Portfolio Optimization with Stochastic Volatilities : A Backward Approach

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Outline

Introduction

Portfolio optimization: a fundamental concern when investors trade between a large number of risky assets

- Merton (1971): Maximizing expected utility of terminal wealth
  He derived a closed formula.
- Many generalizations: maximizing under Bankruptcy constraint, maximizing with transaction costs, maximizing in pure-jump Lévy model.

Maximizing utility problems lead to **HJB** Equations
An efficient algorithm: **Howard algorithm** computing two sequences the optimal strategy and the value function. **Not possible when risky assets number** > 3 **.**

Introduction

- Zariphopoulou (2001): maximizing utility in a market with one risky asset and a stochastic volatility model. Using a power transformation: the value function is a solution of a linear P.D.E
- Pham (2002): multidimensional model, diffusion process, constraints. A power transformation leads to a **semi linear parabolic equation**.
- Mnif ((2007): multidimensional model, stochastic volatility model, exponential utility, constraints on amounts, jump diffusion process. When there is no correlation between the risky asset and the factor volatility, the optimal investment strategy is determined by solving a static optimization problem.

Problem formulation

$(\Omega, \mathcal{F}, \mathcal{P})$ filtered probability space
financial market: bond S^0 and n risky assets S

$$
\begin{aligned}
S^0 &\equiv 1 \\
dS_t &= diag(S_{t-})\Big(b(\Lambda_t)dt + \sigma(\Lambda_t)dW_t + \bar{\sigma}(\Lambda_t)d\bar{W}_t \\
&+ \int_{R^n\setminus\{0\}} \gamma(\Lambda_t, u)\tilde{\mu}(dt, du)\Big)
\end{aligned}
$$

W d-dimensional standard Brownian motion
$\bar{W}$ m-dimensional standard Brownian motion independent of W
μ a Poisson random measure and
$\tilde{\mu}(dt, du) = (\tilde{\mu}(dt, du))_{1\leq i\leq n} = (\mu_i(dt, du) - q_i(du)dt)_{1\leq i\leq n}$ is the compensated Poisson random measure

Problem formulation

$q_i(du)$ is the Lévy measure

$$\int_{R^n\setminus\{0\}} q_i(du) < \infty.$$

Λ is a d dimensional stochastic factor

$$d\Lambda_t = \eta(\Lambda_t)dt + dW_t,$$

Assumption **(H1)** η is Lipschitz.
We denote

$$\Sigma(\lambda) = (\sigma(\lambda), \bar{\sigma}(\lambda)),$$

$$\alpha(\lambda) = \inf_{\pi\in R^n, \pi\neq 0} \frac{|\Sigma(\lambda)^*\pi|^2}{|\pi|^2}, \;\; \lambda \in R^d,$$

Problem formulation

Assumption **(H2)**There exists some positive constants C such that for a.e. $(\lambda, u) \in R^d \times R^n \backslash \{0\}$:

$$\inf_{\lambda \in R^d} \alpha(\lambda) > 0$$

$$|b(\lambda)| + ||\sigma(\lambda)|| \leq C.$$

$$||\bar{\sigma}(\lambda)|| \leq C(1 + |\lambda|)$$

$$\gamma_{ij}(\lambda, u) > -1 \text{ for all } 1 \leq i, j \leq n$$

$$||\gamma(\lambda, u)|| \leq C$$

Problem formulation

A strategy π is said admissible if it is F-predictable and

$$\pi_t \in \Delta := \{(\pi_1, ..., \pi_n) \in [0,1]^n, \sum_{i=1}^{n} \pi_i \leq 1\} a.s. \text{ for all } 0 \leq t \leq T.$$

$\mathcal{A}$ the set of admissible controls.

$$\begin{aligned} X_t &= x + \int_0^t X_{s^-} \Big(\pi_s^* b(\Lambda_s) ds + \pi_s^* \sigma(\Lambda_s) dW_s + \pi_s^* \bar{\sigma}(\Lambda_s) d\bar{W}_s \\ &+ \int_{R^n \setminus \{0\}} \pi_s^* \gamma(\Lambda_s, u) \tilde{\mu}(ds, du) \Big) \end{aligned}$$

The utility function: $U(x) = \frac{x^\delta}{\delta}$, $x \in R_+$, The objective of the agent

$$v(t, x, \lambda) = \sup_{\pi \in \mathcal{A}} E[U(X_T) | X_t = x, \Lambda_t = \lambda], \; (t, x, \lambda) \in [0, T] \times R_+ \times R^d.$$

