Approximations of boundary conditions for HJB equations by using stochastic algorithms

HUANG Junbo

INRIA Sophia Antipolis-Méditerranée

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Given a stochastic differential equation

$$dX(s) = b(s, X(s), \pi(s))ds + \sigma(s, X(s), \pi(s))dW(s), \quad t \leq s \leq T$$

whose solution is $X(s) = X_s^{t,x,\pi}$ with initial data $X(t) = x$.
Consider the stochastic control problem

$$V(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ \int_t^T L(s, X_s^{t,x,\pi}, \pi(s))ds + \psi(X_T^{t,x,\pi}) \right\}. \quad (1)$$

$V(t,x)$ is the classical or viscosity solution to the *Hamilton-Jacobi-Bellman* (HJB) PDE:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{H}(t, x, D_x V, D_x^2 V) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n, \\ \\ V(T, x) = \psi(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where

$$\mathcal{H}(t, x, p, A) = \sup_{u \in \Pi} \left[b \cdot p + \frac{1}{2} Tr(A\sigma\sigma^T) + L \right].$$

Thus, in order to solve (1) we need to solve (2). To solve (2) numerically, we should consider an HJB equation localized in a finite domain:

$$\begin{cases} \dfrac{\partial \widetilde{V}}{\partial t} + \mathcal{H}(t, x, D_x \widetilde{V}, D_x^2 \widetilde{V}) = 0, \quad (t, x) \in [0, T) \times O, \\ \widetilde{V}(T, x) = \psi(x), \quad x \in \overline{O}, \\ \widetilde{V}(t, x) = \Psi(t, x), \quad (t, x) \in ([0, T) \times \partial O), \end{cases} \tag{3}$$

where $\overline{O}$ is bounded in $\mathbb{R}^n$.

Questions

It raises two questions:

Question 1 What are suitable boundary conditions (i.e. $\Psi(t, x)$) for the localized HJB equations?

Question 2 What is the error of localization (i.e. $|V - \widetilde{V}|$) if the last-mentioned boundary conditions are not given precisely?

Observations

- No “financial laws” (compared with “physical laws”).
- No general methodology to provide suitable boundary conditions. It should be carried out case-by-case.
- Consider the HJB equations derived from the portfolio utility maximization problems, which can also be solved theoretically by the martingale method (or the duality approach) introduced by Karatzas.

A financial market model

"Bond": one riskless asset

$$\frac{dB(s)}{B(s)} = r(s)ds, \quad t \leq s \leq T.$$

"Stocks": *n* risky assets

$$\frac{dP_i(s)}{P_i(s)} = \mu_i(s) + \sum_{j=1}^{n} \sigma_{ij}(s)dW_j(s), \quad t \leq s \leq T.$$

A financial market model

"Bond": one riskless asset

"Stocks": n risky assets

Wealth process

$$dX(s) = \left(X(s) - \sum_{i=1}^{n} \pi_i(s)\right) \frac{dB(s)}{B(s)} + \sum_{i=1}^{n} \pi_i(s) \frac{dP_i(s)}{P_i(s)}, \tag{4}$$

where $\pi_i(s)$ is the amount of money invested in the i^{th} stock.

The portfolio utility maximization problem

- Utility function U: strictly increasing, strictly concave, continuously differentiable.
- Utility maximization:

$$V(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X^{t,x,\pi}(T))\right].$$

- The corresponding HJB equation:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{H} = 0, \quad (t,x) \in [0,T) \times (0,+\infty), \\ V(T,x) = U(x), \quad x > 0. \end{cases} \tag{5}$$

The martingale method

The martingale method shows if

$$V(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X^{t,x,\pi}(T))\right],$$

then

$$V(t,x) = \mathbb{E}[U(I(\lambda^* H_T^{t,1}))] \quad (6)$$

with

$$\mathbb{E}(I(\lambda^* H_T^{t,1}) H_T^{t,1}) = x, \quad (7)$$

where I is the inverse of U' and $H_s^{t,1}$ is the solution to the equation

$$dH(s) = H(s)\left[-r(s)ds - \vartheta^T(s)dW(s)\right], \quad t \leq s \leq T, \qquad H(t) = 1. \quad (8)$$

So, the natural way to approximate $V(t,x)$ is:

- Approximate $H_T^{t,1}$ by the Euler scheme. Get $H_T^{(M)} \to H_T^{t,1}$ as $M \to \infty$.
- For given M, let

$$V_2^{(M)} = \mathbb{E}[U(I(\lambda_1^* H_T^{(M)}))] \tag{9}$$

  with

$$\mathbb{E}(I(\lambda_1^* H_T^{(M)})H_T^{(M)}) = x. \tag{10}$$

  $V_2^{(M)}$ can be considered as an approximation of $V(t,x)$.
- Solve (10) numerically. Get $\lambda_N \to \lambda_1^*$.
- By using λ_N, find $V_3^{(N)}$ such that $V_3^{(N)} \to V_2^{(M)}$, $N \to \infty$ for M given.

