

L_2 -time regularity of BSDEs with irregular terminal functions

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Backward Stochastic Differential Equations (BSDEs)

Example 1: The martingale representation theorem

If $(Y_t)_{0 \leq t \leq T}$ is a Brownian martingale, then $\exists (Z_t)_{0 \leq t \leq T}$ such that

$$Y_t = Y_T - \int_t^T Z_s dW_s.$$

Example 2: The European Call

In the BS model and under the risk-neutral probability \mathbb{Q} : If

$Y_t = C(t, S_t)$ and $Z_t = \sigma(t, S_t) \partial_S C(t, S_t)$, then

$$Y_t = (S_T - K)^+ - \int_t^T rY_s ds - \int_t^T Z_s dW_s.$$

Definition (Markovian BSDE)

The forward component:

$$\begin{cases} X_0 &= x_0, \\ dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t, \end{cases} \quad (1)$$

($X, x_0 \in \mathbb{R}^d$, $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$).

The backward component:

$$\begin{cases} -dY_t &= f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T &= g(X_T). \end{cases} \quad (2)$$

- $g(X_T)$ =terminal condition (T =maturity)
- f =driver, or generator
- Z =martingale integrand, or control variable

Existence and uniqueness

Theorem (Existence and uniqueness)

Let $p > 1$. Suppose

- $g(X_T) + \int_0^T |f(t, 0, 0, 0)| dt \in \mathbf{L}_p$
- f is continuous in t and Lipschitz-continuous in (x, y, z)

Then, BSDE (2) has a unique solution $(Y, Z) \in S^p \times \mathcal{M}^p$, i.e.

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^p \right] + \mathbb{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{p/2} \right] < +\infty.$$

Link with PDEs

Let \mathcal{L}_X be the infinitesimal generator of X :

$$\mathcal{L}_X u(t, x) = \sum_{i=1}^d b_i(t, x) \partial_{x_i} u(t, x) + \frac{1}{2} \sum_{i,j=1}^d [\sigma \sigma^*]_{i,j}(t, x) \partial_{x_i x_j}^2 u(t, x).$$

If u is the smooth solution to

$$\partial_t u(t, x) + \mathcal{L}_X u(t, x) + f(t, x, u(t, x), \nabla_x u(t, x) \sigma(t, x)) = 0 \text{ for } t < T,$$

$$u(T, x) = g(x)$$

then

$$Y_t = u(t, X_t),$$

$$Z_t = \nabla_x u(t, X_t) \sigma(t, X_t).$$

L_2 -time regularity

For a given time mesh $\pi = \{0 = t_0 < \dots < t_i < \dots < t_N = T\}$,

L_2 -time regularity of Z

$$\mathcal{E}(Z, \pi) := \sum_{i=0}^{N-1} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_t - \bar{Z}_{t_i}|^2 dt$$

where $\bar{Z}_{t_i} = \frac{1}{t_{i+1} - t_i} \mathbb{E}^{\mathcal{F}_{t_i}} \int_{t_i}^{t_{i+1}} Z_s ds$.

Purpose: estimate $\mathcal{E}(Z, \pi)$ according to the regularity of g and to $|\pi| = \sup_{0 \leq i < N} (t_{i+1} - t_i)$.

NB. If g is Lipschitz-continuous, $\mathcal{E}(Z, \pi) \leq C|\pi|$ (J.Zhang'04).

Motivation

A common numerical scheme for BSDEs is

$$\begin{cases} Y_{t_i}^\pi &= \mathbb{E}^{\mathcal{F}_{t_i}}(Y_{t_{i+1}}^\pi + (t_{i+1} - t_i)f(t_i, X_{t_i}, Y_{t_{i+1}}^\pi, Z_{t_i}^\pi)), \\ Z_{t_i}^\pi &= \frac{1}{(t_{i+1} - t_i)} \mathbb{E}^{\mathcal{F}_{t_i}}(Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i})^*). \end{cases} \quad (3)$$

It is known that

$$\begin{aligned} e(Y^\pi - Y, Z^\pi - Z, \pi) &:= \sup_{0 \leq i \leq N} \mathbb{E}(Y_{t_i}^\pi - Y_{t_i})^2 + \sum_{i=0}^{N-1} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_{t_i}^\pi - Z_t|^2 dt \\ &\leq C(|\pi| + \mathcal{E}(Z, \pi)). \end{aligned}$$

\implies The \mathbf{L}_2 -time regularity of Z plays a crucial role in the rate of convergence of (3).

