

# *Numerical Methods for Portfolio Selection with Transaction Costs <sup>a</sup>*

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Third Conference on Numerical Methods in Finance, Paris, April 2009

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# Agenda

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- Literature review
- Modelling: singular stochastic control
- Penalty method for variational inequality with gradient constraints
- Convergence analysis
- One stock case: a reduced penalty method
- Numerical results
- Conclusion

# Literature review: Merton's model

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- Merton's model: Portfolio selection without transaction costs
  - Market: a bank account ( $S_0$ ) and a risky asset ( $S_1$ )

$$\begin{aligned}dS_0 &= rS_0dt, \\dS_1 &= S_1(\alpha_1dt + \sigma_1d\mathcal{B}_1).\end{aligned}$$

- A self-financing process  $W_t$ :

$$dW_t = (rW_t + (\alpha_1 - r)Y_t)dt + \sigma_1Y_t d\mathcal{B}_{1t}.$$

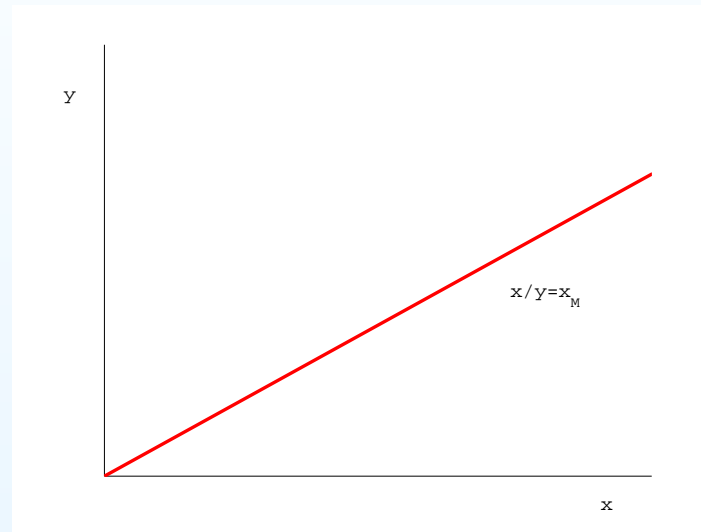
- A CRRA investor's problem:

$$\max_{Y_t} E[u(W_T)],$$

$$\text{where } u(W) = \begin{cases} \frac{1}{\gamma} W^\gamma, & \text{if } \gamma \neq 0, \gamma < 1, \\ \log W, & \text{if } \gamma = 0. \end{cases}$$

# Literature review: drawbacks of Merton's model

- Merton's strategy:  $\frac{X_t}{Y_t} = -\frac{\alpha_1 - r - \sigma_1^2}{\alpha_1 - r} \doteq x_M.$

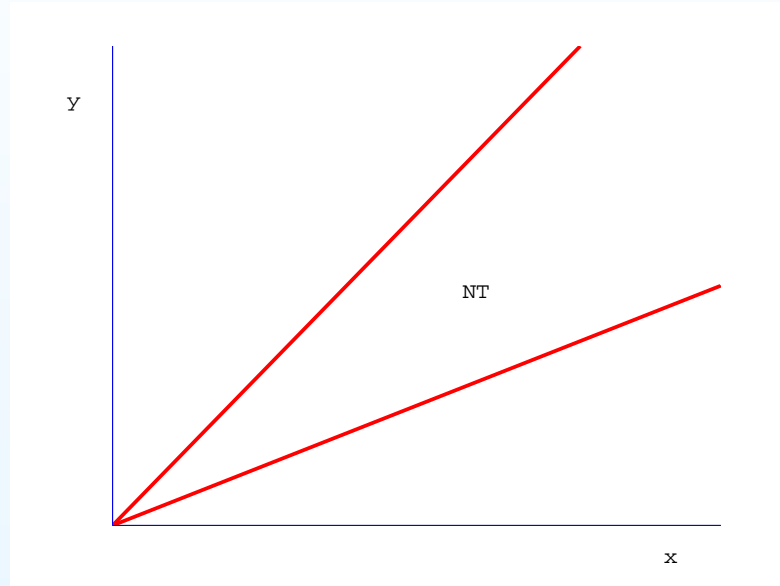


- Drawbacks of Merton's model
  - Incessant trading:
    - Transaction costs incurred are unacceptable in practice
    - It violates the conventional buy-and-hold strategy
  - Horizon independent policy:
    - It is against the conventional wisdom that the younger should allocate a greater share of wealth to stocks than the older

# Literature review: with transaction costs

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- Pioneering work: Magil and Contantitnides (1976). A heuristic argument.



- One risky asset with infinite horizon
  - Davis and Norman (1990): formulation of a free boundary problem.
  - Shreve and Soner (1994): theoretical analysis by viscosity solution.
  - Janecek and Shreve (2004, 2007): asymptotic analysis
  - Muthuraman (2006): numerical scheme
  - Kallsen and Muhle-Karbe (2008): shadow price process

## Literature review (continued)

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- One risky asset with finite horizon
  - Genotte and Jung (1994): dynamic programming method.
  - Liu and Loewenstein (2002): theoretical analysis and numerically solution for optimal strategy, based on Carr (1998).
  - Dai and Yi (2009), Dai et al. (2009): linkage between singular control and optimal stopping
- Multiple risky assets (infinite horizon)
  - Alkian, Menaldi and Sulem (1996): provide a scheme associated with policy iteration and multigrid method for uncorrelated returns.
  - Muthuraman and Kumar (2006): extension of Muthuraman (2006).

