0 1 0 rg 0 1 0 RG

Computation of VaR and CVaR using stochastic approximation and adaptive importance sampling

Thèse à la DRI en collaboration avec:

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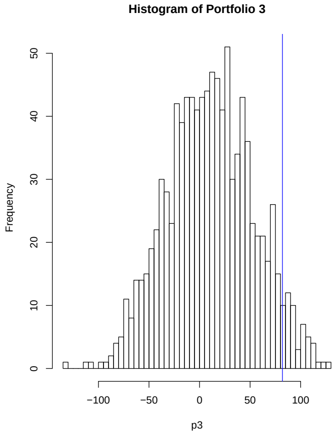
- 1 short position on a power plant that produces electricity from gas.
- 30 long positions (one every day) on calls on electricity.
- Gas and Electricity prices : Black Scholes model with a correlation.
- $X = \mathcal{N}(0, I_d)$ (d=60), loss is a complex random variable which can be written

$$L = \varphi_3(X) = \sum_{t=1}^{T} e^{r(T-t)} \left((S_t^e - h_R S_t^g - C)_+ - (S_t^e - K)_+ \right) + \left(e^{rT} C_0 - P_0^c e^{rT} \right)$$

Worst-case loss level : VaR-CVaR



- What is my worst-case scenario -with a 99% confidence ? VaR_α ξ^* : $\mathbb{P}(L \leq \xi^*) = \alpha = 99\%$

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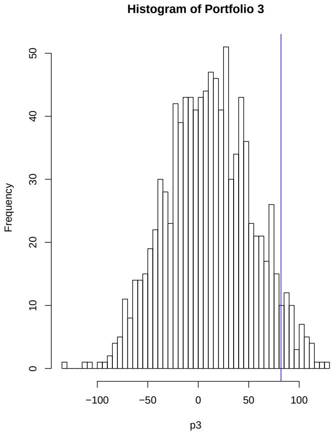

- What is my worst-case scenario -with a 99% confidence ? VaR_α ξ^* : $\mathbb{P}(L \leq \xi^*) = \alpha = 99\%$
- How does my loss look like when I exceed this scenario ? $CVaR_\alpha := \mathbb{E}\left[L | L \geq \xi^*\right]$

Motivation : Which one should we use ?

- VaR_α is widely used. However, it penalizes diversification and does not take into account the structure of the loss beyond the VaR.
- $CVaR_\alpha$ has better properties : it is a coherent risk measure according to Artzner, Delbaen, Eber & Heath (1999).
- Basically related to large losses (rare events), measuring both quantities is a significant challenge.

Methods to measure VaR and CVaR

There are several approaches to approximate VaR and CVaR

- linear or quadratic expansion of the distribution of the loss and joint normal or log-normal distribution assumption : Britten-Jones & al. (1999), Glasserman & al. (2000), (2002), ...
- VaR and CVaR are solution of a convex optimization problem, approximate both by solving a linear programming problem : Rockafellar, Uryasev (2000).
- quantile method based on a (weighted) empirical distribution function + projected importance sampling algorithm : Egloff & al. (2007).

Guidelines

VaR-CVaR : solutions of a convex optimization problem

$L = \varphi(X)$ where X a $\mathbb{R}^d$ *r.v.*, φ a real Borel functions such that : $\varphi(X)$ is continuous and $\varphi \in L^1(\mathbb{P}_X)$. Confidence level $\alpha \in (0, 1)$:

$$\begin{cases} \xi^* = VaR_\alpha(\varphi(X)) \quad \text{if} \quad \mathbb{P}(\varphi(X) \leq \xi^*) = \alpha \\ C^* = CVaR_\alpha(\varphi(X)) := \mathbb{E}\left[\varphi(X) | \varphi(X) \geq \xi^*\right] \end{cases}$$

Solutions of a convex optimization problem

$$\begin{cases} C^* = V(\xi^*) = \min_{\xi \in \mathbb{R}} \left(V(\xi)\right) \\ VaR_\alpha(\varphi(X)) = \xi^* = \arg\min_{\xi \in \mathbb{R}} V(\xi) \end{cases}$$

with
$V : \xi \rightarrow \xi + \frac{1}{1-\alpha}\mathbb{E}[(\varphi(X) - \xi)_+]$ (convex and differentiable)

Stochastic Recursive Zero search by...

