The COS Method in Pricing Bermudan and Barrier options under Heston's model

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Introduction

- The Heston Model:

$$\begin{aligned} dx_t &= \left(\mu - \frac{1}{2}\nu_t\right) dt + \sqrt{\nu_t} dW_{1,t} \\ d\nu_t &= \lambda \left(\bar{\nu} - \nu_t\right) dt + \eta \sqrt{\nu_t} dW_{2,t}, \end{aligned}$$

  where $\lambda \geq 0, \bar{\nu} \geq 0$ and $\eta \geq 0$ denote the speed of the mean reversion, the mean level of variance and the volatility of volatility, respectively. The Brownian motions $W_{1,t}$ and $W_{2,t}$ are correlated with correlation coefficient ρ.

- Pricing Formula for Bermudan Options:

$$\begin{cases} c(x, \nu_{t_{m-1}}, t_{m-1}) = e^{-r\Delta t} \int \int v(y, \nu_{t_m}, t_m) p\left(y, \nu_{t_m} | x, \nu_{t_{m-1}}\right) dy d\nu_{t_m} \\ v(x, t_{m-1}) = \max\left(g(x, t_{m-1}), c(x, t_{m-1})\right) \end{cases}$$

  and

$$v(x, \nu_{t_0}, t_0) = e^{-r\Delta t} \int \int v(y, \nu_{t_1}, t_1) p(y, \nu_{t_1} | x, \nu_{t_0}) dy d\nu_{t_1}.$$

Difficulties in Integration-based Methods

$$c(x, \nu_{t_m}, t_m) = e^{-r\Delta t} \int \int v(y, \nu_{t_{m+1}}, t_{m+1}) p\left(y, \nu_{t_{m+1}} | x, \nu_{t_m}\right) dy d\nu_{t_{m+1}}$$

- Direct application of numerical integration rules would have $O(N^2 M^2)$ complexity, if there are M points in $\nu_{t_{m+1}}$-dimension and N points in y-dimension.
- Neither the joint probability density, $p(y, \nu_{t_{m+1}} \mid x, \nu_{t_m})$, nor its characteristic function is known in closed-form.
- What we do know is the characterisitic function of $p_y(y| \ x, \nu_{t_{m+1}}, \nu_{t_m})$ (the probability density of y given x, $\nu_{t_{m+1}}$ and ν_{t_m}) as well as $p_\nu(\nu_{t_{m+1}}|\nu_{t_m})$ (the probability density of $\nu_{t_{m+1}}$ conditioned on ν_{t_m}). However, both involve the modified Bessel function of the first kind, which overflows on not-too-big y or $\nu_{t_{m+1}}$.

The Densities

- Since neither y nor x is involved in the SDE of $\nu_{t_{m+1}}$, it then follows that

  $$p_\nu\left(\nu_{t_{m+1}}|x,\nu_{t_m}\right) = p_\nu\left(\nu_{t_{m+1}}|\nu_{t_m}\right),$$

  and thus,

  $$p\left(y,\nu_{t_{m+1}} \mid x,\nu_{t_m}\right) = p_y\left(y|\, x,\nu_{t_{m+1}},\nu_{t_m}\right)\cdot p_\nu\left(\nu_{t_{m+1}}|\nu_{t_m}\right).$$

- $p_\nu\left(\nu_{t_{m+1}}|\nu_{t_m}\right)$ has a closed-form expression that involves

  $$I_\alpha\left(2le^{-\frac{1}{2}\lambda\Delta t}\sqrt{\nu_{t_{m+1}}\nu_{t_m}}\right),$$

  where $I_\alpha(x)$ is the modified Bessel function of the first kind.
- $p_y(y|\, x,\nu_{t_{m+1}},\nu_{t_m})$ is not known but we can derive its characteristic function, which again involves $I_\alpha\left(2le^{-\frac{1}{2}\lambda\Delta t}\sqrt{\nu_{t_{m+1}}\nu_{t_m}}\right)$.