# Model formulation

- Market: a riskfree asset ( $S_0$ ) and N risky assets ( $S_i, i = 1, 2, \dots, N$ )

$$\begin{aligned}dS_0 &= rS_0dt, \\dS_i &= S_i (\alpha_i dt + \sigma_i d\mathcal{B}_i),\end{aligned}$$

where  $\alpha_i > r$  and  $E(d\mathcal{B}_i d\mathcal{B}_j) = \rho_{ij} dt$ .

- $X_0(t)$  and  $X_i(t)$ : dollar values in bank and risky assets respectively,

$$\begin{aligned}dX_0 &= (rX_0 - \kappa C(t))dt - \sum_{i=1}^N (1 + \lambda_i) dL_i + \sum_{i=1}^N (1 - \mu_i) dM_i \\dX_i &= \alpha_i X_i dt + \sigma_i X_i d\mathcal{B}_i + dL_i - dM_i,\end{aligned}$$

- $C(t) \geq 0$ , and  $\kappa = 1$  (consumption) or 0 (no consumption)
- $L_i(t)$  and  $M_i(t)$ : cumulative dollar values for buying and selling the  $i^{th}$  risky asset respectively.
- $\lambda_i \in [0, \infty)$  and  $\mu_i \in [0, 1)$ : proportions of transaction costs incurred on purchase and sale of the  $i^{th}$  risky asset respectively.  $\lambda_i + \mu_i > 0$ .

## Model formulation (continued)

- Solvency region:

$$\mathcal{S} = \{x = (x_0, x_1, \dots, x_N) \in \mathcal{R}^{N+1} : x_0 + \sum_{i=1}^N [(1 - \mu_i)x_i^+ - (1 + \lambda_i)x_i^-] > 0\}.$$

- CRRA investors' problem: choosing an admissible strategy  $(L_i(t), M_i(t), C(t))$  so as to maximize

$$E \left[ \int_0^T \kappa e^{-\beta s} u(C(s)) ds + e^{-\beta T} u(W_T) \right]$$

where  $\beta > 0$  is the discount rate.

- Singular stochastic control problem. Define its value function

$$V(x, t) = \sup_{(L_i, M_i, C) \in A} E_t^{x_t=x} \left[ \int_t^T \kappa e^{-\beta(s-t)} u(C(s)) ds + e^{-\beta(T-t)} u(W_T) \right],$$

for  $x_t \in \mathcal{S}$ ,  $t \in [0, T)$ .

## HJB equation [Shreve and Soner (1994)]

$$\max \left\{ V_t + \mathcal{L}_0 V + \kappa u^* \left( \frac{\partial V}{\partial x_0} \right), \max_{1 \leq i \leq N} \mathcal{L}_{0i} V, \max_{1 \leq i \leq N} \mathcal{M}_{0i} V \right\} = 0, \quad \text{in } \mathcal{S}, t \in [0, T),$$

$$V(x, T) = u \left( x_0 + \sum_{i=1}^N [(1 - \mu_i)x_i^+ - (1 + \lambda_i)x_i^-] \right),$$

where  $\mathcal{L}_0 V = \frac{1}{2} \sum_{i,j=1}^N \rho_{ij} \sigma_i \sigma_j x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^N \alpha_i x_i \frac{\partial V}{\partial x_i} + r x_0 \frac{\partial V}{\partial x_0} - \beta V,$

$$\mathcal{L}_{0i} V = -(1 + \lambda_i) \frac{\partial V}{\partial x_0} + \frac{\partial V}{\partial x_i}, \quad \mathcal{M}_{0i} V = (1 - \mu_i) \frac{\partial V}{\partial x_0} - \frac{\partial V}{\partial x_i},$$

$$u^*(\nu) = \max_{c \geq 0} (-c\nu + u(c)) = \begin{cases} (\frac{1}{\gamma} - 1)\nu^{\frac{\gamma}{\gamma-1}}, & \text{if } \gamma \neq 0, \gamma < 1; \\ -\log \nu - 1, & \text{if } \gamma = 0. \end{cases} .$$

Buy regions, sell regions, and no trading regions:

$$BR_i = \{(x, t) \in \mathcal{S}^N \times [0, T) : \mathcal{L}_{0i} V = 0\},$$

$$SR_i = \{(x, t) \in \mathcal{S}^N \times [0, T) : \mathcal{M}_{0i} V = 0\},$$

$$NT_i = (BR_i \cup SR_i)^c,$$

$$NT = \bigcap_{i=1}^N NT_i.$$

## Change of variables

Due to the homogeneity of the utility function, it follows that for any  $\rho > 0$ ,

$$V(\rho x, t) = \begin{cases} \rho^\gamma V(x, t), & \text{if } \gamma \neq 0, \gamma < 1; \\ g(t) \log \rho + V(x, t), & \text{if } \gamma = 0, \end{cases}$$

where  $g(t) = \frac{\kappa(1 - e^{-\beta(T-t)})}{\beta} + e^{-\beta(T-t)}$ . Take

$$\rho = \frac{1}{\sum_{i=1}^N x_i} \text{ and } y_i = \rho x_i, \quad i = 1, 2, \dots, N.$$

Denote  $y = (y_1, y_2, \dots, y_N)$  and  $\varphi(y, t) = V(1 - \sum_{i=1}^N y_i, y_1, y_2, \dots, y_n, t)$ , then

$$V(x, t) = \begin{cases} \rho^\gamma \varphi(y, t), & \text{if } \gamma \neq 0, \gamma < 1; \\ g(t) \log \rho + \varphi(y, t), & \text{if } \gamma = 0. \end{cases}$$