The space $\mathbf{L}_{2,\alpha}$

Definition

For some $\alpha \in (0, 1]$,

$$\mathbf{L}_{2,\alpha} = \left\{ g \text{ s.t. } \mathbb{E}(g(X_T)^2) + \sup_{0 \leq t < T} \frac{\mathbb{E}(g(X_T) - \mathbb{E}^{\mathcal{F}_t}(g(X_T)))^2}{(T-t)^\alpha} < +\infty \right\}.$$

It describes the rate of decreasing of the expected conditional variance of $g(X_T)$ given \mathcal{F}_t as t goes to T .

NB. If X is a Brownian motion and $T = 1$ and $\alpha \neq 1$,
 $\mathbf{L}_{2,\alpha} = \mathbf{B}_{2,\infty}^\alpha$ (Besov space).

Examples

- If g is β -Hölder continuous, then $g \in \mathbf{L}_{2,\beta}$.
- If $d = q = 1$, $X = W$ and $g(x) = \mathbb{1}_{[0,+\infty)}(x)$, then $g \in \mathbf{L}_{2,\frac{1}{2}}$.
- More generally, for an indicator function of a smooth domain, $g \in \mathbf{L}_{2,\frac{1}{2}}$.

- When $\mathbb{E}|g(X_T)|^2 < +\infty$, we set

$$V_{t,T}(g) := \mathbb{E} |g(X_T) - \mathbb{E}^{\mathcal{F}_t}(g(X_T))|^2.$$

- When $g \in \mathbf{L}_{2,\alpha}$, we define $K^\alpha(g)$ as

$$K^\alpha(g) := \mathbb{E}|g(X_T)|^2 + \sup_{t \in [0, T)} \frac{V_{t,T}(g)}{(T-t)^\alpha}.$$

Assumption ($A_{b,\sigma}$)

- b and σ are bounded and twice continuously differentiable with respect to x , with uniformly bounded and γ -Hölder continuous derivatives, for some $\gamma \in (0, 1]$.
- b and σ are $\frac{1}{2}$ -Hölder continuous in time.
- σ is uniformly elliptic: $\exists \delta > 0$ such that, $\forall (t, x) \in [0, T] \times \mathbb{R}^d$, $[\sigma\sigma^*](t, x) \geq \delta I_d$.

Assumption (A_g)

- g is bounded (only for this section).

Under $(\mathbf{A}_{b,\sigma})$ and (\mathbf{A}_g) , the PDE

$$\begin{aligned} \partial_t u(t, x) + \mathcal{L}_X u(t, x) &= 0 \text{ for } t < T, \\ u(T, x) &= g(x) \end{aligned}$$

has a unique smooth solution u (on $[0, T) \times \mathbb{R}^d$).

Let $(y_t, z_t)_{0 \leq t \leq T}$ be the solution of the linear BSDE:

$$y_t = g(X_T) - \int_t^T z_s dW_s.$$

Then

$$y_t = u(t, X_t), \quad z_t = \nabla_x u(t, X_t) \sigma(t, X_t).$$

The time nets

We shall use the following net ($\beta \in (0, 1]$)

$$\pi^{(\beta)} := \left\{ t_k^{(N,\beta)} := T - T \left(1 - \frac{k}{N}\right)^{\frac{1}{\beta}}, 0 \leq k \leq N \right\}.$$

NB.

