbol{x}^i)\}$. Let then $b \in \operatorname{supp}(\boldsymbol{x}^i) \setminus \{a\}$.

Since $\operatorname{supp}(\boldsymbol{x}^i) \subseteq \operatorname{supp}(\boldsymbol{x}^j)$ according to Lemma 6.18, a and $b \in \operatorname{supp}(\boldsymbol{x}^j)$. Summing Equations (6.7) for both players, we get

$$2c_a(x_a) + x_a c_a'(x_a) = 2c_b(x_b) + x_b c_b'(x_b). \tag{6.12}$$

According to the definition of a, we have $c_a(x_a) \geq c_b(x_b)$ and then

$$c_a(x_a) + x_a c_a'(x_a) \leq c_b(x_b) + x_b c_b'(x_b), \text{ i.e. } \frac{\partial Q_a}{\partial x_b}(\boldsymbol{x}^{-a}) \geq 0. \tag{6.13}$$

Furthermore, Corollary 6.23 gives that $y_b < x_b$. □

We can now prove the result.

Proposition 6.26. *Suppose that there are two arcs or two players. The cost at equilibrium* $\bar{Q}$ *is nonincreasing on the set of transfers such that the support of each player remains constant after the transfer.*

Proof. Let then δ_0 such that the players keep the same support after a transfer of δ_0. We prove that $\bar{Q}'(\delta) \leq 0$ for any $\delta \in [0, \delta_0]$.

When there are two arcs, the result is an immediate corollary of Proposition 6.20 or Lemmas 6.19 and 6.24.

When there are two players, if they have the same support, the result is a corollary of Proposition 6.20. If they have a different support, let $a \in \operatorname{supp}(\boldsymbol{x}^i)$ as in Lemma 6.25. Then, we prove that Equation (6.9) is satisfied. Indeed, we use Lemma 6.24 for the sum over arcs in $\operatorname{supp}(\boldsymbol{x}^j) \setminus \operatorname{supp}(\boldsymbol{x}^i)$, Lemma 6.25 for the sum over arcs in $\operatorname{supp}(\boldsymbol{x}^i) \setminus \{a\}$. The sum over arcs not in $\operatorname{supp}(\boldsymbol{x}^j)$ is zero since there is no flow on these arcs before and after the transfer. We can conlude with Lemma 6.19. □

Note that when there are two arcs, Q_a is a function of only one variable. We can directly take its derivative, without using Lemma 6.19. Then, we do not need to use the differentiability of $\vec{\boldsymbol{x}}^{NE}$. The result holds then even if we do not assume that the cost functions are twice continuously differentiable.

6.6.4 Proof of Theorem 6.3

According to Theorem 6.2, and more precisely Corollary 6.16, the support of player i is nonincreasing with respect to the inclusion, and the support of player j is nondecreasing. Hence, when δ goes from 0 to d^i, the two players change supports only a finite number of times. According to Proposition 6.26, the equilibrium cost $\bar{Q}$ is nonincreasing on a finite number of sub-intervals of $[0, d^i]$ whose closure is $[0, d^i]$ itself.

Since $\bar{Q}$ is continuous, using Theorem 6.1, we can conclude that $\bar{Q}$ is nonincreasing on $[0, d^i]$. It proves Theorem 6.3.

6.7 Discussion

A natural question is whether Theorems 6.2 and 6.3 are valid when there are three arcs or three players. Another question is whether Theorem 6.3 remains valid when we allow player-specific cost functions. The answer is “no” for Theorem 6.3 and partially for Theorem 6.2, as shown by the following examples.

6.7.1 When there are three arcs and three players

We introduce the example of Huang (2011).

Consider a graph with three arcs a, b and c, and three players $1, 2$ and 3. Suppose that the players have the same cost functions

$$c_a(x) = 20x + 5000, \quad c_b(x) = x^2 + 500, \quad \text{and} \quad c_c(x) = x^{11}.$$

When the vector of demand is $\boldsymbol{d} = (0.1, 20.9, 200)$, the (rounded) flows at equilibrium are in the following table and the equilibrium cost is 1 558 627.

Arc		a	b	c
Flow	player 1	0	0	0.1
	player 2	0	20.18	0.72
	player 3	152.50	46.38	1.12
Total flow		152.50	66.55	1.95

After the transfer of $\delta = 0.1$ from player 3 to player 2 we have a vector of demand $\boldsymbol{d}_\delta = (0, 21, 200)$. The (rounded) flows at equilibrium are in the following table and the equilibrium cost is 1 558 633. In particular, contrary to Theorem 6.3, the cost has increased after the transfer.

