Pricing convertible bonds with call protection

JF Chassagneux

ENSAE - CREST, Paris 7 - PMA

Joint work with S. Crépey and A. Rahal

17 avril 2009

Callable American Option

- Game Option or Dynkin Game: σ stopping time of the seller, τ stopping time of the buyer. The seller pays to the buyer:

$$h(X_\sigma)\mathbf{1}_{\{\sigma<\tau\leq T\}} + l(X_\tau)\mathbf{1}_{\{T>\sigma\geq\tau\}} + g(X_T)\mathbf{1}_{\{\sigma=\tau=T\}} \ .$$

- Call protection: an increasing finite sequence of stopping time $(\theta_k)_{k\geq 0}$, American option callable on $[\theta_{2l+1}, \theta_{2l+2})$.

Option pricing using BSDEs

- SDE for X: $X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$
- European option $\longleftrightarrow$ BSDEs

$$Y_t = g(X_T) + \int_t^T f(X_u, Y_u, Z_u)\mathrm{d}u - \int_t^T Z_u \mathrm{d}W_u$$

- American option $\longleftrightarrow$ simply reflected BSDEs

$$Y_t = g(X_T) + \int_t^T f(X_u, Y_u, Z_u)\mathrm{d}u - \int_t^T Z_u \mathrm{d}W_u + \int_t^T \mathrm{d}K_u^+$$

$$Y_t \geq l(X_t) \ , \quad t \leq T \quad \text{and} \quad \int_0^T (\, Y_u - l(X_u)\,)\mathrm{d}K_u^+ \; = \; 0$$

Doubly Reflected BSDEs

- Game Option $\longleftrightarrow$ doubly reflected BSDEs

$$Y_t = g(X_T) + \int_t^T f(X_u, Y_u, Z_u)\mathrm{d}u - \int_t^T Z_u \mathrm{d}W_u + \int_t^T \mathrm{d}K_u^+ - \int_t^T \mathrm{d}K_u^-$$

$$h(X_t) \geq Y_t \geq l(X_t),\ \int_0^T (Y_u - l(X_u))\mathrm{d}K_u^+ = \int_0^T (\ Y_u - h(X_u)\ )\mathrm{d}K_u^- = 0\ ,$$

- With call protection $\longleftrightarrow$ Intermittent upper barrier

$$U_t = \infty \sum_{l \geq 0} \mathbf{1}_{[\theta_{2l}, \theta_{2l+1})} + h(X_t) \sum_{l \geq 0} \mathbf{1}_{[\theta_{2l+1}, \theta_{2l})}$$

$$U_t \geq Y_t \geq l(X_t),\ \int_0^T (Y_{u-} - l(X_u))\mathrm{d}K_u^+ = \int_0^T (\ Y_{u-} - U_{u-}\ )\mathrm{d}K_u^- = 0\ ,$$

$\hookrightarrow$ Y may be discontinuous !

Approximation of the forward process

- SDE X: $X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u$
- Euler scheme of X, grid $\pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\}$:

$$\begin{cases} X_0^\pi & = & X_0 \\ X_t^\pi & = & X_{t_i}^\pi + b(X_{t_i}^\pi)(t - t_i) + \sigma(X_{t_i}^\pi)(W_t - W_{t_i}), \ t \in (t_i, t_{i+1}] \end{cases}$$

- Error (b, σ Lipschitz)

$$\mathcal{E}rr(X, X^\pi) := \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - X_t^\pi|^2\right]^{\frac{1}{2}} \leq \frac{C}{\sqrt{n}}$$

$(\max_i |t_{i+1} - t_i| \leq \frac{C}{n})$

Approximation scheme for the BSDE

- Key idea

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(X_u, Y_u, Z_u)\mathrm{d}u - \int_{t_i}^{t_{i+1}} (Z_u)'\mathrm{d}W_u$$
$$Y^\pi_{t_i} \simeq Y^\pi_{t_{i+1}} + (t_{i+1} - t_i) f(X^\pi_{t_i}, Y^\pi_{t_i}, \bar{Z}^\pi_{t_i}) - (\bar{Z}^\pi_{t_i})'(W_{t_{i+1}} - W_{t_i})$$