## Change of variables (continued)

For  $\gamma \neq 0$  and  $\gamma < 1$ , [cf. Davis and Norman (1990), Shreve and Soner (1994), Akian et al. (1996) or Muthuraman and Kuman (2006)]

$$\max \left\{ \varphi_t + \mathcal{L}_1 \varphi + \kappa \left( \frac{1}{\gamma} - 1 \right) \left( \gamma \varphi - \sum_{i=1}^N y_i \frac{\partial \varphi}{\partial y_i} \right)^{\frac{\gamma}{\gamma-1}}, \max_{1 \leq i \leq N} \mathcal{L}_{i1} \varphi, \max_{1 \leq i \leq N} \mathcal{M}_{i1} \varphi \right\} = 0,$$

$$\varphi(y, T) = \frac{(1 - \sum_{i=1}^N (\mu_i y_i^+ + \lambda_i y_i^-))^\gamma}{\gamma}, \quad \text{in } y \in \Omega^N, t \in [0, T),$$

where  $\Omega^N = \left\{ y = (y_1, y_2, \dots, y_N) \in \mathcal{R}^N : 1 - \sum_{i=1}^N (\mu_i y_i^+ + \lambda_i y_i^-) > 0 \right\}$ ,

$$\mathcal{L}_1 \varphi = \sum_{k,l=1}^N a_{k,l} \frac{\partial^2 \varphi}{\partial y_k \partial y_l} + \sum_{k=1}^N b_k \frac{\partial \varphi}{\partial y_k} - \theta \gamma \varphi$$

$$\mathcal{L}_{i1} \varphi = \sum_{k=1}^N (\delta_{ik} + \lambda_i y_k) \frac{\partial \varphi}{\partial y_k} - \lambda_i \gamma \varphi, \quad \mathcal{M}_{i1} \varphi = \sum_{k=1}^N (-\delta_{ik} + \mu_i y_k) \frac{\partial \varphi}{\partial y_k} - \mu_i \gamma \varphi$$

$$a_{k,l} = y_k y_l \sum_{i,j=1}^N \frac{1}{2} \rho_{ij} \sigma_i \sigma_j (\delta_{il} - y_i) (\delta_{jk} - y_j)$$

$$b_k = y_k \sum_{i=1}^N (\delta_{ik} - y_i) \left[ (\alpha_i - r) + \sum_{j=1}^N (\gamma - 1) \rho_{ij} \sigma_i \sigma_j y_j \right]$$

$$\theta = \frac{\beta}{\gamma} - \left( r + \sum_{i=1}^N y_i (\alpha_i - r - \frac{1-\gamma}{2} \sum_{j=1}^N \rho_{ij} \sigma_i \sigma_j y_j) \right)$$

Here  $\delta_{ij} = 1$  if  $i = j$ , and  $\delta_{ij} = 0$  otherwise.

## Change of variables (continued)

- We adopt a further transformation:  $W(y, t) = \frac{1}{\gamma} \log(\gamma\varphi)$ . Then

$$\begin{cases} \max \{W_t + \mathcal{L}W + \kappa f(W), \max_{1 \leq i \leq N} \mathcal{L}_i W, \max_{1 \leq i \leq N} \mathcal{M}_i W\} = 0, \\ W(y, T) = \log \left( 1 - \sum_{i=1}^N (\mu_i y_i^+ + \lambda_i y_i^-) \right), \text{ in } \Omega^N, t \in [0, T), \end{cases}$$

where  $\mathcal{L}W = \sum_{k,l=1}^N a_{k,l} \left( \frac{\partial^2 W}{\partial y_k \partial y_l} + \gamma \frac{\partial W}{\partial y_k} \frac{\partial W}{\partial y_l} \right) + \sum_{k=1}^N b_k \frac{\partial W}{\partial y_k} - \theta$   
 $\mathcal{L}_i W = \sum_{k=1}^N (\delta_{ik} + \lambda_i y_k) \frac{\partial W}{\partial y_k} - \lambda_i$ ,  $\mathcal{M}_i W = \sum_{k=1}^N (-\delta_{ik} + \mu_i y_k) \frac{\partial W}{\partial y_k} - \mu_i$ ,  
 $f(W) = \left( \frac{1}{\gamma} - 1 \right) e^{\frac{\gamma}{\gamma-1} W} \left( 1 - \sum_{i=1}^N y_i \frac{\partial W}{\partial y_i} \right)^{\frac{\gamma}{\gamma-1}}$

$$a_{k,l} = y_k y_l \sum_{i,j=1}^N \frac{1}{2} \rho_{ij} \sigma_i \sigma_j (\delta_{il} - y_i) (\delta_{jk} - y_j)$$

$$b_k = y_k \sum_{i=1}^N (\delta_{ik} - y_i) \left[ (\alpha_i - r) + \sum_{j=1}^N (\gamma - 1) \rho_{ij} \sigma_i \sigma_j y_j \right]$$

- For  $\gamma = 0$ , let  $W(y, t) = \frac{\varphi(y, t)}{g(t)}$ . satisfying the same equation with

$$f(W) = -(1 + \log g(t) + \log(1 - \sum_{i=1}^N y_i \frac{\partial W}{\partial y_i}) + W) / g(t)$$

$$\theta = -(r + \sum_{i=1}^N y_i (\alpha_i - r - \frac{1}{2} \sum_{j=1}^N \rho_{ij} \sigma_i \sigma_j y_j))$$

# Penalty approximation: a heuristic derivation of HJB

## equation

Confined to a class of restricted policies (Davis and Norman (1990)):

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$$L_{it} = \int_0^t l_{is} ds, \quad M_{it} = \int_0^t m_{is} ds, \quad 0 \leq l_{is}, \quad m_{is} \leq K, \quad \text{for } i = 1, 2, \dots, N.$$

It follows ( $\gamma \neq 0$  for example)

$$\max_{(l_i, m_i, C)} \left\{ \bar{V}_t + \mathcal{L}_0 \bar{V} + \kappa \left( \frac{C^\gamma}{\gamma} - C \frac{\partial \bar{V}}{\partial x_0} \right) + \sum_{i=1}^N (l_i \mathcal{L}_{0i} \bar{V} + m_i \mathcal{M}_{0i} \bar{V}) \right\} = 0.$$