$V_3^{(N)}$ is then a suitable approximation of boundary conditions.

We shall prove

Result 1

- $|V(t,x) - V_2^{(M)}| \leq C\frac{1}{M^{\rho_0}}$, where C, ρ_0 are constants and C depends on x and T.

Result 2

- For given M,
  $\mathbb{P}\left(|V_3^{(N)} - V_2^{(M)}| \leq CN^{-\frac{1}{4}}\right) \geq 4\Phi(N^{\frac{1}{4}}) - 3 - C'N^{-\frac{\delta}{2}}$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function, C and C' are constants and $\delta > 0$.

$V_3^{(N)}$ will be specified later.

Approximation by the Euler scheme

Note that $H_T^{t,1} = \exp\{-h(T)\}$, where $h(\cdot)$ satisfies

$$\begin{cases} dh(s) = -(r(s) + \frac{1}{2}|\vartheta(s)|^2)ds - \vartheta^T(s)dW(s), \\ h(t) = 0. \end{cases} \tag{11}$$

The Euler scheme of (11) is defined by

$$\begin{cases} h_0^{(M)} = 0, \\ h_i^{(M)} = h_{i-1}^{(M)} - (r(t_{i-1}) + \frac{1}{2}|\vartheta(t_{i-1})|^2)\Delta^M t \\ \qquad - \vartheta^T(t_{i-1})\Delta^M W_i, \qquad i = 1, 2, \dots, M. \end{cases} \tag{12}$$

Then the Euler approximation of $H_T^{t,1}$ is

$$H_T^{(M)} = \exp\{-h_M^{(M)}\}.$$

Approximation by the Euler scheme

(12) can be written in a "continuous" form

$$h^{(M)}(s) = -\int_t^s [r(m_{\tilde{s}}^{(M)}) + \frac{1}{2}|\vartheta(m_{\tilde{s}}^{(M)})|^2] d\tilde{s} - \int_t^s \vartheta^T(m_{\tilde{s}}^{(M)}) dW(\tilde{s}), \tag{13}$$

where

$$m_s^{(M)} = \max\{t_i,\ i = 0, \ldots, M \text{ s.t. } t_i \leq s\}.$$

Another market model

Let

$$\tilde{r}(s) = r(m_s^{(M)}), \quad \tilde{\mu}(s) = \mu(m_s^{(M)}), \quad \tilde{\sigma}(s) = \sigma(m_s^{(M)}). \tag{14}$$

We then can define the similar "Bond" process $\widetilde{B}(\cdot)$, the "Stock" process $\widetilde{P}_i(\cdot)$ and the wealth process $\widetilde{X}(\cdot)$.

A key observation

If $V_2^{(M)}$ is defined by

$$V_2^{(M)} = \mathbb{E}[U(I(\lambda_1^* H_T^{(M)}))]$$

with

$$\mathbb{E}(I(\lambda_1^* H_T^{(M)}) H_T^{(M)}) = x,$$

where $H_T^{(M)}$ is the Euler approximation of $H_T^{t,1}$, then

$$V_2^{(M)} = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(\widetilde{X}^{t,x,\pi}(T))\right]$$

according to the martingale method.

Therefore,

$$V(t,x) - V_2^{(M)} = \sup_{\pi} \mathbb{E}\left[U(X^{t,x,\pi}(T))\right] - \sup_{\pi} \mathbb{E}\left[U(\widetilde{X}^{t,x,\pi}(T))\right].$$

Since X and $\widetilde{X}$ are the solutions of two SDEs, the error between X and $\widetilde{X}$ is controlled by the coefficients of the SDEs. Thus it is not difficult to prove

Result 1

$|V(t,x) - V_2^{(M)}| \leq C\frac{1}{M^{\rho_0}}$, where C, ρ_0 are constants and C depends on x and T.

Approximation by the RM algorithm

We need to solve numerically

$$\mathbb{E}(I(\lambda_1^* H_T^{(M)})H_T^{(M)}) = x \tag{15}$$

to find λ_1^*. λ_1^* can be regarded as a zero of the following function

$$f(\theta) = \mathbb{E}(I(\theta H_T^{(M)})H_T^{(M)}) - x, \quad \theta > 0. \tag{16}$$

We then can invoke the so-called Robbins-Monro (RM) algorithm.

