Connecting Discrete and Continuous Lookback and Hindsight Options Under Exponential Lévy Model

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Plan

- Introduction
- Continuous vs Discrete Supremum
- Financial Applications
- Simulations

Introduction

Let X be a real Lévy process

Lévy-Khinchin representation

X is a Lévy process with generating triplet (γ, σ, ν) :

$$\mathbb{E}e^{iuX_s} = e^{s\varphi(u)}$$
$$\varphi(u) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx)$$

Where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and ν is Radon measure on $\mathbb{R}^*$ satisfying :

$$\int_{\mathbb{R}} 1 \wedge x^2 \nu(dx) < \infty$$

Finite activity Lévy process

It means : $\nu(\mathbb{R}) < \infty$. Hence

$$X_s = \gamma_0 s + \sigma B_s + \sum_{i=1}^{N_s} Y_i$$

where B is a standard Brownian motion, N is a Poisson process with parameter $\lambda = \nu(\mathbb{R})$, $(Y_i)_{i\geq 1}$ are i.i.d. r.v. with a common law $\frac{\nu(dx)}{\nu(\mathbb{R})}$, and

$$\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx)$$

Infinite activity Lévy process

It means $\nu(\mathbb{R}) = +\infty$.

Finite variation Lévy process

It means $\sigma = 0$ and

$$\int_{|x|\leq 1} |x|\nu(dx) < \infty$$

We define

$$
\begin{aligned}
M_T &= \sup_{0 \leq s \leq T} X_s, \quad M_T^n = \max_{0 \leq k \leq n} X_{\frac{kT}{n}} \\
m_T &= \inf_{0 \leq s \leq T} X_s, \quad m_T^n = \min_{0 \leq k \leq n} X_{\frac{kT}{n}}
\end{aligned}
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Our goal is to study this expression :

$$M_T - M_T^n$$

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$$M_T - M_T^n$$

Referenes

1. Asmussen, S., Glynn, P. et Pitman, J. (1995).
2. Broadie, M., Glasserman, P. et Kou, S. G.(1999).

Proposition

Let X a real integrable Lévy process, then

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{\mathbb{E}X^{+}_{k\frac{T}{m}}}{k}=\int_0^T\frac{\mathbb{E}X_s^{+}}{s}ds$$

$$\mathbb{E}M_T^n=\sum_{k=1}^{n}\frac{\mathbb{E}X^{+}_{k\frac{T}{n}}}{k}$$

$$\mathbb{E}M_T=\int_0^T\frac{\mathbb{E}X_s^{+}}{s}ds$$

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$$\mathbb{E}M_T^n=\sum_{k=1}^{n}\frac{\mathbb{E}X^+_{k\frac{T}{n}}}{k}$$

$$\mathbb{E}M_T=\int_0^T\frac{\mathbb{E}X_s^+}{s}ds$$

Hence we have

$$\mathbb{E}\left(M_T-M_T^n\right)=\int_0^T\frac{\mathbb{E}X_s^+}{s}ds-\sum_{k=1}^{n}\frac{\mathbb{E}X^+_{k\frac{T}{n}}}{k}$$

Theorem

Let X be a real integrable Lévy process with generating triplet (γ, σ, ν) satisfying $\nu(\mathbb{R}) < \infty$, $T > 0$, and $n \in \mathbb{N}$.
*If $\sigma > 0$

$$\mathbb{E}\left(M_T - M_T^n\right) = -\frac{\sigma\sqrt{T}\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi n}} + \frac{\theta_+}{2n} + O\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

Where ζ is the Riemann's zeta function and

$$\begin{aligned} \theta_+ &= \frac{\gamma_0 t}{2} + \lambda T \mathbb{E} Y_1^+ - \sigma\sqrt{T}\mathbb{E}\phi\left(\frac{\gamma_0}{\sigma}\sqrt{T} + \frac{\sum_{i=1}^{N_T} Y_i}{\sigma\sqrt{T}}\right) \\ &\quad -\mathbb{E}\left(\gamma_0 T + \sum_{i=1}^{N_T} Y_i\right)\Phi\left(\frac{\gamma_0}{\sigma}\sqrt{T} + \frac{\sum_{i=1}^{N_T} Y_i}{\sigma\sqrt{T}}\right) \end{aligned}$$

Theorem (Next)

