Recent Developments on Sensitivity Analysis for Pure-Jump Infinite-Activity Lévy processes

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$$\frac{\partial}{\partial S_0}\mathbb{E}\left[\Phi(S_T)\right].$$

By the martingale representation theorem, it holds that

$$\Phi(S_T) = \mathbb{E}\left[\Phi(S_T)\right] + \int_0^T \frac{\partial}{\partial S_t}\mathbb{E}\left[\Phi(S_T)|\mathscr{F}_t\right] dS_t, \quad a.s.$$

Estimating Derivatives

To compute this sensitivity, a standard method is a finite difference approximation;

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}\left[\Phi((1+\varepsilon)S_T) - \Phi((1-\varepsilon)S_T)\right]}{2\varepsilon S_0} = \frac{\partial}{\partial S_0}\mathbb{E}\left[\Phi(S_T)\right],$$

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Unless $\varepsilon = 0$, this method is biased!

Malliavin Calculus Approach (Fournié et al [1999])

Using some (Gaussian) Malliavin calculus formulas: the integration by parts;

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Therefore, we get an estimator;

$$\frac{\partial}{\partial S_0}\mathbb{E}\left[\Phi(S_T)\right] = \frac{1}{S_0}\mathbb{E}\left[\Phi'(S_T)S_T\right] = \frac{1}{S_0\sigma T}\mathbb{E}\left[\Phi(S_T)W_T\right].$$

Features of Malliavin Formulas

Compared with a finite difference estimator

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2. requires no computation of additional points $\Phi((1+\varepsilon)S_T)$ and $\Phi((1-\varepsilon)S_T)$!
3. possibly reduces variance in Monte Carlo simulation!

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For example, a Malliavin calculus has been developed with an integration-by-parts formula

$$\mathbb{E}\left[\Phi(Y_T)\int_0^T\int_{\mathbb{R}_0} X(t,z)(\mu-\nu)(dz,dt)\right] = \mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0} X(t,z)\left(D_{t,z}\Phi(Y_T)\right)\nu(dz)dt\right],$$

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$$D_{t,z}\Phi(Y_T) = \Phi(Y_T + D_{t,z}Y_T) - \Phi(Y_T),$$

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which are unfortunately not useful in the sensitivity analysis framework. (Related Works; (i) simple Poisson processes by El-Khatib and Privault [2004], (ii) jump-diffusion by Davis and Johansson [2006] and Cass and Friz [2007].)

1: Lévy Processes as Time-Changed Brownian Motion

(This part is joint work with Arturo Kohatsu-Higa.)

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1. $\{Y_t : t \geq 0\}$ is a one-sided positive Lévy process,
2. $\{W_t : t \geq 0\}$ is a standard Brownian motion,
3. $\{Y_t : t \geq 0\}$ and $\{W_t : t \geq 0\}$ are independent.

Set a time-changed Brownian motion

$$X_t := \mu t + \theta Y_t + \sigma W_{Y_t},$$

and consider an asset price dynamics model

$$S_t := S_0 \frac{e^{X_t}}{\mathbb{E}\left[e^{X_t}\right]}.$$

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For example,

1. $\{Y_t : t \geq 0\}$ is a gamma process $\rightarrow$ $\{X_t : t \geq 0\}$ is called a variance gamma (Lévy) process,

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Conditional on $\{Y_t : t \geq 0\}$ (due to the independence), the (Gaussian) Malliavin calculus approach is almost directly applicable;

$$\begin{aligned}\mathbb{E}\left[\Phi'(S_T)S_T \middle| Y_T\right] &= \mathbb{E}\left[\Phi(S_T)\delta\left(\frac{S_T}{\int_0^{Y_T} D_v S_T dv}\right)\middle| Y_T\right] \\ &= \mathbb{E}\left[\Phi(S_T)\frac{W_{Y_T}}{\sigma Y_T}\middle| Y_T\right],\end{aligned}$$

and thus we get a delta formula

$$\frac{\partial}{\partial S_0}\mathbb{E}\left[\Phi(S_T)\right]=\frac{1}{S_0}\mathbb{E}\left[\Phi'(S_T)S_T\right]=\frac{1}{\sigma S_0}\mathbb{E}\left[\Phi(S_T)\frac{W_{Y_T}}{Y_T}\right].$$

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In a similar manner, we can derive other sensitivities

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4. Delta, Gamma and correlation for Basket.