In order to compute ξ^*,we have to find the zero of V'

$$\xi^* = \arg\min_{\xi\in\mathbb{R}} V(\xi) \in \{\xi \in \mathbb{R} | V'(\xi) = 0\}$$

$$V(\xi) = \mathbb{E}[v(\xi, X)] \Rightarrow \; V'(\xi) = \mathbb{E}\left[\frac{\partial v}{\partial \xi}(\xi, X)\right] = \mathbb{E}\left[H(\xi, X)\right]$$

Local gradient :

$$H(\xi, x) := \frac{\partial v}{\partial \xi}(\xi, x) = 1 - \frac{1}{1-\alpha}\mathbf{1}_{\{\varphi(x)\geq\xi\}}$$

...a Robbins Monro procedure.

Let $X_n \overset{i.i.d}{\approx} X$ *and the step sequence* $(\gamma_n)_{n\geq 1}$ *:*

$$\sum_{n\geq 1} \gamma_n = +\infty \text{ and } \sum_{n\geq 1} \gamma_n^2 < +\infty$$

Set

$$\xi_n = \xi_{n-1} - \gamma_n H(\xi_{n-1}, X_n), n \geq 1, \xi_0 \in L^1(\mathbb{P})$$

There exists $\xi^* : (\Omega, \mathcal{A}, \mathbb{P}) \to \{V' = 0\}, \xi^* \in L^2(\mathbb{P})$ *such that*

$$\xi_n \xrightarrow{a.s.} \xi^*, \quad n \to +\infty$$

The convergence also holds in $L^p(\mathbb{P}), \quad p \in (0, 2).$

Why ?

1. Key 1 : $\mathcal{F}_n := \sigma\left(\xi_0, X_1, ..., X_n\right)$

$$\Delta M_n := H\left(\xi_{n-1}, X_n\right) - V'\left(\xi_{n-1}\right) = H\left(\xi_{n-1}, X_n\right) - \mathbb{E}\left[H\left(\xi, X\right)\right]_{\xi=\xi_{n-1}}$$

so that $$\underbrace{\xi_n = \xi_{n-1} - \gamma_n V'\left(\xi_{n-1}\right)}_{Newton-Raphson\ \ likealgo.} - \underbrace{\gamma_n \Delta M_n}_{Disturbance\ \ term}.$$

2. Key 2 : If $\|H(\xi, X)\|_2 \leq C(1 + |\xi|)$ and $\sum_{n\geq 1} \gamma_n^2 < +\infty$, then

$$\sup_n < M >_n := \sum_{n\geq 1} \gamma_n^2 \mathbb{E}\left[\Delta M_n^2 | \mathcal{F}_{n-1}\right] < \infty$$

so that $$M_n \xrightarrow{a.s.} M_\infty \in L^2\left(\mathbb{P}\right).$$

Companion procedure for the CVaR

$$C^* = V(\xi^*) = \mathbb{E}[v(\xi^*, X)]$$

Replace ξ^* by ξ_n + Compute empirical mean

$$C_n := \frac{1}{n}\sum_{k=1}^{n} v\left(\xi_{k-1}, X_k\right) = C_{n-1} - \frac{1}{n}\left(C_{n-1} - v(\xi_{n-1}, X_n)\right)$$

$$\Rightarrow C_n = \frac{1}{n}\sum_{k=1}^{n} v(\xi_{k-1}, X_k) = \underbrace{\frac{1}{n}\sum_{k=1}^{n} V(\xi_k)}_{\xrightarrow{a.s.} V(\xi^*)} + \underbrace{\frac{1}{n}\sum_{k=1}^{n} v(\xi_{k-1}, X_k) - V(\xi_{k-1})}_{\textit{Martingale disturbance term}: \frac{N_n}{n} \xrightarrow{a.s.} 0}$$