*If $\sigma = 0$

$$\mathbb{E}\left(M_T - M_T^n\right) = \frac{1}{2n}\left(\gamma_0^+ T + \lambda T \mathbb{E} Y_1^+ - E X_T^+\right) + o\left(\frac{1}{n}\right)$$

*If $\sigma = 0$ and Y_1 has a density function $f_{Y_1} \in C(\mathbb{R})$

$$\mathbb{E}\left(M_T - M_T^n\right) = \frac{1}{2n}\left(\gamma_0^+ T + \lambda T \mathbb{E} Y_1^+ - \mathbb{E} X_T^+\right) + O\left(\frac{1}{n^2}\right)$$

*If $\sigma = 0$ and $\gamma_0 = 0$

$$\mathbb{E}\left(M_T - M_T^n\right) = \frac{1}{2n}\left(\lambda T \mathbb{E} Y_1^+ - \mathbb{E} X_T^+\right) + O\left(\frac{1}{n^2}\right)$$

Theorem

Let X be a real Lévy process integrable with generating triplet $(\gamma, 0, \nu)$. Assume that $\int_{|x|\leq 1} |x| \left|\log\left(|x|\right)\right| \nu(dx) < \infty$, then

$$\mathbb{E}\left(M_T - M_T^n\right) = \frac{1}{2n}\left(\left(\gamma_0^+ + \int_{\mathbb{R}} x^+ \nu(dx)\right) T - \mathbb{E}X_T^+\right) + o\left(\frac{1}{n}\right)$$

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Define $\delta = \frac{T}{n}$, we have

$$\mathbb{E}\left(M_T - M_T^n\right) = \int_0^\delta \left(\frac{\mathbb{E}X_s^+}{s} - \frac{\mathbb{E}X_\delta^+}{\delta}\right) ds + \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} \left(\frac{\mathbb{E}X_s^+}{s} - \frac{\mathbb{E}X_{k\delta}^+}{k\delta}\right) ds$$

Theorem

Let X be a real Lévy process integrable with generating triplet (γ, σ, ν).

1. If $\sigma > 0$

$$\mathbb{E}\left(M_T - M_T^n\right) = O\left(\frac{1}{\sqrt{n}}\right)$$

2. If $\sigma = 0$

$$\mathbb{E}\left(M_T - M_T^n\right) = o\left(\frac{1}{\sqrt{n}}\right)$$

3. If $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$

$$\mathbb{E}\left(M_T - M_T^n\right) = O\left(\frac{\log(n)}{n}\right)$$

Compound Poisson Process Generalization

Let X a real finite activity Lévy process with generating triplet $(\gamma, 0, \nu)$.

$$X_s = \gamma_0 s + \sum_{i=1}^{N_s} Y_i$$

Theorem

Let $f \in C_b^1$, then

$$\mathbb{E}\left(f(M_T) - f(M_T^n)\right) = \frac{T}{2n}\left(|\gamma_0|\mathbb{E}f^{'}(M_T)Z + C\right) + o\left(\frac{1}{n}\right)$$

Where the r.v. Z and the constant C can be derived explicitly.

Asmussen et al. Theorem's Extension

Let X be a Lévy process with generating triplet (γ, σ, ν) satisfying $\sigma > 0$, and we define

$$W = \min_{n \in \mathbb{Z}} R\left(U + n\right)$$

where U is uniform on $(0, 1)$ and R is a two-sided version three-dimensional Bessel process.

Theorem (Asmussen, Glynn, Pitman (1995))

If $\nu = 0$, the sequence $\left(\sqrt{n}\left(M_T - M_T^n\right), X\right)$ converges in distribution to $\left(\sigma\sqrt{T}W, X\right)$, where W and X are independent.

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Theorem

If X is a finite activity Lévy process, the sequence $\left(\sqrt{n}\left(M_T - M_T^n\right), M_T\right)$ converges in distribution to $\left(\sigma\sqrt{T}W, M_T\right)$, where W and M_T are independent.

The Exponential Lévy Model

The dynamic

The behaviour of the underlying asset is modeled as :

$$S_\tau = S_0 e^{X_\tau}$$

Where X is a Lévy process. And we assume that

$$\left(e^{-(r-\delta)\tau} S_\tau\right)_{\tau \in [0,T]}$$

is a martingale. With

- r : the continuously compounded interest rate.
- δ : the continuously compounded dividend rate.

Lookback Option

Let $\Delta = \frac{T}{n}$.