The optimal strategies are  $C = \left( \frac{\partial \bar{V}}{\partial x_0} \right)^{\frac{1}{\gamma-1}}$ ,

$$l_i = \begin{cases} K, & \text{if } \mathcal{L}_{0i} \bar{V} \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad m_i = \begin{cases} K, & \text{if } \mathcal{M}_{0i} \bar{V} \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

which yields

$$\bar{V}_t + \mathcal{L}_0 \bar{V} + \kappa u^* \left( \frac{\partial \bar{V}}{\partial x_0} \right) + K \sum_{i=1}^N \left[ (\mathcal{L}_{0i} \bar{V})^+ + (\mathcal{M}_{0i} \bar{V})^+ \right] = 0.$$

## Penalty method (continued)

- Penalty approximation to the transformed equation:

$$-W_t - \mathcal{L}W - \kappa f(W) = K \sum_{i=1}^N [(\mathcal{L}_i W)^+ + (\mathcal{M}_i W)^+]$$

where  $K$  is a positive constant to be chosen big enough.

- Historical work of penalty method
  - Forsyth and Vetzal (2002): American option pricing.

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda(\phi - V)^+ = 0$$

- Dai, Kwok and You (2007): linked with intensity-based framework.
- Dai, Kwok and Zong (2008): the pricing of GMWBs.

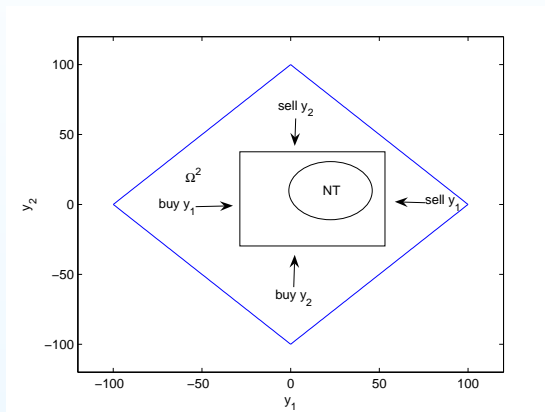
# Advantages of penalty method

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- no prior knowledge of free boundaries required;
- robustness: any discretization, dimension, and unstructured mesh;
- Newton iteration can handle penalty terms and other nonlinearities;
- solve the Jacobian matrix by standard sparse matrix software.

# Penalty method with difference discretization

- Solution domain and boundary conditions



Solution domain when  $N = 2$

- Discretization
  - Fully implicit scheme
  - The penalty terms are linearized by non-smooth Newton iteration [cf. Forsyth and Vetzal (2002) and Dai et al. (2007)]:

$$K \left( \mathcal{L}_i W^{,k} \right)^+ = \begin{cases} K \mathcal{L}_i W^{,k}, & \text{if } \mathcal{L}_i W^{,k-1} \geq 0, \\ 0, & \text{if } \mathcal{L}_i W^{,k-1} < 0, \end{cases}$$

Here the upwind scheme should be applied for  $\frac{\partial W}{\partial y_j}$  in  $\mathcal{L}_i W$  and  $\mathcal{M}_i W$ .

# The penalty algorithm

$$\left\{ \begin{array}{l} (FW^n)_i = P_{1i}^n \left( \frac{(E^+ W^n)_i}{h} - \left( \frac{\lambda}{1+\lambda y} \right)_i \right) + P_{2i}^n \left( \left( \frac{-\mu}{1-\mu y} \right)_i - \frac{(E^- W^n)_i}{h} \right) \\ W_i^{\bar{n}} = \log(1 - (\mu y^+ + \lambda y^-))_i, \text{ for } i = \underline{m} + 1, \dots, \bar{m} - 1, n = 0, \dots, \bar{n} - 1 \end{array} \right.$$

where

$$(E^+ W^n)_i = W_{i+1}^n - W_i^n, \quad (E^- W^n)_i = W_i^n - W_{i-1}^n,$$

$$P_{1i}^n = \begin{cases} K \Delta t, & \text{if } \frac{(E^+ W^n)_i}{h} > \left( \frac{\lambda}{1+\lambda y} \right)_i, \\ 0, & \text{otherwise.} \end{cases} \quad P_{2i}^n = \begin{cases} K \Delta t, & \text{if } \left( \frac{-\mu}{1-\mu y} \right)_i > \frac{(E^- W^n)_i}{h}, \\ 0, & \text{otherwise.} \end{cases}$$

We aim to show that as  $K \rightarrow +\infty$ , the scheme converges to

$$(FW^n)_i \geq 0, \quad \left( \frac{\lambda}{1+\lambda y} \right)_i - \frac{(E^+ W^n)_i}{h} \geq 0, \quad \frac{(E^- W^n)_i}{h} - \left( \frac{-\mu}{1-\mu y} \right)_i \geq 0$$

$$(FW_i^n = 0) \vee \left( \left( \frac{\lambda}{1+\lambda y} \right)_i = \frac{(E^+ W^n)_i}{h} \right) \vee \left( \frac{(E^- W^n)_i}{h} = \left( \frac{-\mu}{1-\mu y} \right)_i \right)$$

$$W_i^{\bar{n}} = \log(1 - (\mu y^+ + \lambda y^-))_i.$$

# Convergence analysis ( $\kappa = 0$ )

- Error in Penalty formulation:

- If  $\frac{\Delta t}{h} < \text{const.}$ , as  $\Delta t, h \rightarrow 0$ , then the penalty method solves

$$FW_i^n \geq 0, \quad \left(\frac{\lambda}{1 + \lambda y}\right)_i - \frac{(E^+ W^n)_i}{h} \geq -\frac{C}{K\Delta t}, \quad \frac{(E^- W^n)_i}{h} - \left(\frac{-\mu}{1 - \mu y}\right)_i \geq -\frac{C}{K\Delta t}$$

$$(FW_i^n = 0) \vee \left( \left| \left(\frac{\lambda}{1 + \lambda y}\right)_i - \frac{(E^+ W^n)_i}{h} \right| \leq \frac{C}{K\Delta t} \right) \vee \left( \left| \frac{(E^- W^n)_i}{h} - \left(\frac{-\mu}{1 - \mu y}\right)_i \right| \leq \frac{C}{K\Delta t} \right)$$

$$W_i^{\bar{n}} = \log(1 - (\mu y^+ + \lambda y^-))_i,$$

where  $C > 0$  is independent of  $K, \Delta t, h$ .

