An Improved Convolution Algorithm for Discretely Sampled Asian Options

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Overview

- Introduction to Asian options
- Pricing arithmetic Asian options
    - Underlying model
    - Review of previous pricing methods
- The backward price convolution algorithm
- Contribution - filling the gaps

Asian options

- Payoff (path-)dependence on the average price of some asset, over a pre-determined period of time.
- Raised popularity of Asian options
  - Reduced risk of market manipulation (average price is harder to manipulate).
  - Less exposed to extreme movements at settlement (averaging reduces the volatility of the underlying asset); hence, lower option prices.
- Prevalent case: Discrete sampling and arithmetic average
  - For example, call option with fixed strike price K has terminal payoff

$$\max\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K, 0\right) := \left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)^{+}.$$

- Lack of analytical tractability; need to apply numerical techniques to calculate $\mathbb{E}\left(\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)^{+}\right)$.

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Preliminaries

- 2-D framework:
  - Set of equidistant sampling points $\mathcal{T} := \{t_k\}_{k=0}^{n}$; $n \in \mathbb{N}$, $t_k - t_{k-1} = \delta t$; $1 \leq k \leq n$.
  - Carverhill-Clewlow-Hodges factorization: Generalized **Markov** process Y such that

    $$\begin{aligned} Y_k &:= \ln\left(e^{Y_{k-1}} + \lambda_{n+1-k}\right) + Z_{n+1-k};\ 0 < k \leq n, \\ Y_1 &:= \ln \lambda_n + Z_n;\ k = 1, \end{aligned}$$

    where $Z_k := \ln\left(\frac{S_k}{S_{k-1}}\right)$ for $1 \leq k \leq n$, $S_0 > 0$, is a sequence of independent random variables and λ some general deterministic process. For fixed-strike Asian call: $\lambda_0 := -\frac{K}{S_0}$ and $\lambda_1 = \ldots = \lambda_n := \frac{1}{n}$.
  - By recursive substitution, we get

    $$\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)^{+} = S_0\left(e^{Y_n} + \lambda_0\right)^{+}.$$

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Review of previous transform pricing approaches

- Key idea: $\ln\left(e^{Y_{k-1}} + \lambda_{n+1-k}\right) \perp Z_{n+1-k}$; hence, for densities h_k, $g_{k-1}$and $f_k$of Y_k, $\ln\left(e^{Y_{k-1}} + \lambda_{n+1-k}\right)$and Z_{n+1-k}respectively, we have

$$\begin{aligned} h_k &= g_{k-1} * f_k, \\ \mathcal{F}(h_k) &= \mathcal{F}(g_{k-1})\mathcal{F}(f_k). \end{aligned}$$

    - Evaluate the unconditional density of Y_k by a forward propagation in time until maturity ($k = n$) via FFT means.
    - Difficulty: The density h spreads out as k increases.
- Original application.Carverhill and Clewlow (1990): Use the same large equidistantly spaced grid for all variables Y_k.
- First extension.Benhamou (2002): Re-centre random variables: $Y_k - \mathbb{E}\left(Y_k\right)$, and evaluate on common grid.
    - Imperfect re-centring (non-trivial for high volatilities).
    - Slow convergence: variable precision 1-3 d.p.

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The Fusai-Meucci (2008) approach

- Second extension.Fusai and Meucci (2008): Use a fixed grid which is not equidistant and take advantage of Gaussian numerical integration.
  - Observed oscillatory convergence with the number of grid points.
    - Difficult to gauge precision of the scheme.
    - Convergence is exacerbated further by Lévy log-returns.
    - Variable precision 3-4 d.p.

Backward price convolution

The algorithm

- We replace forward density convolution by backward price convolution.
  - Computation of the density of Y_k is bypassed altogether.
- Smoothness of density of Z_{n+1-k} is not required anymore.
- Step 1. Initialize: $p_n(y) := (e^y + \lambda_0)^+$; $k = n$.
- Step 2. Risk-neutral valuation: $q_{k-1}(x) := \int_{\mathbb{R}} p_k(x+z) f_k(z)\, dz$; $0 < k \leq n$.
- Step 3. Take the Fourier transform: $\mathcal{F}(q_{k-1}) = \mathcal{F}(p_k)\bar{\phi}_k$.
- Step 4. Invert: $q_{k-1}(x) = \mathcal{F}^{-1}(\mathcal{F}(p_k)\bar{\phi}_k)$.
- Step 5. Function composition: $p_{k-1}(y) = q_{k-1}(\ln(e^y + \lambda_{n+1-k}))$
- Step 6. Repeat steps 2-5 until present time. Price: $P_0 = e^{-rt_n} S_0 q_0(\ln \lambda_n)$.

Contribution - filling the gaps

- For first time in the literature: Derivation of explicit bounds for the pricing error from the curtailment of the integration range.
- Regular second-order convergent numerical scheme (in the number of grid points) **independent** of model assumptions.
- 5 d.p. precision in less than 1 sec (1 year to maturity, weekly sampling)
    - Fusai and Meucci (2008): 3-4 d.p. in 5 sec on faster PC.
    - Monte Carlo with geometric Asian control variate: 3 d.p. (at 99% conf interval) in 19 sec.
    - Večeř's (2002) PDE worked out with the Crank-Nicolson FD scheme: 3-5 d.p. across strikes and volatilities in 1.1 sec per price.
- Higher precision (8-10 d.p.) possible for greater number of grid points (subject to linearly increasing computational effort).

References

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