Two components algorithm

$$\begin{cases} \xi_n = \xi_{n-1} - \gamma_n H(\xi_{n-1}, X_n), \quad \xi_0 \in L^1(\mathbb{P}) \\ C_n = C_{n-1} - \frac{1}{n}(C_{n-1} - v(\xi_{n-1}, X_n)), \; n \geq 1, C_0 \in L^1(\mathbb{P}) \end{cases}$$

Why $\frac{1}{n}$ in the second component ? Two different step sizes (see e.g. Mokkadem, Pelletier (2006)).

$$\Rightarrow \begin{cases} \xi_n = \xi_{n-1} - \gamma_n H(\xi_{n-1}, X_n), \; \xi_0 \in L^1(\mathbb{P}) \\ C_n = C_{n-1} - \gamma_n(C_{n-1} - v(\xi_{n-1}, X_n)), \; n \geq 0, \; C_0 \in L^1(\mathbb{P}) \end{cases}$$

Choice of step size $(\gamma_n)_{n\geq 1}$ according :

$$\sum_{n\geq 1} \gamma_n = +\infty \;\text{ and }\; \sum_{n\geq 1} \gamma_n^2 < +\infty$$

Optimal Step size for the algorithm

- $(\gamma_n) \equiv (Cn^{-1})$, $C > 0$: optimal rate of convergence (a.s convergence, CLT : weak rate $\sqrt{n}$, LIL, ...) but condition on C involving $f_{\varphi(X)}(\xi^*)$. Thus, difficult to handle.

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- Solution : Ruppert Polyak averaging principle provides the best asymptotic behaviour.

Ruppert and Polyak Averaging principle

Compute the regular $\mathbb{R}^d$-value Robbins-Monro procedure with $\gamma_n = \frac{\gamma_1}{n^p}$ with $\frac{1}{2} < p < 1$

$$Z_n = Z_{n-1} - \gamma_n H(Z_{n-1}, X_n), \; n \geq 1, Z_0 \in L^1(\mathbb{P})$$

$$+$$

Compute the Césaro mean of Z_n which can be written recursively

$$\bar{Z}_n = \bar{Z}_{n-1} + \frac{1}{n}(Z_{n-1} - \bar{Z}_{n-1}), \; n \geq 0, \; \bar{Z}_0 = 0.$$

Then, under some classical assumptions, **Gaussian CLT :**

$$\sqrt{n}\left(\bar{Z}_n - z^*\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

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A first CLT

Proposition 1 (Bardou, Frikha, Pagès, (2008))
Let $(\gamma_n)_{n\geq 1} \equiv (\gamma_1 n^{-p})$ with $\frac{1}{2} < p < 1$, $\gamma_1 > 0$. Suppose $f_{\varphi(X)} > 0$.

If $\exists a > 1 : \ \varphi(X) \in L^{2a}(\mathbb{P}), \varphi(X)$ is continous, increasing.

Then,

$$\sqrt{n}\begin{pmatrix} \overline{\xi}_n - \xi^* \\ \overline{C}_n - C^* \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Sigma)$$

where Σ is the asymptotic covariance matrix given by

$$\begin{cases} \Sigma_{1,1} = \frac{\alpha(1-\alpha)}{f_{\varphi(X)}(\xi^*)} \\ \Sigma_{1,2} = \Sigma_{2,1} = \frac{\alpha}{(1-\alpha)} \frac{1}{f_{\varphi(X)}(\xi^*)} \mathbb{E}[(\varphi(X) - \xi^*)_+] \\ \Sigma_{2,2} = \text{Var}\left((\varphi(X) - \xi^*)_+\right) \end{cases}$$

Important issues

- Slow and Chaotic convergence since $\mathbb{P}(\varphi(X) \geq VaR_\alpha) = 1 - \alpha \approx 0$ (rare events)

$$\xi_n = \xi_{n-1} - \gamma_{n-1}\left(1 - \frac{1}{1-\alpha}\mathbf{1}_{\{\varphi(X_n)\geq\xi_{n-1}\}}\right)$$

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- For pratical implementation, the above procedure must be combined with variance reduction techniques : through Importance Sampling, we can give greater probability to *important scenarios* in which $\varphi(X)$ exceeds ξ.