- Key proof (stability): Suppose  $W_i^n$  is the solution, then

$$|W_i^n| \leq \|\theta\|_\infty T + \|W(T, y)\|_\infty, \quad \text{for all } n, i$$

- Convergence of the Penalty Iteration:

- The iterations converge monotonically, i.e.  $W^{n,k+1} \geq W^{n,k}$  for  $k \geq 1$ .
- The nonlinear iteration is convergent for any initial guess  $W^{n,0}$ .

## Numerical results: $N=2$

Test of varying the penalty parameter  $K$  on the gradient constraint problem ( $N = 2$ ).

$K$	$\varphi(y_{M_1}, y_{M_2}, 0)$
0.1	-7.104291
0.5	-7.097533
$10^2$	-7.097435
$10^6$	-7.097435
benchmark	-7.097434

Penalty method for gradient constraints with Fully Implicitity and Upwind scheme ( $N = 2$ ).

$\bar{n}$	$N_{y_1}$	$N_{y_2}$	$\ \epsilon\ _\infty$	Ratio
50	10	10	3.07e-03	-
100	20	20	1.50e-03	2.05
200	40	40	6.99e-04	2.14
400	80	80	2.99e-04	2.34

## Case of one stock

- Singular control: variational inequality with gradient constraints

$$\min \left\{ -W_t - \mathcal{L}W - \kappa f(W), \frac{\lambda}{1 + \lambda y} - W_y, W_y + \frac{\mu}{1 - \mu y} \right\} = 0$$

or equivalently,

$$-W_t - \mathcal{L}W - \kappa f(W) = 0 \quad \text{if } -\frac{\mu}{1 - \mu y} \leq W_y \leq \frac{\lambda}{1 + \lambda y}$$

$$-W_t - \mathcal{L}W - \kappa f(W) \geq 0 \quad \text{if } W_y = -\frac{\mu}{1 - \mu y}$$

$$-W_t - \mathcal{L}W - \kappa f(W) \geq 0 \quad \text{if } W_y = \frac{\lambda}{1 + \lambda y}$$

- Let  $v = W_y$ , ( $\gamma = 0$  for illustration)

$$\frac{\partial}{\partial y} \mathcal{L}W = \mathcal{T}v, \quad \frac{\partial}{\partial y} f(W) = \bar{f}(v) \doteq \frac{y}{g(t)(1 - yv)} (v^2 + v_y)$$

## Case of one stock: a double obstacle problem

- A double obstacle problem (optimal stopping):

$$\begin{aligned} -v_t - \mathcal{T}v - \kappa \bar{f}(v) &= 0 & \text{if } -\frac{\mu}{1 - \mu y} \leq v \leq \frac{\lambda}{1 + \lambda y} \\ -v_t - \mathcal{T}v - \kappa \bar{f}(v) &\geq 0 & \text{if } v = -\frac{\mu}{1 - \mu y} \\ -v_t - \mathcal{T}v - \kappa \bar{f}(v) &\leq 0 & \text{if } v = \frac{\lambda}{1 + \lambda y} \end{aligned}$$

or equivalently,

$$\min \left\{ \max \left\{ -v_t - \mathcal{T}v - \kappa \bar{f}(v), v - \frac{\lambda}{1 + \lambda y} \right\}, v + \frac{\mu}{1 - \mu y} \right\} = 0,$$

- Connection of singular control and optimal stopping
  - Dai and Yi (2009): no consumption case, JDE.
  - Dai et al. (2009): consumption case, SIAM J. Contr. & Opt.

# Case of one stock: a reduced penalty method

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- Standard penalty approximation ( $\gamma = 0$  for illustration):

$$-v_t - \mathcal{T}v - \kappa \bar{f}(v) = -K\left(v - \frac{\lambda}{1 + \lambda y}\right)^+ + K\left(-\frac{\mu}{1 - \mu y} - v\right)^+$$

- Discretization: Crank-Nicolson scheme (second order of accuracy)