- A backward scheme

$$Y^\pi_{t_i} := \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, Y^\pi_{t_i}, \bar{Z}^\pi_{t_i})$$
$$\bar{Z}^\pi_{t_i} := (t_{i+1} - t_i)^{-1}\, \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})(Y^\pi_{t_{i+1}})' \mid \mathcal{F}_{t_i}\right]$$

$\hookrightarrow$ terminal condition $Y^\pi_T := g(X^\pi_T)$.

Error to control

- continuous version of the scheme (Y^π, Z^π):

$$Y^\pi_{t_{i+1}} = \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \int_{t_i}^{t_{i+1}} (Z^\pi_u)' \mathrm{d}W_u$$

$$Y^\pi_t = Y^\pi_{t_{i+1}} + (t_{i+1} - t) f(X^\pi_{t_i}, Y^\pi_{t_i}, \bar{Z}^\pi_{t_i}) - \int_t^T (Z^\pi_u)' \mathrm{d}W_u \,.$$

- Error:

$$\mathcal{E}rr(Y, Y^\pi) := \sup_{t \in [0,T]} \mathbb{E}\left[|Y_t - Y^\pi_t|^2\right]^{\frac{1}{2}}$$

$$\mathcal{E}rr(Z, \bar{Z}^\pi) := \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_t - \bar{Z}^\pi_{t_i}|^2 \mathrm{d}t\right]^{\frac{1}{2}} = ||Z - \bar{Z}^\pi||_{\mathcal{H}^2}$$

Regularity and convergence

- (Bouchard-Touzi 2004):

$$\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, \bar{Z}^\pi) \leq C\mathcal{E}rr(X, X^\pi) + C||Z - P^\pi Z||_{\mathcal{H}^2}$$

with $V \in \mathcal{H}^2$

$$P^\pi V := \sum_{i=0}^{n-1} \bar{V}_{t_i} \mathbf{1}_{[t_i, t_{i+1})} \, , \; \bar{V}_{t_i} := \frac{1}{t_{i+1} - t_i} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} V_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right] \, .$$

- Need "regularity" on Z. (Zhang 2001)

$$||Z - P^\pi Z||_{\mathcal{H}^2} \leq \frac{C}{\sqrt{n}}$$

- Convergence rate

$$\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, \bar{Z}^\pi) \leq \frac{C}{\sqrt{n}}.$$

Known results in the simply reflected case

- Bally and Pages (2002): f does not depend on Z, l, b, σ lipschitz.
  $\hookrightarrow$ bound $\frac{C}{\sqrt{n}}$.
- Ma and Zhang (2005): f depends on Z, b, σ is C_b^1 and σ elliptic, l is C_b^2
  $\hookrightarrow$ bound $\frac{C}{n^{\frac{1}{4}}}$
- our goal:
  - extension to doubly reflected BSDEs
  - in a setting convenient for financial application

Framework: No call protection

- Assumptions
  - On the boundaries l, $-h$ are semi-convex (OK for financial application)
  - SDE coefficients are lipschitz, no ellipticity condition on σ
  - f depends on z.

- Remark:

The reflection term is a problem!

Discretely reflected BSDE

Given a time grid $\Re = \{0 = r_0 < ... < r_k < ... < r_m = T\}$.
A triplet $(Y^d, \tilde{Y}^d, Z^d)$ satisfying

$$Y_T^d = \tilde{Y}_T^d := g(X_T)$$

and, for $j \leq m-1$ and $t \in [r_j, r_{j+1})$,

$$\begin{cases} \tilde{Y}_t^d &= Y_{r_{j+1}}^d + \int_t^{r_{j+1}} f(X_u, \tilde{Y}_u^d, Z_u^d)\mathrm{d}u - \int_t^{r_{j+1}} (Z_u^d)'\mathrm{d}W_u \quad , \\ Y_t^d &= \tilde{Y}_t^d \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X_t, \tilde{Y}_t^d)\mathbf{1}_{\{t \in \Re\}} \end{cases}$$