Finite-dimensional setting : translation of the mean

Computation of the mean

$F(X)$ with $F : \mathbb{R}^d \to \mathbb{R}$, $\mathbb{P}_X(dx) = p(x)\lambda_d(dx)$, for every $\theta \in \mathbb{R}^d$

$$\mathbb{E}[F(X)] = \mathbb{E}\left[F(X+\theta)\frac{p(X+\theta)}{p(X)}\right]$$

Select the one with lowest variance, *i.e* the one with lowest quadratic norm :

$$Q(\theta) := \mathbb{E}\left[F^2(X+\theta)\frac{p^2(X+\theta)}{p^2(X)}\right] = \mathbb{E}\left[F^2(X)\frac{p(X)}{p(X-\theta)}\right], \quad \theta \in \mathbb{R}^d.$$

Mean translation for log-concave p.d.f.

We make the following assumptions

$$\text{p is log-concave and } \lim_{|x|\to+\infty} p(x) = 0$$

and

$$\exists b \in [1,2] \text{ such that } \begin{cases} (i) & \frac{\|\nabla p\|}{p}(x) = O(||x||^{b-1}) \quad \text{as} \quad \|x\| \to \infty \\ (ii) & \exists \rho > 0, \log\left(p(x)\right) + \rho \left\|x\right\|^{b} \quad \text{is convex.} \end{cases}$$

Under this assumptions, Q is finite, convex and goes to infinity at infinity so that

$$\arg\min_{\theta\in\mathbb{R}^d} Q(\theta) = \left\{\theta \in \mathbb{R}^d \middle| \nabla Q(\theta) = 0\right\}$$

How to select the optimal parameter ?

- Arouna (Gaussian framework), uses the representation of the gradient :

$$\nabla Q(\theta) = \mathbb{E}\left[F^2(X)\frac{p(X)}{p^2(X-\theta)}\nabla p(X-\theta)\right]$$

However, it leads to the explosion of the procedure. Sub-linear growth assumption is not fulfilled :

$$||H(\theta, X)||_2 \leq C(1+|\theta|)$$

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- Remedy : algorithm with repeated projections “a la Chen” (repeated trials). A.s. convergence holds with a CLT (Lelong 2007) once stabilization has occured.

New approach (Lemaire, Pagès (2008)) :

third change of variable to plug θ in the payoff F !

$$\nabla Q(\theta) = \mathbb{E}\left[\underbrace{F(X-\theta)\frac{p^2(X-\theta)}{p(X)p(X-2\theta)}\frac{\nabla p(X-2\theta)}{p(X-2\theta)}}_{H(\theta,X)}\right],$$

and, Growth control of F at infinity

$$\forall x \in \mathbb{R}^d,\ |F(x)| \leq C\ e^{a\|x\|}.$$

$$\theta_n = \theta_{n-1} - \gamma_n e^{-2a(|\theta|^2+1)-2\rho||\theta||^b} H(\theta_{n-1}, X_n),\ \ n \geq 1$$

a.s. converges toward $\theta^* \in \arg\min_{\theta \in \mathbb{R}^d} Q(\theta)$.

Variance reduction using Unconstrained Adaptive Importance Sampling

Twist the distribution of $\varphi(X)$ to minimize the quadratic norm of the two components in the CLT :

$$\begin{cases} Q_1(\theta) := \mathbb{E}\left[\mathbf{1}_{\{\varphi(X)\geq\xi^*\}}\frac{p(X)}{p(X-\theta)}\right] \\ Q_2(\mu) := \mathbb{E}\left[(\varphi(X)-\xi^*)_+^2\,\frac{p(X)}{p(X-\mu)}\right] \end{cases}$$

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- **Minimizing two variances :**

$$\Rightarrow \begin{cases} \theta_n = \theta_{n-1} - \gamma_n H_3\left(\xi_{n-1}, \theta_{n-1}, X_n\right) \\ \mu_n = \mu_{n-1} - \gamma_n H_4\left(\xi_{n-1}, \mu_{n-1}, X_n\right) \\ (\theta_0, \mu_0) \in \mathbb{R}^d \times \mathbb{R}^d \end{cases}$$

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- $\theta_n \xrightarrow{a.s.} \theta^* = \arg\min Q_1$ and $\mu_n \xrightarrow{a.s.} \mu^* = \arg\min Q_2$.