2: Density Transform Approach based on Gamma Processes

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The gamma process is a one-sided pure-jump Lévy process in $[0,+\infty)$ whose Lévy measure is given in the form

$$\nu(dz) = a\frac{e^{-bz}}{z}dz, \quad z \in (0,+\infty),$$

where $a > 0$ and $b > 0$. For each (time) $t > 0$, its marginal has the gamma distribution with the characteristic function

$$\mathbb{E}_{\mathbb{P}}\left[e^{iyY_t}\right] = \exp\left[t\int_{(0,+\infty)}\left(e^{iyz}-1\right)\nu(dz)\right] = \left(1-\frac{iy}{b}\right)^{-at},$$

while the density function $f_t^{\mathbb{P}}$ of the marginal distribution at time t under $\mathbb{P}$ is given in closed form by

$$f_t^{\mathbb{P}}(y) = \frac{b^{at}}{\Gamma(at)}y^{at-1}e^{-by}, \quad y \in [0,+\infty).$$

Let $(\mathscr{F}_t)_{t\in[0,T]}$ be the natural filtration of $\{Y_t : t \in [0,T]\}$.
Define a probability measure $\mathbb{Q}_\lambda$ via the so-called Esscher transform

$$\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}\Big|_{\mathscr{F}_T} = \frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}\left[e^{\lambda Y_T}\right]} = \exp\left[\lambda Y_T - \varphi_T(\lambda)\right],$$

where

$$\varphi_T(\lambda) := Ta \ln \frac{b}{b-\lambda}.$$

The probability measures $\mathbb{P}$ and $\mathbb{Q}_\lambda$ are mutually absolutely continuous.

Scaling Property of Gamma Process

Lemma
It holds that

$$\mathscr{L}_{\mathbb{Q}_\lambda}\left(\{Y_t : t \geq 0\}\right) = \mathscr{L}_{\mathbb{P}}\left(\left\{\frac{b}{b-\lambda} Y_t : t \geq 0\right\}\right).$$

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Sketch of Proof

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Sketch of Proof

1. Both are Lévy processes.
2. The marginal laws are identical:

$$\mathbb{Q}_\lambda(Y_t \leq y) = \int_0^y f_{Y_t}^{\mathbb{Q}_\lambda}(x)dx = \int_0^{\frac{b-\lambda}{b}y} f_{Y_t}^{\mathbb{P}}(x)dx = \mathbb{P}\left(\frac{b}{b-\lambda} Y_t \leq y\right).$$

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Theorem

$$\frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}\left[\Phi(S_T)\right] = \frac{1}{S_0} \mathbb{E}_{\mathbb{P}}\left[\Phi'(S_T) S_T\right] = \frac{1}{S_0} \mathbb{E}_{\mathbb{P}}\left[\Phi(S_T) \frac{bY_T - aT + 1}{Y_T}\right].$$

Sketch of Proof

On one hand, by applying the scaling property, we have

$$\mathbb{E}_{\mathbb{Q}_\lambda}\left[\frac{\Phi(S_T^{(0)})}{Y_T}\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{\Phi(S_T^{(\lambda)})}{\frac{b}{b-\lambda}Y_T}\right].$$

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On the other hand, by the definition of the Esscher transform, we have

$$\mathbb{E}_{\mathbb{Q}_\lambda}\left[\frac{\Phi(S_T^{(0)})}{Y_T}\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}\Big|_{\mathscr{F}_T}\frac{\Phi(S_T^{(0)})}{Y_T}\right] = \mathbb{E}_{\mathbb{P}}\left[e^{\lambda Y_T - \varphi_T(\lambda)}\frac{\Phi(S_T^{(0)})}{Y_T}\right].$$

Hence, we get

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By taking derivative at $\lambda = 0$, we get

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Therefore,

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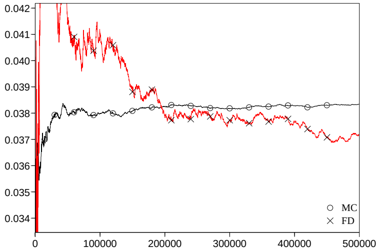
2. "Delta and Gamma" for Asians;

$$\mathbb{E}_{\mathbb{P}}\left[\Phi\left(\frac{1}{T}\int_0^T S_u du\right)\right],\ \mathbb{E}_{\mathbb{P}}\left[\Phi\left(N^{-1}\sum_{k=1}^{N} S_{t_k}\right)\right],$$

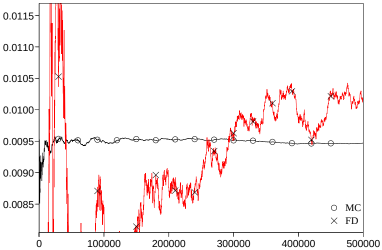
and

$$\mathbb{E}_{\mathbb{P}}\left[\Phi\left(\sqrt[N]{S_{t_1}S_{t_2}\cdots S_{t_N}}\right)\right].$$

Numerical Illustrations



$\frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}[\mathbb{1}(S_T > K)]$, vratio$= 35$.



$\frac{\partial^2}{\partial S_0^2} \mathbb{E}_{\mathbb{P}}[\mathbb{1}(S_T > K)]$, vratio$= 840$.

Conclusion

1. For the first part, we make use of the "independence" of the Brownian motion and the time-changing process.
2. For the second part, we make use of the "scaling property of gamma process" with respect to Esscher transform.
3. Our formulas are unbiased and often reduce Monte Carlo variance.
4. Both are interesting enough; our models can be both of "pure-jump type" and of "infinite-activity type."

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5. Future directions
   (i) general SDE form?
   (ii) Gaussian and Jump simultaneously?