## Numerical results: $N=1$

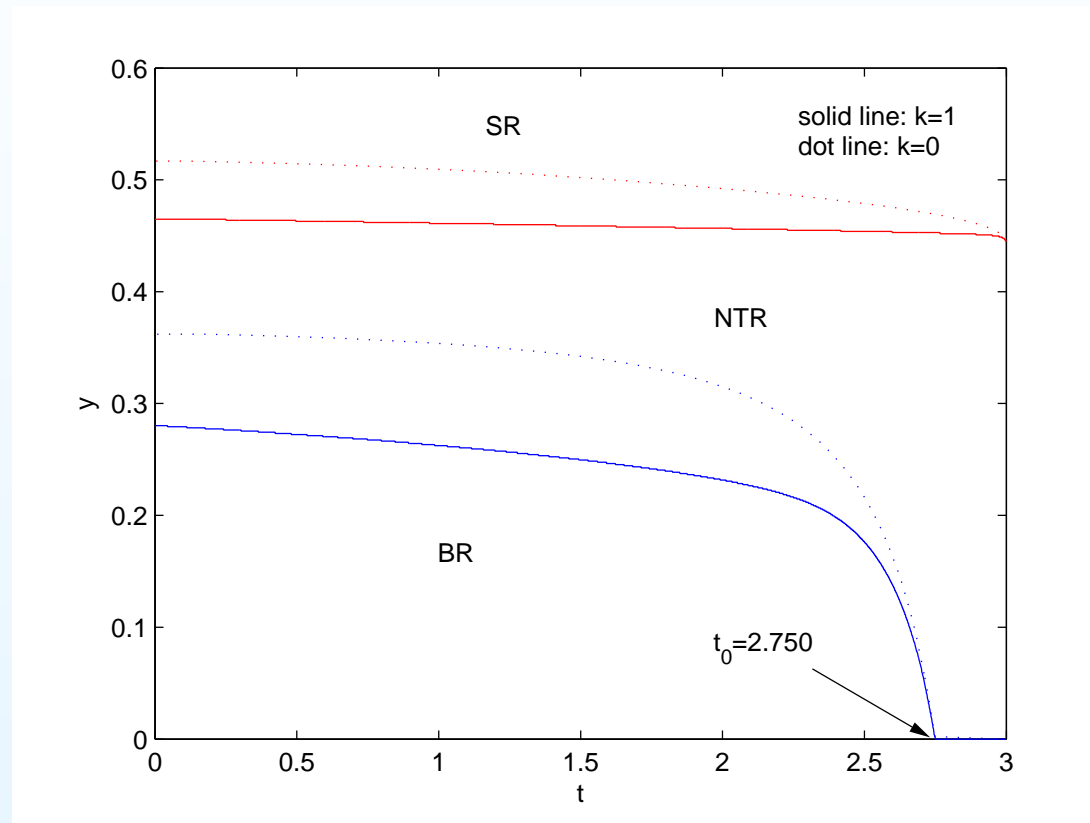
Test of varying the penalty parameter  $K$  on the double obstacle problem ( $N = 1$ ).

$K$	$\varphi(y_M, 0)$
10	-7.123355
$10^4$	-7.123784
$10^5$	-7.123796
$10^6$	-7.123798
benchmark	-7.123798

Penalty method for double obstacle problem with Crank Nicolson scheme ( $N = 1$ ).

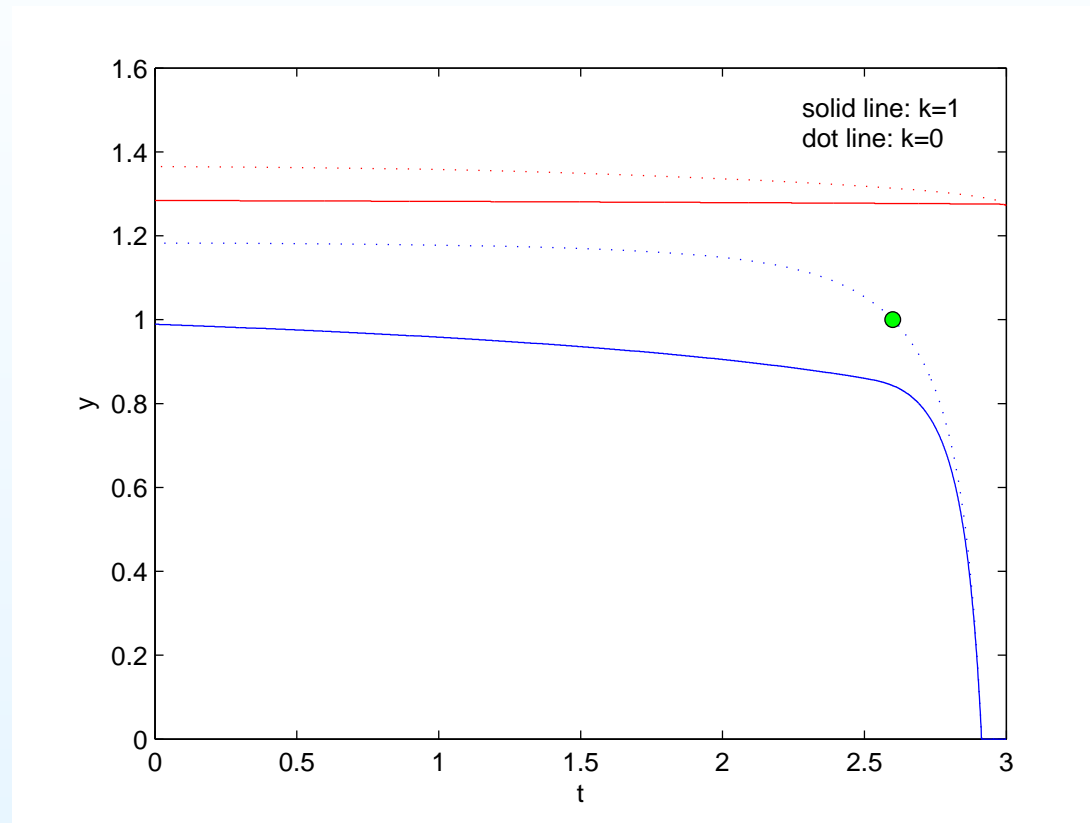
$N_t$	$N_y$	$\ \epsilon\ _{\infty, \kappa = 0}$	ratio	$\ \epsilon\ _{\infty, \kappa = 1}$	ratio
1200	200	1.29e-05	-	3.12e-05	-
2400	400	3.46e-06	2.6	2.70e-06	11.6
4800	800	8.70e-07	4.0	1.11e-06	2.4
9600	1600	2.02e-07	4.3	3.57e-07	3.1