Now, plug back adaptively the IS parameters into the VaR-CVaR algorithm :

$$\begin{cases} L_1(\xi,\theta,x) := e^{-\rho||\theta||^b}\left(1 - \frac{1}{1-\alpha}\mathbf{1}_{\{\varphi(x+\theta)\geq\xi\}}\frac{p(x+\theta)}{p(x)}\right) \\ L_2(\xi,C,\mu,x) := C - v(\xi,\mu,x) \\ v(\xi,\mu,x) := \xi + \frac{1}{1-\alpha}(\varphi(x+\mu)-\xi)_+\frac{p(x+\mu)}{p(x)} \end{cases}$$

New VaR-CVaR algorithm :

$$\begin{cases} \xi_n = \xi_{n-1} - \gamma_n L_1(\xi_{n-1},\theta_{n-1},X_n), \ \xi_0 \in L^1(\mathbb{P}), \\ C_n = C_{n-1} - \gamma_n L_2(\xi_{n-1},C_{n-1},\mu_{n-1},X_n), \ C_0 \in L^1(\mathbb{P}) \end{cases}$$

$$\sqrt{n}\begin{pmatrix} \overline{\xi}_n - \xi^* \\ \overline{C}_n - C^* \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0,\Sigma^*)$$

Gaussian case

Suppose $X \stackrel{d}{=} \mathcal{N}(0, I_d)$,then

$$\frac{p^2(X-\theta)}{p(X)p(X-2\theta)} \frac{\nabla p(X-2\theta)}{p(X-2\theta)} = e^{||\theta||^2}(2\theta - X)$$

then,

$$\begin{cases} H_3(\xi,\theta,X) = \mathbf{1}_{\{\varphi(X-\theta)\geq\xi\}}(2\theta - X) \\ H_4(\xi,\mu,X) = e^{-2a(||\mu||^2+1)}\left(\varphi(X-\mu)-\xi\right)_+^2(2\mu - X). \end{cases}$$

Serious issue : $\varphi(X-\theta) \geq \xi$ involves rare event to update the IS parameters θ_n, μ_n. In practice, we need to control the growth of the IS parameters at the beginning of the procedure.

Control the move to the critical risk area

In order to control the growth of θ_n and μ_n, it is possible to move slowly the *critical risk area* in which $\varphi(X)$ exceeds ξ by replacing α by a deterministic sequence α_n that converges to α.

$$\begin{cases} \hat{\xi}_{n+1} = \hat{\xi}_n - \gamma_{n+1} H_n(\hat{\xi}_n, X_{n+1}), n \geq 0 \\ H_n(\hat{\xi}_n, X_{n+1}) := \frac{1}{1-\alpha_n} \mathbf{1}_{\{X_{n+1} \geq \hat{\xi}_n\}} \\ \hat{\xi}_0 \in L^1(\mathbb{P}) \end{cases}$$

Under the previous assumptions and if $\sum \gamma_n |\alpha_n - \alpha| < \infty$, $\hat{\xi}_n$ converges towards ξ^*. New IS algorithm :

$$\begin{cases} \hat{\theta}_n = \hat{\theta}_{n-1} - \gamma_n H_3(\hat{\xi}_{n-1}, \hat{\theta}_{n-1}, X_n) \ \ n \geq 1, \ \ \hat{\theta}_0 \in \mathbb{R}^d \\ \hat{\mu}_n = \hat{\mu}_{n-1} - \gamma_n H_4(\hat{\xi}_{n-1}, \hat{\mu}_{n-1}, X_n), \ \ n \geq 0, \ \ \hat{\mu}_0 \in \mathbb{R}^d \end{cases}$$

Numerical examples

Computation of VaR_α, $CVaR_\alpha$ in a normal distribution framework : $X = \mathcal{N}(0, I_d)$.