- $\pi^{(1)} = (t_k^{(N,1)})$ = the equidistant net.
- For $\beta < 1$, the points in $\pi^{(\beta)}$ are more concentrated near T .

Estimate of $\mathcal{E}(z, \pi)$

With the equidistant time net

Theorem

Assume $(\mathbf{A}_{b,\sigma})$ and (\mathbf{A}_g) . Assume moreover that $g \in \mathbf{L}_{2,\alpha}$, for some $\alpha \in (0, 1]$. Then, with the choice of the equidistant time net,

$$\mathcal{E}(z, \pi^{(1)}) = \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k^{(N,1)}}^{t_{k+1}^{(N,1)}} \left| z_s - \bar{z}_{t_k^{(N,1)}} \right|^2 ds \leq CK^\alpha(g) \left(\frac{T}{N} \right)^\alpha$$

(where C does not depend on N).

NB. We find again the rate $\frac{1}{N}$ for a Lipschitz-continuous g .

With the non-equidistant time net

Theorem

Assume $(\mathbf{A}_{b,\sigma})$, (\mathbf{A}_g) , and that $g \in \mathbf{L}_{2,\alpha}$, for some $\alpha \in (0, 1]$. Take β such that $\beta = 1$, if $\alpha = 1$, and $\beta < \alpha$ otherwise. Then, with the choice $\pi^{(\beta)}$ (defined in (1))

$$\mathcal{E}(z, \pi^{(\beta)}) = \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k^{(N,\beta)}}^{t_{k+1}^{(N,\beta)}} |z_s - \bar{z}_{t_k^{(N,\beta)}}|^2 ds \leq CK^\alpha(g) \frac{T^\alpha}{N}$$

(where C does not depend on N).

Using the PDE representation for z_t and Itô's formula for $\nabla_x u$, we can show that

$$\begin{aligned} \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k}^{t_{k+1}} |z_s - \bar{z}_{t_k}|^2 ds &\leq c |\pi| + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E} |\nabla_x u(t_k, X_{t_k})|^2 dr \\ &\quad + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t_{k+1} - r) \mathbb{E} |D^2 u(r, X_r)|^2 dr. \end{aligned}$$

\implies we need estimates for the derivatives of u .

Lemma

Assume $(\mathbf{A}_{b,\sigma})$ and (\mathbf{A}_g) . Then, there exists a positive constant C , such that, for all $t \in [0, T)$,

$$\begin{aligned}\mathbb{E}|u(t, X_t)|^2 &\leq \mathbb{E}|g(X_T)|^2, \\ \mathbb{E}|\nabla_x u(t, X_t)|^2 &\leq C \frac{V_{t,T}(g)}{T-t}, \\ \mathbb{E}|D^2 u(t, X_t)|^2 &\leq C \frac{V_{t,T}(g)}{(T-t)^2}.\end{aligned}$$

Idea of the proof: By Malliavin calculus,

$$\nabla_x u(t, X_t) = \mathbb{E}^{\mathcal{F}_t}[(g(X_T) - \mathbb{E}^{\mathcal{F}_t} g(X_T))H_{t,T}^{(1)}],$$

with $\mathbb{E}^{\mathcal{F}_t}(H_{t,T}^{(1)}) = 0$ and $\mathbb{E}^{\mathcal{F}_t}|H_{t,T}^{(1)}|^2 \leq c \frac{1}{T-t}$.

Similarly for $D^2 u(t, X_t)$.

Lemma

Let $\alpha \in (0, 1]$, and assume $(\mathbf{A}_{b,\sigma})$ and (\mathbf{A}_g) . Then the three following assertions are **equivalent**:

(i) $g \in L_{2,\alpha}$.