# Numerical results: optimal trading strategies ( $N = 1$ )



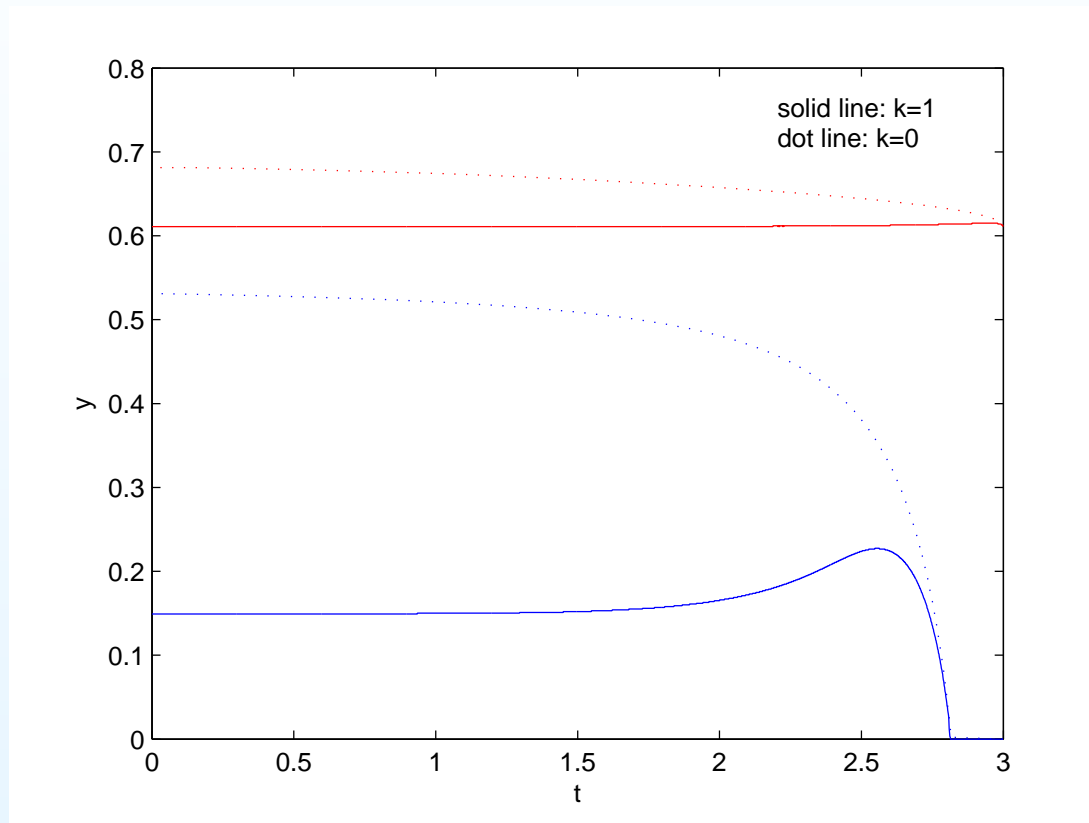
Compare consumption and nonconsumption. This figure shows the optimal fraction of wealth in the risky asset will decrease when consumption is involved. The case  $\alpha - r - (1 - \gamma)\sigma^2 < 0$ . Parameter default values:  $\alpha = 0.15$ ,  $r = 0.07$ ,  $\sigma = 0.3$ ,  $\gamma = -1$ ,  $\beta = 0.1$ ,  $\lambda = \mu = 0.01$ ,  $T = 3$ .

## Numerical results: optimal trading strategies ( $N = 1$ )



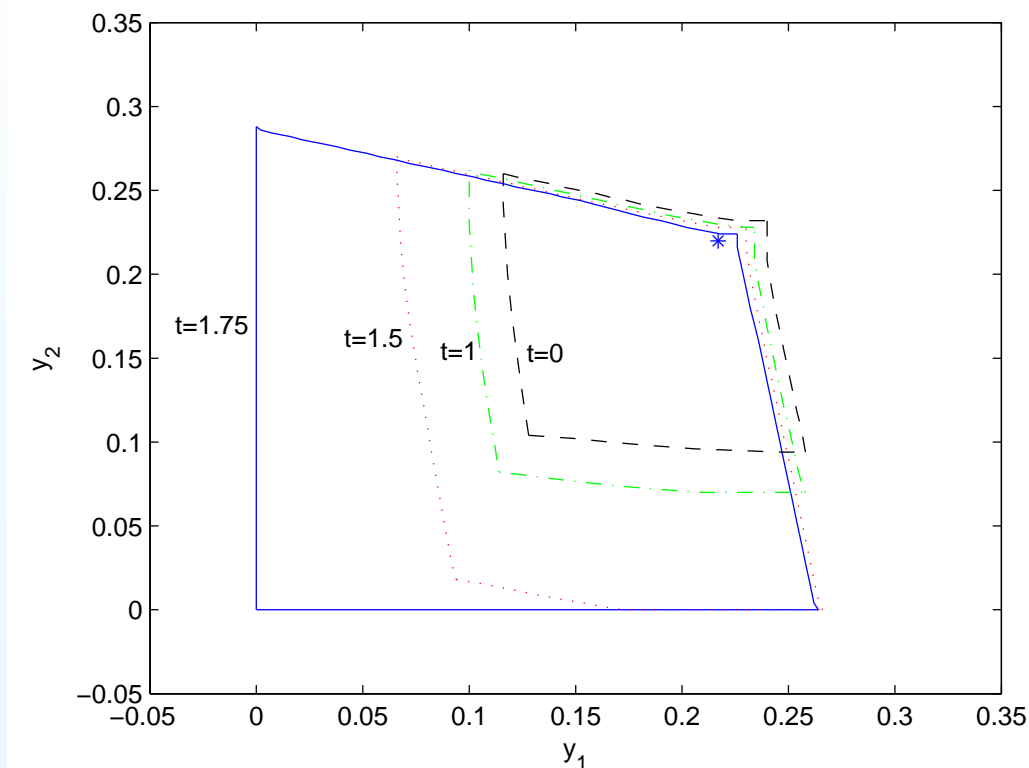
The case  $\alpha - r - (1 - \gamma)\sigma^2 > 0$ . This figure shows that the leverage is possible when the excess return  $(\alpha - r)$  is large enough. Parameter default values:  $\alpha = 0.30$ ,  $r = 0.07$ ,  $\sigma = 0.3$ ,  $\gamma = -1$ ,  $\beta = 0.1$ ,  $\lambda = \mu = 0.01$ ,  $T = 3$ .