- (1)Short position in 1 put : $X = \mathcal{N}(0,1)$:
$$\varphi_1(X) := (K - S_T)_+ - e^{rT} P_0$$
- (2)Short positions in 10 calls and 10 puts. $X = \mathcal{N}(0, I_d)$ (d=5) : $\varphi_2(X) := \sum_{i=1}^{5} \sum_{j=1}^{10} (K_p^j - S_T^i)_+ - e^{rT} P_0^{i,j} + (S_T^i - K_c^j)_+ - e^{rT} C_0^{i,j}$
- (3)Short position in a power plant that produces electricity and 30 long positions in calls on electricity day-ahead price $X = \mathcal{N}(0, I_d)$ (d=60) :

$$\varphi_3(X) = \sum_{t\geq 1}^{T} \left(e^{r(T-t)} \left(S_t^e - h_R S_t^g - C \right)_+ - P_0^c e^{rT} \right) + \left(e^{rT} C_0 - e^{r(T-t)} \left(S_t^e - K \right)_+ \right)$$

Portfolio 1

TABLE: Portfolio 1 Results

Number of steps	α	VaR_α	$CVaR_\alpha$	RV_{VaR}	RV_{CVaR}
10 000	95%	24.6	29.9	5.5	30.5
	99%	34.4	37.5	11.1	125.3
	99.5%	37.8	41.4	13.4	192.9
100 000	95%	24.6	30.4	6.6	32.2
	99%	34.18	37.9	11.5	127.9
	99.5%	37.3	40.7	15.1	185
500 000	95%	24.6	30.3	7.7	31.3
	99%	34.2	38	14.6	118.4
	99.5%	37.3	40.5	15.5	184

Portfolio 2

TABLE: Portfolio 2 Results

Number of steps	α	VaR_α	$CVaR_\alpha$	RV_{VaR}	RV_{CVaR}
10 000	95%	339	440.5	6.5	14.9
	99%	493.1	561.4	10.1	24.3
	99.5%	540.1	606.4	18.2	37.9
100 000	95%	349.8	439.7	6.7	17
	99%	495.7	563.8	11.3	28.6
	99.5%	544.8	607.8	18.9	40.3
500 000	95%	352.4	439.6	6.8	17.3
	99%	495.2	563	11.1	27.7
	99.5%	545.3	608.4	19.2	37

Portfolio 3

TABLE: Portfolio 3 Results

Number of steps	α	VaR_α	$CVaR_\alpha$	RV_{VaR}	RV_{CVaR}
10 000	95%	115.7	150.5	3.4	6.8
	99%	169.4	196	8.4	12.9
	99.5%	186.3	213.2	13.5	20.3
100 000	95%	118.7	150.5	4.5	8.7
	99%	169.4	195.4	12.6	17.5
	99.5%	188.8	212.9	15.6	29.5
500 000	95%	119.2	150.4	5	9.2
	99%	169.8	195.7	13.1	18.6
	99.5%	188.7	212.8	17	29

Conclusion 1/2

- Fast convergence of VaR_α and $CVaR_\alpha$ and great reduction of variance especially when $\mathbb{P}(\varphi(X) \geq \xi^*) = 1 - \alpha$ is close to zero.

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- Fast convergence of VaR_α and $CVaR_\alpha$ and great reduction of variance especially when $\mathbb{P}(\varphi(X) \geq \xi^*) = 1 - \alpha$ is close to zero.
- Fast convergence of importance sampling procedures : 10,000-20,000 steps to obtain good estimate of θ^* and μ^*.
- Variance reduction achieved in estimating the *Conditional Value-at-Risk* is greater than the one achieved in estimating the *Value-at-Risk*. The factor $\frac{1}{f_{\varphi(X)}(\xi^*)}$ in the asymptotic variance of VaR_α.

Conclusion 2/2

- Esscher Transform applied to non gaussian r.v.
- Quasi-Monte Carlo applied to the VaR-CVaR algorithm.
- Infinite dimensional setting : VaR-CVaR and variance reduction (Girsanov Theorem) for loss that depends on the path of a process $X = (X_t)_{t\in[0,T]}$. (preprint to appear soon)