Continuous prices

The continuous prices at date $\tau \in [0, T]$:

$$V(S_-) = e^{-r(T-\tau)}\mathbb{E}\left(S_0 e^{X_T} - \min\left(S_-, S_0 e^{m_T}\right)\right), \quad call$$
$$V(S_+) = e^{-r(T-\tau)}\mathbb{E}\left(\max\left(S_+, S_0 e^{M_T}\right) - S_0 e^{X_T}\right), \quad put$$

Discrete prices

The discrete prices at date $k\Delta$ with $k \in \{0, \dots, n\}$:

$$V^n(S_-) = e^{-r\Delta(n-k)}\mathbb{E}\left(S_0 e^{X_{n\Delta}} - \min\left(S_-, S_0 e^{m_T^n}\right)\right), \quad call$$
$$V^n(S_+) = e^{-r\Delta(n-k)}\mathbb{E}\left(\max\left(S_+, S_0 e^{M_T^n}\right) - S_0 e^{X_{n\Delta}}\right), \quad put$$

Hindsight Option

Continuous prices

The continuous prices at date $\tau \in [0, T]$:

$$V(S_+, K) = e^{-r(T-\tau)} \mathbb{E}\left(\max\left(S_+, S_0 e^{M_T}\right) - K\right)^+, \ \textit{call}$$
$$V(S_-, K) = e^{-r(T-\tau)} \mathbb{E}\left(K - \min\left(S_-, S_0 e^{m_T}\right)\right)^+, \ \textit{put}$$

Discrete prices

The discrete prices at date $k\Delta$ avec $k \in \{0, \ldots, n\}$:

$$V^n(S_+, K) = e^{-r\Delta(n-k)} \mathbb{E}\left(\max\left(S_+, S_0 e^{M_T^n}\right) - K\right)^+, \ \textit{call}$$
$$V^n(S_-, K) = e^{-r\Delta(n-k)} \mathbb{E}\left(K - \min\left(S_-, S_0 e^{m_T^n}\right)\right)^+, \ \textit{put}$$

Assumptions

- $(H1)$ X is a finite activity Lévy process, integrable, satisfying $\sigma > 0$ et $\exists \alpha > 0$ / $\mathbb{E}e^{(1+\alpha)M_T} < \infty$.
- $(H2)$ X is a finite activity Lévy process, integrable, satisfying $\sigma > 0$.
- $(H3)$ X is a compound Poisson process (with a drift) integrable and satisfying $\gamma_0 = 0$ or Y_1 have a bounded density.
- $(H4)$ X is a integrable infinite activity Lévy process.

Finite Activity Case

Proposition

The price of a discrete lookback option at the k^{th} fixing date and its continuous version at time $\tau = k\Delta t$ satisfy

$$V_n(S_\pm) = e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} V\left(S_\pm e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}\right) \pm \left(e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} - 1\right) e^{-\delta(T-\tau)} S_\tau + o\left(\frac{1}{\sqrt{n}}\right)$$

$$V(S_\pm) = e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} V_n\left(S_\pm e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}\right) \pm \left(e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} - 1\right) e^{-\delta(T-\tau)} S_\tau + o\left(\frac{1}{\sqrt{n}}\right)$$

where in $\pm$ and $\mp$, the top case applies to the put and the bottom to the call,

$$\beta_1 = -\frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}}$$

and ζ is the Rieman-zeta function. The put relations are true under $H1$, and those for the call under $H2$.

Finite Activity Case

Proposition

The price of a discrete hindsight option at the k^{th} fixing date and its continuous version at time $\tau = k\Delta t$ satisfy

$$V_n(S_\pm, K) = e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} V\left(S_\pm e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}, Ke^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}\right) + o\left(\frac{1}{\sqrt{n}}\right)$$

$$V(S_\pm, K) = e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} V_n\left(S_\pm e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}, Ke^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}\right) + o\left(\frac{1}{\sqrt{n}}\right)$$

where in $\pm$ and $\mp$, the top case applies to the call and the bottom to the put. The call relations are true under $H1$, and those for the put under $H2$.

Finite Activity Case

Proposition

Under $H3$, the price of a discrete lookback option at the k^{th} fixing date and its continuous version at time $\tau = k\Delta t$ satisfy

1. for the call

$$V_n(S_-) = V(S_-) + \frac{\alpha}{n} + o\left(\frac{1}{n}\right)$$

   where α is a constant which we can derive explicitly.