with $\mathcal{P}(x, y) = (h(x) \wedge y) \vee l(x)$

First results

- "Good" approximation of the reflected BSDE:

$$\mathcal{E}rr(Y, Y^d) := \sup_{t\in[0,t]} \mathbb{E}\Big[|Y_t - Y_t^d|^2\Big]^{\frac{1}{2}} \leq \frac{C}{\sqrt{m}}$$

$$\mathcal{E}rr(Z, Z^d) := ||Z - Z^d||_{\mathcal{H}^2} \leq \frac{C}{\sqrt{m}}.$$

$(\max_j |r_{j+1} - r_j| \leq \frac{C}{m})$

- Continuous w.r.t. the parameters f, g, X (thus σ, b)
  $\hookrightarrow$ In the proofs, regularization of the parameters.

Euler scheme for discretely RBSDEs

Given the grids $\Re \subset \pi$

- Starting from the terminal condition $\widetilde{Y}^\pi_T = Y^\pi_T := g(X^\pi_T)$
- compute at each step

$$\begin{cases} \bar{Z}^\pi_{t_i} &= (t_{i+1} - t_i)^{-1}\, \mathbb{E}\left[Y^\pi_{t_{i+1}} (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i} \right] \\ \widetilde{Y}^\pi_{t_i} &= \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, Y^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\ Y^\pi_{t_i} &= \widetilde{Y}^\pi_{t_i} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X^\pi_{t_i}, \widetilde{Y}^\pi_{t_i}) \mathbf{1}_{\{t_i \in \Re\}} \end{cases}$$

$Y^\pi \leftrightarrow Y^d$, $\widetilde{Y}^\pi \leftrightarrow \tilde{Y}^d$, $\bar{Z}^\pi \leftrightarrow Z^d$

Convergence error

- As in the non-reflected case,

$$\mathcal{E}rr(Y^d, Y^\pi) + \mathcal{E}rr(Z^d, \bar{Z}^\pi) \leq C\mathcal{E}rr(X, X^\pi) + C||Z^d - P^\pi Z^d||^2_{\mathcal{H}^2}$$

- Regularity

$$||Z^d - P^\pi Z^d||_{\mathcal{H}^2} \leq C(\sqrt{\frac{m}{n}} + \frac{1}{n^{\frac{1}{4}}})$$

- thanks to a representation of Z^d $(f = f(Z))$
$(Z^d_t)' = \mathbb{E}\left[D_t Y^d_{r_{j+1}} + \int_t^{r_{j+1}} \nabla f(Z^d_u) D_t Z^d_u \mathrm{d}u \mid \mathcal{F}_t\right].$

Continuously reflected case: convergence and regularity

- We use the scheme of the discretely reflected BSDE!
  $\hookrightarrow$ we control the error between (Y, Z) and (Y^d, Z^d), and between (Y^d, Z^d) and $(Y^\pi, \bar{Z}^\pi)$
- We chose convenient grids $\Re$ and π $(m \sim \sqrt{n})$
  $\hookrightarrow$ Convergence of the scheme

$$\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, \bar{Z}^\pi) \leq \frac{C}{n^{\frac{1}{4}}} .$$

$\hookrightarrow$ Regularity on Z : $||Z - P^\pi Z||_{\mathcal{H}^2} \leq \frac{C}{n^{\frac{1}{4}}}$

Different kind of improvement

- Weaker assumption on the boundaries: "only" lipschitz
  $\hookrightarrow$ convergence rate controled only by $\frac{C}{n^{\frac{1}{6}}}$
- Is it possible to obtain better convergence rate ?
  - on the Z part, no...
  - on the Y part, yes! stronger assumption
  boundaries are C_b^2 with lipschitz second derivatives and strictly separated.
  b, σ are C_b^1 with lipschitz first derivatives.