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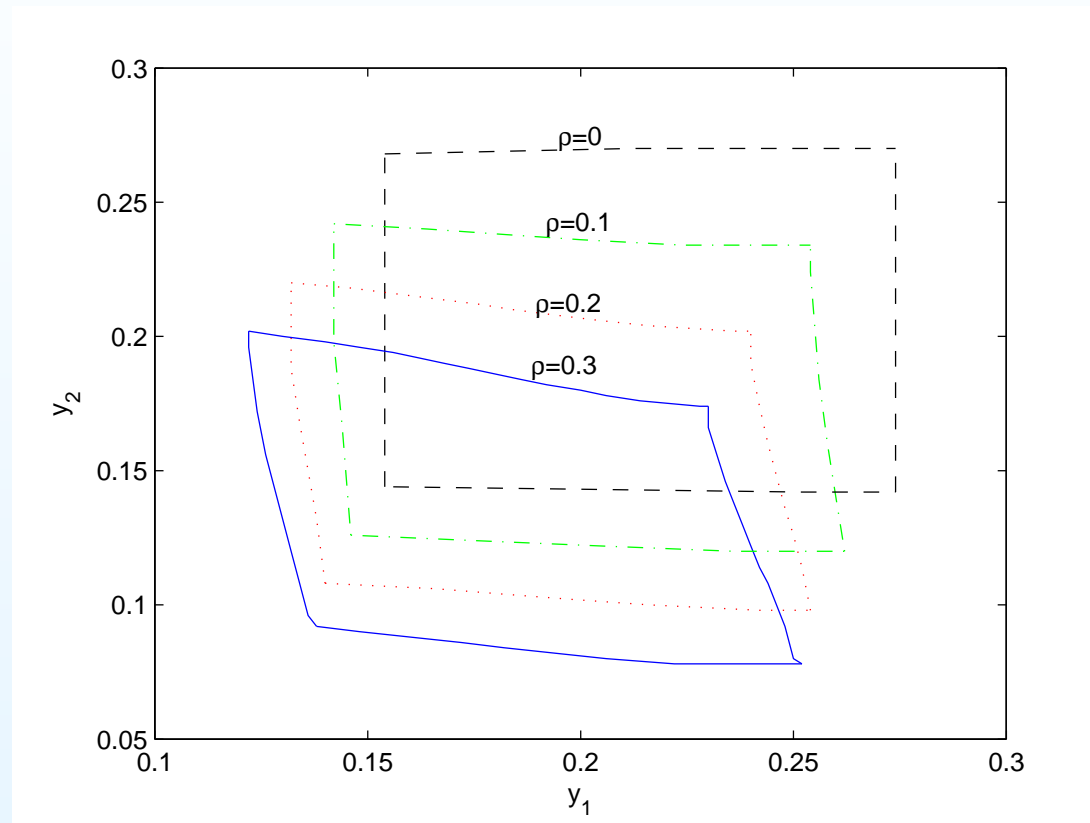
An example of nonmonotone boundaries for consumption case when  $\beta$  is large. Parameter default values:  $\alpha = 0.18$ ,  $r = 0.07$ ,  $\sigma = 0.3$ ,  $\gamma = -1$ ,  $\beta = 7$ ,  $\lambda = \mu = 0.01$ ,  $T = 3$ .

## Numerical results: optimal trading strategies ( $N = 2$ )



The no transaction region with respect to time  $t$ . From this figure, we can see the buy boundaries for both  $y_1$  and  $y_2$  equal zero at  $t = 1.75$ , close to the expire date  $T = 2$ . This illustrates that it is never optimal to buy risky assets when the time horizon is close to the expire date since there is not enough time to recover the transaction costs. Parameter default values:  $\alpha_1 = 0.15$ ,  $\alpha_2 = 0.12$ ,  $r = 0.07$ ,  $\sigma_1 = 0.4$ ,  $\sigma_2 = 0.3$ ,  $\rho = 0.2$ ,  $\gamma = -1$ ,  $\beta = 0.1$ ,  $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$ ,  $T = 2$ ,  $\kappa = 1$ .

## Numerical results: optimal trading strategies ( $N = 2$ )



The no transaction region at  $t = 0$ : the impact of positive correlation  $\rho > 0$ .  
Parameter default values:  $\alpha_1 = 0.14$ ,  $\alpha_2 = 0.11$ ,  $r = 0.07$ ,  $\sigma_1 = 0.4$ ,  $\sigma_2 = 0.3$ ,  
 $\gamma = -1$ ,  $\beta = 0.1$ ,  $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$ ,  $T = 2$ ,  $\kappa = 0$ .

# Conclusion and future work

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- Conclusion
  - A unified framework of penalty method
  - A convergence analysis of penalty method
  - A reduced penalty method based on the connection of singular control and optimal stopping
  - Numerical results
- Future work
  - Numerical methods for  $N \geq 4$
  - Fixed transaction costs: a quasi-linear variational inequality

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# *Questions and Comments*