(ii)

$$\int_0^t \mathbb{E} \left| D^2 u(s, X_s) \right|^2 ds \leq \frac{CK^\alpha(g)}{(T-t)^{1-\alpha}}.$$

(iii)

$$\mathbb{E} \left| \nabla_x u(t, X_t) \right|^2 \leq \frac{CK^\alpha(g)}{(T-t)^{1-\alpha}}.$$

If $\alpha < 1$ (resp. $\alpha = 1$), the previous three assertions are also equivalent to (resp. lead to) the following one:

(iv)

$$\mathbb{E} \left| D^2 u(t, X_t) \right|^2 \leq \frac{K^\alpha(g)}{(T-t)^{2-\alpha}}$$

NB. Geiss and Hujo ('07), showed that (iii) in the previous Lemma is equivalent to $\mathcal{E}(z, \pi) \leq \frac{1}{N^\alpha}$ (for all time nets) when X is either the Brownian motion or the geometric Brownian motion.

\implies the choice of $\mathbf{L}_{2,\alpha}$ is natural in our framework.

Estimate of $Z_t - z_t$ when g is bounded**Estimate of $Z_t - z_t$ when g is bounded****Theorem**

Assume $(\mathbf{A}_{b,\sigma})$, (\mathbf{A}_g) and (\mathbf{A}_f) . Assume moreover that $g \in \mathbf{L}_{2,\alpha}$, for some $\alpha \in (0, 1]$. Then, $d\mathbb{P} \otimes dt$ - a.s., one has

$$Z_t - z_t = U_t \sigma(t, X_t),$$

where $(U_t, V_t)_{t \in [0, T]}$ satisfies a linear BSDE, with

$$\mathbb{E} \left[\sup_{r \in [0, T]} |U_r|^2 + \int_0^T |V_r|^2 dr \right] \leq_c T^\alpha K^\alpha(g) + T^2.$$

NB. The proof uses the link between $Z_t - z_t$ and the Malliavin derivative of $Y_t - y_t$, which satisfies a linear BSDE.

Now, we suppose that g is not necessarily bounded (and g belongs to $\mathbf{L}_{2,\alpha}$).

For $M > 0$, we set

$$g_M := -M \vee g \wedge M.$$

With some intermediary stability results when $M \rightarrow +\infty$, and noticing that $K^\alpha(g_M) \leq K^\alpha(g)$, we use the previous theorem to derive the following \mathbf{L}_2 -regularity results for the general BSDE (Y, Z) .

Theorem

Assume $(\mathbf{A}_{b,\sigma})$, (\mathbf{A}_f) and that $g \in \mathbf{L}_{2,\alpha}$ (*not necessarily bounded*), for some $\alpha \in (0, 1]$. Then, there is a positive constant C such that, for any time net $\pi = \{t_k : k = 0 \dots N\}$

$$\begin{aligned} \mathcal{E}(Z, \pi) &= \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k}^{t_{k+1}} |Z_s - \bar{Z}_{t_k}|^2 ds \\ &\leq C\mathcal{E}(z, \pi) + C(K^\alpha(g)T^\alpha + T^2)|\pi|. \end{aligned}$$

Theorem

Assume $(\mathbf{A}_{b,\sigma})$, (\mathbf{A}_f) and that $g \in \mathbf{L}_{2,\alpha}$ (*not necessarily bounded*), for some $\alpha \in (0, 1]$. Then, there is a positive constant C (which does not depend on N) such that

a) for the choice of the equidistant time net $\pi^{(1)}$,

$$\mathcal{E}(Z, \pi^{(1)}) \leq C \frac{K^\alpha(g) T^\alpha + T^3 N^{-1+\alpha}}{N^\alpha};$$

b) for the choice of $\pi^{(\beta)}$, with $\beta = 1$ if $\alpha = 1$, and $\beta < \alpha$ if $\alpha < 1$,

$$\mathcal{E}(Z, \pi^{(\beta)}) \leq C \frac{K^\alpha(g) T^\alpha + T^3}{N}.$$

Conclusion

We can adapt the time mesh size $|\pi|$ (the number N of points being fixed) according to the regularity of the terminal function g in order to have an \mathbf{L}_2 -regularity of Z (and thus a discretization scheme error) of order $|\pi|$.