Arc		a	b	c
Flow	player 1	0	0	0
	player 2	0	20.24	0.76
	player 3	152.49	46.33	1.18
Total flow		152.49	66.57	1.94

Moreover, the part concerning players that keep the same demand in Theorem 6.2 does not hold, since $(y_a^3 - x_a^3)(y_a - x_a) > 0$, where $\vec{\boldsymbol{x}}$ (resp. $\vec{\boldsymbol{y}}$) is the equilibrium flow before (resp. after) the transfer. However the other part of Theorem 6.2 still holds, namely the part stating that on each arc the flow of the player giving his demand decreases or remains constant equal to zero, and the flow of the other player increases or remains constant equal to zero.

Whether this result holds in general for graphs with more than two arcs and two players is an open question.

6.7.2 When we allow player-specific cost functions

We introduce an example that contradicts Theorem 6.3 when we allow player-specific cost functions.

Consider a graph with two arcs a and b, and two players 1 and 2.

Let the cost on arc b for player 1 (resp. on arc a for player 2) be prohibitively high, in such a way that at equilibrium, for every repartition of the demand, player 1 (resp. 2) puts all his demand on arc a (resp. b). Let the total demand equal 5, and the costs $c_a^1(x) = 2x$, $c_b^2(x) = x$. When $d^1 = 3$ and $d^2 = 2$, the cost at equilibrium is 8, while after a transfer of 1, i.e. when $d^1 = 4$ and $d^2 = 1$, the cost at equilibrium is 9. The result of Theorem 6.3 does not hold.

6.7.3 Consequences on the price of anarchy

A consequence of Theorem 6.3 is that for a fixed total demand, the repartition giving the worst Nash equilibrium is when every player has the same demand. More precisely, let d be the fixed total demand and denote by $\triangle d$ the set of repartitions of demand satisfying this total demand:

$$\triangle d = \left\{ \boldsymbol{d} \in \mathbb{R}_+^K, \quad \sum_{k \in K} d^k = d \right\}.$$

We define then for every $\boldsymbol{d} \in \triangle d$, the cost at equilibrium $Q^{NE}(\boldsymbol{d})$. We define furthermore the symmetric repartition $\bar{\boldsymbol{d}}$ such that $\bar{d}^k = \frac{D}{K}$ for every $k \in K$. Then, we have

$$\max_{\boldsymbol{d} \in \triangle d} Q^{NE}(\boldsymbol{d}) = Q^{NE}(\bar{\boldsymbol{d}}).$$

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Abstract

We consider congestion games on graphs. In *nonatomic games*, we are given a set of infinitesimal players. Each player wants to go from one vertex to another by taking a route of minimal cost, the cost of a route depending on the number of players using it. In *atomic splittable games*, we are given a set of players with a non-negligible demand. Each player wants to ship his demand from one vertex to another by dividing it among different routes. In these games, we reach a Nash equilibrium when every player has chosen a minimal-cost strategy.

The existence of a Nash equilibrium is ensured under mild conditions. The main issues are the uniqueness, the computation, the efficiency and the sensitivity of the Nash equilibrium. Many results are known in the specific case where all players are impacted in the same way by the congestion. The goal of this thesis is to generalize these results in the case where we allow player-specific cost functions.

We obtain results on uniqueness of the equilibrium in nonatomic games. We give two algorithms able to compute a Nash equilibrium in nonatomic games when the cost functions are affine. We find a bound on the price of anarchy for some atomic splittable games, and prove that it is unbounded in general, even when the cost functions are affine. Finally we find results on the sensitivity of the equilibrium to the demand in atomic splittable games.

Résumé

Nous considérons des jeux de congestion sur des graphes. Dans les jeux *non-atomiques*, nous considérons un ensemble de joueurs infinitésimaux. Chaque joueur veut aller d'un sommet à un autre en choisissant une route de coût minimal. Le coût de chaque route dépend du nombre de joueur la choisissant. Dans les jeux *atomiques divisibles*, nous considérons un ensemble de joueurs ayant chacun une demande à transférer d'un sommet à un autre, en la subdivisant éventuellement sur plusieurs routes. Dans ces jeux, un équilibre de Nash est atteint lorsque chaque joueur a choisi une stratégie de coût minimal.

L'existence d'un équilibre de Nash est assurée sous de faibles hypothèses. Les principaux sujets sont l'unicité, le calcul, l'efficacité et la sensibilité de l'équilibre de Nash. De nombreux résultats sont connus dans le cas où les joueurs sont tous affectés de la même façon par la congestion. Le but de cette thèse est de généraliser ces résultats au cas où les joueurs ont des fonctions de coût différentes.

Nous obtenons des résultats sur l'unicité de l'équilibre dans les jeux non-atomiques. Nous donnons deux algorithmes capables de calculer un équilibre dans les jeux non-atomiques lorsque les fonctions de coût sont affines. Nous obtenons une borne sur le prix de l'anarchie pour certains jeux atomiques divisibles et prouvons qu'il n'est pas borné en général, même lorsque les fonctions sont affines. Enfin, nous prouvons des résultats sur la sensibilité de l'équilibre par rapport à la demande dans les jeux atomiques divisibles.