Our Method

- Main Idea:
  - Applying the COS method (a method based on Fourier-cosine series expansion) in the log-stock dimension.
  - Using $\ln(\nu_t)$ instead of ν_t to obtain exponential error convergence in the Trapezoidal rule.
  - Modifying the algorithm that computes modified Bessel functions of the first kind to let it return $e^{-x}I_\alpha(x)$, so as to solve the overflow problem.
- Performance of the Method:
  - Exponential error convergence and almost linear computational complexity in the log-stock dimension.
  - Exponential error convergence in the log-volatlity dimension.
  - Free from overflow problem.

The COS Method for Density Recovery

- Cosine expansions of densities on a finite interval $[a, b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

- F_n has close relation to ch.f., $\phi(\xi) := \int_{\mathbb{R}} f(x) e^{i\xi x} dx$: if $\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$,

$$\begin{aligned} F_n \approx A_n : &= \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \frac{2}{b-a} \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a}\right) \exp\left(-i\frac{ka\pi}{b-a}\right)\right\}. \end{aligned}$$

The COS Formula for Density Recovery

- Replace F_n by A_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} {\sum_{n=0}^{N-1}}' \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}; x\right) \exp\left(in\pi \frac{-a}{b-a} \right) \right\} \cos\left(n\pi \frac{x-a}{b-a} \right),$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a, b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

Exponential error convergence in N!

Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(x,t_0) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(y,T)|x\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(y,T)f(y|x)dy.$$

- Truncate the integration range:

$$v(x,t_0) = e^{-r\Delta t}\int_{[a,b]} v(y,T)f(y|x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange the summation and the integration:

$$\hat{v}(x,t_0) = e^{-r\Delta t}{\sum_{n=0}^{N-1}}{}' \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a};x\right)e^{-in\pi\frac{a}{b-a}}\right\}V_n,$$

  where V_n, the series coefficients of the payoff, have analytical solutions.
- Simplified formula for Heston and Lévy processes

$$v(\mathbf{x},t_0) \approx \mathbf{K}e^{-r\Delta t}\cdot\mathrm{Re}\left\{{\sum_{n=0}^{N-1}}{}'\varphi\left(\frac{n\pi}{b-a}\right)U_n\cdot e^{in\pi\frac{\mathbf{x}-a}{b-a}}\right\},$$

  where $\varphi(\xi) := \phi(\xi;0)$

Numerical Results

Pricing for 21 strikes $K = 50, 55, 60, \cdots, 150$ under Heston's model. Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \lambda = 1.5768, \eta = 0.5751, \bar{u} = 0.0398, u_0 = 0.0175, \rho = -0.5711$.

	N	32	64	96	128	160
COS	(msec.)	0.852	1.446	2.039	2.641	3.220
	max. abs. err.	1.43e-01	6.75e-03	4.52e-04	2.61e-05	$4.40e-06$
	N	512	1024	2048	4096	8192
Carr-Madan	(msec.)	7.436	12.84	20.36	37.69	76.02
	max. abs. error	4.70e+06	6.69e+01	2.61e-01	$2.15e-03$	2.08e-07

Pricing Bermudan Options under Lévy Processes

- Starting point: the pricing formulae

$$\begin{cases} c(x, t_{m-1}) &= e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_m) f(y|x) dy \\ v(x, t_{m-1}) &= \max\left(g(x, t_{m-1}), c(x, t_{m-1})\right) \end{cases}$$

  and

$$v(x, t_0) = e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_1) f(y|x) dy.$$

- Step 1: use Newton's method to locate the early exercise point x_m^*, which is the root of

$$g(x, t_m) - c(x, t_m) = 0.$$

- Step 2: Replace $c(y, t_{m+1})$ by the COS representation then interchange the summation and integration to yield

$$\mathbf{V}(t_m) = \mathrm{M}\,\Lambda \mathbf{V}(t_{m+1}) + \mathbf{G}(t_m).$$

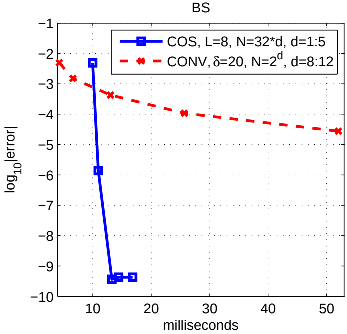
  FFT algorithm can be applied since M has a special structure.