The RM algorithm

Consider a function $f(\theta) = \mathbb{E}(Y(X,\theta))$, where X is a random variable. Suppose $f(\theta^*) = 0$. The Robbins-Monro algorithm is usually defined as

$$\theta_n = \theta_{n-1} + \gamma_n Y(X_n, \theta_{n-1}), \tag{17}$$

where $\{\gamma_n\}$ is a suitable sequence satisfying

$$\gamma_n > 0, \quad \gamma_n \xrightarrow[n\to\infty]{} 0, \quad \sum_{n=1}^{\infty} \gamma_n = \infty, \tag{18}$$

and $\{X_n, n \in \mathbb{N}\}$ is i.i.d. and each X_n has the same probability distribution as X.

The RM algorithm

Many authors have proved that under certain conditions θ_n converges to θ^* a.s. and $\frac{\theta_n - \theta^*}{\sqrt{\gamma_n}}$ is asymptotically normal. We, however, have provided a Berry-Esseen type rate of convergence for the RM algorithm.

Theorem

For any $\nu > 0$,

$$\mathbb{P}\left(\frac{|\theta_n - \theta^*|}{\sqrt{\gamma_n}} \leq C\nu\right) \geq 2\Phi(\nu) - 1 - C'\gamma_n^{\delta},$$

where C and C' are constants, $\delta > 0$ and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

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where C *and* C' *are constants,* $\delta > 0$ *and* $\Phi(\cdot)$ *is the standard normal cumulative distribution function.*

Go back to our problem:

$$V_2^{(M)} = \mathbb{E}[U(I(\lambda_1^* H_T^{(M)}))]$$

with

$$f(\lambda_1^*) = \mathbb{E}(I(\lambda_1^* H_T^{(M)}) H_T^{(M)}) - x = 0. \tag{19}$$

Apply the RM algorithm to (19) and obtain a RM sequence $\{\theta_n\}$ such that $\theta_n \to \lambda_1^*$. Let

$$V_3^{(N)} = \frac{1}{N} \sum_{i=1}^{N} U(I(\theta_{i-1} X_i)), \tag{20}$$

where each X_n has the same probability distribution as $H_T^{(M)}$ and $\{X_n, n \in \mathbb{N}\}$ is the same sequence used in the RM algorithm.

By the Berry-Esseen type result for the RM algorithm, we then can show

Result 2

- For given M, $\mathbb{P}\left(|V_3^{(N)} - V_2^{(M)}| \leq CN^{-\frac{1}{4}}\right) \geq 4\Phi(N^{\frac{1}{4}}) - 3 - C'N^{-\frac{\delta}{2}}$, where $\Phi(\cdot)$ is the standard normal cumulative distribution function, C and C' are constants and $\delta > 0$.

Questions

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In a complete market

We have

Theorem

$$|V(t,x) - \widetilde{V}(t,x)| \leq \sup_{0 \leq s \leq T} |V(s,\beta) - \widetilde{V}(s,\beta)| F(\beta)$$

for $t \in [0,T]$, $x \in (0,\beta)$. *Especially, when* r, μ_i *and* σ_{ij} *are constants, for* $t \in [0,T]$, $x \in (0, \beta e^{-r(T-t)})$,

$$F(\beta) = \Phi\left(\Phi^{-1}\left(\frac{x}{\beta}e^{r(T-t)}\right) + |\theta|\sqrt{T-t}\right),$$

where θ *is a known constant and* $\Phi(\cdot)$ *is the cumulative normal distribution function.*

Error estimates in high-dimensional space

Given a $\mathbb{R}^n$-valued state process $X(s)$ evolving as follows

$$dX(s) = b(s, X(s), \pi(s))ds + \sigma(s, X(s), \pi(s))dW(s), \quad t \leq s \leq T.$$

Denote this process by $X_s^{t,x}$ if $X(t) = x$. And assume

$$|b(t, x, u) - b(t, y, u)| + \|\sigma(t, x, u) - \sigma(t, y, u)\| \leq K|x - y|,$$

$$|b(t, x, u)| + \|\sigma(t, x, u)\| \leq K(1 + |x|),$$

for some constant $K > 1$.

Error estimates in high-dimensional space

Theorem

$$|V(t,x) - \widetilde{V}(t,x)|$$
$$\leq \sup_{0\leq s\leq T,\,|y|=R} |V(s,y) - \widetilde{V}(s,y)| 2e^{-\frac{9}{2}K^2(T-t)}$$
$$\left(\frac{1+|x|^2}{1+R^2}\right)^{\frac{1}{18K^2(T-t)}\ln\frac{1+R^2}{1+|x|^2}-1}$$

Thanks!