Path regularity and explicit convergence rate of truncated quadratic growth BSDE

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Outline

The basics

What is a Backward Stoch. Diff. eq. or BSDE?

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Filtration $(\mathcal{F}_t)_{t\in[0,T]}$.
A BSDE is an equation of the type:

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds$$

- T, deterministic terminal time
- ξ, the terminal condition. An $\mathcal{F}_T$ adapted integrable R.V.
- $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ we call generator

The basics
Applications

- Mathematical Finance
- Stochastic Control

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- PDEs through Feynman-Kac

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El Karoui & Peng & Quenez: "‘BSDE in Finance"’ (1997)

The basics

A particular setting: Quadratic growth BSDE (qgBSDE)

- Our setting is the following:

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T \left[f(s, Y_s, Z_s) + \alpha Z_s^2 \right] ds, \quad (\Omega, \mathcal{F}, \mathbb{P})$$

with $\xi \in L^\infty$ and $f(s, y, z)$ is Lipschitz.

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- Kobylanski (2000)
  - unique solutions (Y, Z) exist
  - Y is bounded
  - Z is square integrable

Our problem
Numerics for qgBSDE

- Given a qgBSDE

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- Find a numerical scheme to approximate the BSDE!
  - Do usual discretization work?

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- Find a numerical scheme to approximate the BSDE!
  - Do usual discretization work?
  - Can one prove the *path regularity Theorem* in this setting?
  - Would a truncation procedure work?

Tools - BMO property

A hidden result in Kobylanski (2000):
There exists a $C > 0$ for all $t \in [0, T]$ such that

$$\mathbb{E}\Big[\int_t^T Z_s^2 ds | \mathcal{F}_t\Big] \leq C$$

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- the smallest C is the BMO norm
- C depends on: T, bound of ξ, Lipschitz constant and α

Tools: What can BMO do for you?

(Kazamaki, 1994): For $\int_0^{\cdot} ZdW \in BMO$

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(Kazamaki, 1994): For $\int_0^\cdot ZdW \in BMO$

- $\mathcal{E}(\int ZdW)$ is uniformly integrable (Girsanov Theorem)
- There exists a number $\bar{r} \in (1, \infty)$ such that
  - $\mathcal{E}(\int ZdW) \in L^{\bar{r}}$
  - $\bar{r}$ is determined by the BMO norm
  - define q such that: $1/\bar{q} + 1/\bar{r} = 1$

(Allows use of Hölder inequality to return to $\mathbb{P}$)

A priori estimate

$$
Y_t = \xi_1 - \int_t^T Z_s dW_s + \int_t^T f_1(s, Y_s, Z_s) ds
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\tilde{Y}_t = \xi_2 - \int_t^T \tilde{Z}_s dW_s + \int_t^T f_2(s, \tilde{Y}_s, \tilde{Z}_s) ds
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then, we have

$$\mathbb{E} \sup_t |Y_t - \tilde{Y}_t|^{2p} + \mathbb{E} \left(\int_0^T |Z_s - \tilde{Z}_s|^2 ds \right)^p$$
$$\leq \mathbb{E} \left[|\xi_1 - \xi_2|^{2p\bar{q}^2} + \left(\int_0^T |f_1(Z) - f_2(Z)|^2 ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}$$

Our Result / Contribution

Truncation
Part 1 - The idea

$$Y_t^n = \xi - \int_t^T Z_s^n dW_s + \int_t^T f(s, Y_s^n, Z_s^n) + \alpha(h_n(Z_s^n))ds$$

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then the new BSDE is Lipschitz and we can use any existing numerical scheme!

$$\text{Error}(Y - Y^{n,\pi}) \leq \text{Error}(Y - Y^n) + \text{Error}(Y^n - Y^{n,\pi})$$

The problem is how to control the error coming from the truncation itself!

Truncation

Part 2 - Explicit convergence rate

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T \Big[f(s, Y_s, Z_s) + \alpha Z_s^2 \Big] ds$$

Define a family of truncation functions:

$$h_n(z) = \begin{cases} z^2 & \text{, if } |z| \leq n \\ \frac{2n|z| - n^2}{z} & \text{, if } |z| > n \end{cases}$$

We obtain a family of BSDEs

$$Y_t^n = \xi - \int_t^T Z_s^n dW_s + \int_t^T \Big[f(s, Y_s^n, Z_s^n) + \alpha h_n(Z_s^n) \Big] ds$$

Truncation
Part 3 - The conditions

The inequality is reached by properly estimating

$$\mathbb{E}\Big(\int_0^T |Z-h_n(Z)|^2 ds\Big)^\gamma \leq \mathbb{E}\Big(\int_0^T |Z|^2 \mathbf{1}_{\{|Z|>n\}} ds\Big)^\gamma$$

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- Chebyshev inequality applied to Z
- Implies estimating $\mathbb{E}\sup_t |Z_t|^p$
- implies estimating $D_u Y_t$ and establish continuity

Truncation
Part 4 - The Rate

We need the following conditions:

- drift and volatility are C_b^1; the volatility is non-degenerate
- ξ is bounded. f is continuous and $|f(t,x,y,z)| \leq M(1+|y|+|z|^2)$ with $M > 0$
- f is differentiable such that

$$|\nabla_x f| \leq M(1+|y|+|z|^2), \quad |\nabla_y f| \leq M, \quad |\nabla_z f| \leq M(1+|z|)$$

Then

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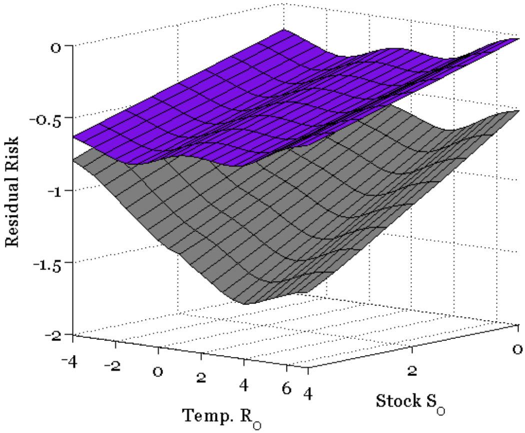
Then

$$\boxed{\mathbb{E}\Big[\sup_t |Y_t - Y_t^n|^{2p}\Big] + \mathbb{E}\Big[\Big(\int (Z_s - Z_s^n)^2 ds\Big)^p\Big] \leq C_p \sqrt[2\bar{q}]{\frac{1}{n}}}$$

An example

$$\frac{1}{2}[(\theta^S)^2-(z^2)^2-2\theta^S z^1]$$



The path regularity theorem

Let π be a uniform partition of $[0, T]$ and define

$$\bar{Z}_{t_i} = \frac{1}{h}\mathbb{E}[\int_{t_i}^{t_{i+1}} Z_s ds|\mathcal{F}_{t_i}].$$

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Then under the same conditions as above

$$\sum_{i=0}^{N-1} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}|^2 ds\Big] \le C|\pi|$$

Sum up

- Explicit convergence rate of truncated qgBSDE $\left(\sqrt[2\bar{q}]{\frac{1}{n}}\right)$
- Added the path regularity theorem to the qgBSDE toolbox

Thank you!

Thank you very much!

For Further Reading I

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Mathematical Finance, Vol.7 (No. 1):1-71, 1997.

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Preprint 2009