2. for the put, if it exists $\beta > 1$ such that $\mathbb{E}e^{\beta M_T} < \infty$, then

$$V_n(S_+) = V(S_+) + O\left(\frac{1}{n^{\frac{\beta-1}{\beta}}}\right)$$

Infinite Activity Case

Proposition

Under $H4$, the price of a discrete call lookback option at the k^{th} fixing date and its continuous version at time $\tau = k\Delta t$ satisfy

$$V_n(S_-) = V(S_-) + o\left(\frac{1}{\sqrt{n}}\right)$$

If $\int_{|x|\leq 1} |x|\nu(dx) < \infty$

$$V_n(S_-) = V(S_-) + O\left(\frac{\log(n)}{n}\right)$$

If $\int_{|x|\leq 1} |x| \left|\log(|x|)\right| \nu(dx) < \infty$

$$V_n(S_-) = V(S_-) + O\left(\frac{1}{n}\right)$$

Proposition

We assume $H4$ and that exists $\beta > 1$ such that $\mathbb{E}e^{\beta M_T} < \infty$. The price of a discrete put lookback option at the k^{th} fixing date and its continuous version at time $\tau = k\Delta t$ satisfy, $\forall \epsilon > 0$

$$V_n(S_+) = V(S_+) + o\left(\frac{1}{n^{\frac{\beta-1}{2\beta}-\frac{\epsilon}{2}}}\right)$$

If $\int_{|x|\leq 1} |x|\nu(dx) < \infty$

$$V_n(S_+) = V(S_+) + O\left(\left(\frac{\log(n)}{n}\right)^{\frac{\beta-1}{\beta}-\epsilon}\right)$$

If $\int_{|x|\leq 1} |x| \left|\log(|x|)\right| \nu(dx) < \infty$

$$V_n(S_+) = V(S_+) + O\left(\frac{1}{n^{\frac{\beta-1}{\beta}-\epsilon}}\right)$$

Infinite Activity Case

Corollary

The results found for lookback under $H3$ $H4$ and $H5$ are true for hindsight, by interchanging "call" and "put".

Double Exponential Jump Diffusion Model

We have

$$X_s = \gamma_0 s + \sigma B_s + \sum_{i=1}^{N_s} Y_i$$

N is a poisson process with parameter λ, and Y_1 follows an asymmetric double exponential distribution of density :

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} \mathbb{1}_{\{y < 0\}}$$

where $p,q \geq 0$ are constants, $p + q = 1$, $\eta_1 > 1$ and $\eta_2 > 0$.

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$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbb{1}_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} \mathbb{1}_{\{y < 0\}}$$

where $p,q \geq 0$ are constants, $p + q = 1$, $\eta_1 > 1$ and $\eta_2 > 0$.

Parameters of simulations

The parameters are : $\sigma = 0.3$, $p = 0.6$, $\lambda = 7$, $\eta_1 = 50$ and $\eta_2 = 25$.

Call Case

n	approx. relative error	discrete relative error
1	-1.56%	-57.88%
2	-0.95%	-39.98%
3	-0.66%	-32.27%
4	-0.60%	-27.72%
5	-0.52%	-24.65%
6	-0.48%	-22.39%
7	-0.42%	-20.65%
8	-0.40%	-19.26%
9	-0.38%	-18.13%
10	-0.38%	-17.18%

Table: Performance of the propositions of approximation in the double exponential jump diffusion model. The parameters are : $S_0 = 100$, $r = 0.05$, $\delta = 0$, $T = 1$, $S_- = 90$. The true continuous call price is 26.1436 and the relative Monte Carlo error is 0.01%.

Put Case

n	approx. relative error	discrete relative error
1	9.75%	−55.64%
2	2.97%	−42.93%
3	0.82%	−36.49%
4	0.37%	−32.35%
5	0.34%	−29.40%
6	0.29%	−27.18%
7	0.23%	−25.42%
8	0.17%	−24.00%
9	0.12%	−22.80%
10	0.06%	−21.80%

Table: Performance of the propositions of approximation in the double exponential jump diffusion model. The parameters are : $S_0 = 100$, $r = 0.05$, $\delta = 0$, $T = 1$, $S_+ = 110$. The true continuous put price is 26.0449 and the relative Monte Carlo error is 0.01%.

Thank you for your attention!