New representation for the Z^d $(g = l,\, f = f(Z))$

- Recall: $(Z^d_t)' = \mathbb{E}\left[D_t Y^d_{r_{j+1}} + \int_t^{r_{j+1}} \nabla_z f(Z^d_u) D_t Z^d_u \mathrm{d}u \mid \mathcal{F}_t\right]$.
- New representation of Z^d using stopping times

$$\tau_j = \inf\{t \in \Re \mid t \geq r_{j+1},\ \tilde{Y}^d_t \notin (l(X_t), h(X_t))\} \wedge T\ .$$

New representation for the Z^d ($g = l$, $f = f(Z)$) *bis*

- New representation of Z^d:

$$(Z_t^d)' \;=\; \mathbb{E}\left[\nabla\phi(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} \mid \mathcal{F}_t\right] \,,\; t \in [r_j, r_{j+1})\,.$$

- where

$$\nabla\phi_r = \nabla l(X_r)\mathbf{1}_{\{l(X_r) > \tilde{Y}_r^d\}} + \nabla h(X_r)\mathbf{1}_{\{h(X_r) < \tilde{Y}_r^d\}}\;.$$

and

$$\Lambda_t^s \;:=\; \exp\left\{\int_s^t \nabla_z f(Z_u^d)' \mathrm{d}W_u - \int_s^t \left(\frac{1}{2}|\nabla_z f(Z_u^d)|^2\right)\mathrm{d}u\right\}\;.$$

Convergence results

- Regularity for Z^d

$$||Z^d - P^\pi Z^d||_{\mathcal{H}^2} \quad \leq \quad C\frac{\mathbf{m}^{\frac{1}{4}}}{\sqrt{n}} \; .$$

- We then obtain

$$\mathcal{E}rr(Y^d, Y^\pi) \leq C\frac{m^{\frac{1}{4}}}{\sqrt{n}} \quad \text{et} \quad \mathcal{E}rr(Y, Y^\pi) \leq \frac{C}{n^{\frac{1}{3}}}.$$

- key assumption: boundaries are strictly separated.

Conclusion

- Efficient method (even works for RBSDE in convex domain of $\mathbb{R}^d$)
- Improvement of known results
- Computation of the Delta

$$(Z_t^d)' \;=\; \mathbb{E}\left[\nabla\phi(X_{\tau_j})\Lambda^t_{\tau_j}\nabla X_{\tau_j} \mid \mathcal{F}_t\right](\nabla X_t)^{-1}\sigma(X_t)\,,\; t\in[r_j,r_{j+1})\,.$$

- Can we apply it to Game Option with call protection ?

Assumptions

- X unidimensionnal (for this talk)
- σ smooth and invertible around a given level S.
- f does not depend on z ! (working on it...)

Definition of Call protection period

- Protection monitoring times: Fixed grid
  $\mathcal{T} := \{T_0 = 0 < \cdots < T_l < \cdots < T_N = T\}$
- Auxiliary process H with value in a finite set $\mathcal{K}$ and depending on the past value of X
- Activation/Deactivation times for the upper boundary ($K \subset \mathcal{K}$)

$$\theta_{2l+1} = \inf\{t > \theta_{2l}\, ;\, H_t \notin K\} \wedge T$$
$$\theta_{2l+2} = \inf\{t > \theta_{2l+1}\, ;\, H_t \in K\} \wedge T\ .$$

- Definition of H:

$$H_{T_l} = \kappa^1(H_{T_{l-1}})\mathbf{1}_{\{X_{T_l-} \geq S\}} + \kappa^2(H_{T_{l-1}})\mathbf{1}_{\{X_{T_l-} < S\}}$$