- Step 3: Repeat step 1 to 2 recursively until we obtain $\mathbf{V}(t_1)$; Insert it into the COS formula for European options to get $v(x, t_0)$.

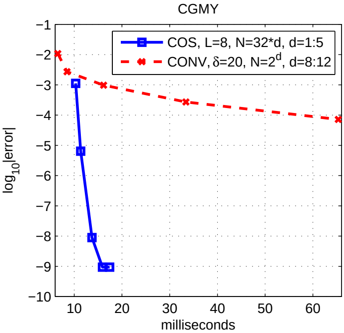
Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

Test No.	Model	S_0	K	T	r	σ	Other Parameters
2	BS	100	110	1	0.1	0.2	–
3	CGMY	100	80	1	0.1	0	$C = 1, G = 5, M = 5, Y = 1.5$



(a) BS



(b) CGMY with $Y = 1.5$

Pricing Bermudan Options under Heston's Model

- Starting point: the pricing formulae

$$\begin{cases} c(x, \ln(\nu_{t_{m-1}}) t_{m-1}) & = & e^{-r\Delta t} \int\int v(y, \ln(\nu_{t_m}), t_m) p_y(y|\ln(\nu_{t_m}), x) \\ & & \cdot p_v\left(\ln(\nu_{t_m})|\ln(\nu_{t_{m-1}})\right) dyd\ln(\nu_{t_m}) \\ v(x, t_{m-1}) & = & \max\left(g(x, t_{m-1}), c(x, t_{m-1})\right) \end{cases}$$

- Step 1: Modify the algorithm of the modified Bessel functions to let it return $e^{-x} I_\alpha(x)$ instead; Modify the formula of the ch.f. of $p_y(y|\ln(\nu_{t_m}), x)$ and the formula of $p_v\left(\ln(\nu_{t_m})|\ln(\nu_{t_{m-1}})\right)$ accordingly.
- Step 2: Apply Trapezoidal rule on the integral over $\ln(\nu_{t_m})$; Recover $p_y(y|\ln(\nu_{t_m}), x)$ from its ch.f. by the COS method.
- Step 3: Use Newton's method to locate early-exercise point.
- Step 4: Derive the recursive formula for Cosine coefficients:

$$\mathbf{V}(t_m; \ln(\nu)[k]) = \mathrm{M}\ \Lambda \mathbf{V}(t_{m+1}; \ln(\nu)[k]) + \mathbf{G}(t_m).$$

  FFT algorithm can be applied since M has a special structure.

- Step 5: Repeat step 3 to 4 recursively until we obtain $\mathbf{V}(t_1; \ln(\nu)[k])$. Insert $\mathbf{V}(t_1)[k]$ into the COS formula for European options and then compute the Trapezoidal Sum to get $v(x, \ln(\nu_{t_0}), t_0)$.

Bermudan puts with 5 early-exercise dates

Table: Error Convergence and the CPU times

$J = 200$			$N = 64$		
N	error	CPU time	J	error	CPU time (sec.)
8	-2.0e-1	0.11	80	-4.5e-2	0.19
16	-7.6e-3	0.22	120	3.9e-3	0.38
32	-1.5e-6	0.45	160	-3.7e-5	0.62
64	5.2e-9	0.88	200	5.2e-9	0.88

ref=1.125668694931144; $\lambda = 5; \eta = 0.1; \bar{\nu} = 0.16; \nu_{t_0} = 0.25; \rho = 0.1; r = 0.04$ and $T = 0.5$; Other parameters: $S_0 = 10, K = 10$ and $q = 0$.

Conclusions

- We presented the COS method for pricing Bermudan options under Heston's model.
- The computational speed is fast due to exponential error converges in both dimensions so that with very small N and J one can already have satisfactory accuracy.
- The method is stable and free from overflow problem.
- The overall computational complexity is $O(MNJ^2 \log_2(N))$.
- The same idea can be applied to pricing Barrie options as well. It would be even faster without the employment of the Newton's method.