H constant on $[T_{l-1}, T_l)$, $H_{-1} \in \mathcal{K}$

Examples

- Activation possible at one date:
  $\mathcal{T} := \{T_0 = 0 < T_1 < T_2 = T\}$
  $\mathcal{K} = \{0,1\}$, $K = \{0\}$
  $H_0 = 0$, $H_{T_1} = \mathbf{1}_{\{X_{T_1} \geq S\}}$, $H_{T_2} = H_{T_1}$
- Activation/Deactivation
  $\mathcal{T} := \{T_0 = 0 < \cdots < T_l < \cdots < T_N = T\}$
  $\mathcal{K} = \{0,1\}$, $K = \{0\}$
  $H_{T_l} = \mathbf{1}_{\{X_{T_l-} \geq S\}}$

Another example

- "l out of d" : Call is possible if $X \geq S$ on at least l of the last d monitoring times.
  $\mathcal{K} = \{0,1\}^d$, $K = \{k \in \mathcal{K} | |k| < l\}$ where $|k| = \sum_{i=1}^{d} k^i$
  And $H_{T_l} = (\mathbf{1}_{\{X_{T_{l-d}} \geq S\}}, ..., \mathbf{1}_{\{X_{T_l} \geq S\}})$.

Approximation of Activation/Deactivation times

- H^π is defined defined using X^π instead of X
- $\bar{\theta}_I$ approximation of θ_I defined using H^π instead of H
- we use the assumption on σ: smooth and invertible around S.

$$\hookrightarrow \mathbb{E}\big[|\bar{\theta}_I - \theta_I|\big] \le C_\epsilon |\pi|^{\frac{1}{2}-\epsilon} .$$

Discrete-time scheme

Given the grids $\mathcal{T} \subset \Re \subset \pi$

- Starting from the terminal condition $\widetilde{Y}^\pi_T = Y^\pi_T := g(X^\pi_T)$
- compute at each step

$$\begin{cases} \bar{Z}^\pi_{t_i} & = & (t_{i+1} - t_i)^{-1}\, \mathbb{E}\left[Y^\pi_{t_{i+1}} (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i} \right] \\ \widetilde{Y}^\pi_{t_i} & = & \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X^\pi_{t_i}, Y^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \\ Y^\pi_{t_i} & = & \widetilde{Y}^\pi_{t_i} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}^{\bar{\theta}}(t_i, X^\pi_{t_i}, \widetilde{Y}^\pi_{t_i}) \mathbf{1}_{\{t_i \in \Re\}} \end{cases}$$

$$\mathcal{P}^{\bar{\theta}}(t,x,y) = y + [l(x) - y]^+ - \sum_{l \geq 0} [y - h(x)]^+ \mathbf{1}_{\{\bar{\theta}_{2l+1} \leq t \leq \bar{\theta}_{2l+2}\}}$$

Convergence results

- Once again use of discretely reflected BSDEs
- Control the error between
  $\hookrightarrow (Y_-, Y, Z)$ and $(Y^{d,\theta}, \tilde{Y}^{d,\theta}, Z^{d,\theta})$
  $\hookrightarrow (Y^{d,\theta}, \tilde{Y}^{d,\theta}, Z^{d,\theta})$ and $(Y^{d,\bar{\theta}}, \tilde{Y}^{d,\bar{\theta}}, Z^{d,\bar{\theta}})$
  $\hookrightarrow (Y^{d,\bar{\theta}}, \tilde{Y}^{d,\bar{\theta}}, Z^{d,\bar{\theta}})$ and $Y^\pi, \widetilde{Y}^\pi, \bar{Z}^\pi$
- to obtain

$$\mathcal{E}rr(Y_-, Y^\pi) + \mathcal{E}rr(Y, \widetilde{Y}^\pi) + \mathcal{E}rr(Z, Z^\pi) \leq C_L^\epsilon |\pi|^{\frac{1}{4}-\epsilon}$$

Conclusion

- New difficulties, Y discontinuous: discrete monitoring of the protection,
- approximation of stopping times: quantity of interest $\mathbb{E}\big[|\bar{\theta}_I - \theta_I|\big]$
- Main open question: if f depends on Z... Gobet-Makhlouf ?