Semi-linear PDE

$$\begin{aligned} \frac{\partial v}{\partial t} & +\eta(\lambda)^* D_\lambda v + \frac{1}{2}\triangle_\lambda v + \max_{\pi\in\Delta}\Bigg\{ x\pi^* b(\lambda)\frac{\partial v}{\partial x} \\ + & \quad x^2\frac{1}{2}|\Sigma(\lambda)^*\pi|^2\frac{\partial^2 v}{\partial x^2} + x\pi^*\sigma(\lambda)D^2_{x\lambda}v \\ + & \quad \sum_{i=1}^n \int_{R^n\setminus\{0\}} (v(t, x(1+\pi^*\gamma_i(\lambda,u)),\lambda) \\ - & \quad v(t,x,\lambda) - x\pi^*\gamma_i(\lambda,u)\frac{\partial v}{\partial x})q_i(du)\Bigg\} = 0, \end{aligned}$$

for a.e. $(t, x, \lambda) \in [0, T) \times R_+ \times R^d$

$$v(T, x, \lambda) = \frac{x^\delta}{\delta},$$

Semi-linear PDE

Due to the power utility function, we look for a candidate of HJB equation in the form:

$$v(t, x, \lambda) = \frac{x^\delta}{\delta} \exp\left(-\phi(t, \lambda)\right).$$

By differentiation, HJB implies

$$-\frac{\partial \phi}{\partial t} - \frac{1}{2}\triangle\phi + H(\lambda, D\phi) = 0,\ (t, \lambda) \in [0, T) \times R^d, , \tag{1}$$

with terminal condition

$$\phi(T, \lambda) = 0,\ y \in R^d, \tag{2}$$

Semi-linear PDE

H is defined on $R^d \times R^d$ by

$$
\begin{aligned}
H(\lambda, p) &= \frac{1}{2}|p|^2 - p^* \eta(\lambda) \\
+ & \max_{\pi \in \Delta} \Big\{ \delta(\pi^* b(\lambda) - \pi^* \sigma(\lambda) p) - \frac{1}{2}\delta(1-\delta)|\Sigma(\lambda)^* \pi|^2 \\
+ & \sum_{i=1}^{n} \int_{R^n \setminus \{0\}} \Big((1 + \pi^* \gamma_i(\lambda, u))^\delta - 1 - \delta \pi^* \gamma_i(\lambda, u) \Big) q_i(du) \Big\}.
\end{aligned}
$$

Semi-linear PDE

Theorem

Let assumptions **(H1)** *and* **(H2)** *hold. Suppose that there exists a solution* $\phi \in C^{1,2}([0,T) \times R^d) \cap C^0([0,T] \times R^d)$ *to the semi-linear equation* (1) *with the terminal condition* (2). *We also assume that* $|D\phi(t,\lambda)|$ *has a linear growth condition in* λ *uniformly in* t. *Then the value function of* (1) *is given by:*

$$v(t,x,\lambda) = \frac{x^\delta}{\delta} \exp\left(-\phi(t,\lambda)\right), \;\; (t,x,\lambda) \in [0,T] \times R \times R^d.$$

Theorem

The optimal portfolio is given by the Markov control $\{\hat{\pi}_t = \hat{\pi}(t, \Lambda_t), 0 \leq t \leq T\}$ *where*

$$\begin{aligned}
\hat{\pi}(t,\lambda) &\in \arg\min_{\pi \in \Delta} \Big\{ -\delta(\pi^* b(\lambda) - \pi^* \sigma(\lambda) D\phi(t,\lambda)) \\
&+ \frac{1}{2}\delta(1-\delta)|\Sigma(\lambda)^* \pi|^2 \\
&- \sum_{i=1}^{n} \int_{R^n \setminus \{0\}} \Big((1 + \pi^* \gamma_i(\lambda, u))^{\delta} - 1 - \delta \pi^* \gamma_i(\lambda, u) \Big) q_i(du) \Big\}.
\end{aligned}$$

Regularity of the value function

(H3)(i) $\inf_{\lambda \in R^d} \alpha(\lambda) > 0$ (uniform elliptic volatility).
(ii) η is C^1 and Lipschitz, b is C^1 and bounded and Db is bounded.
(iii)$\Sigma\Sigma^*$ is C^1 and bounded and $||D(\Sigma\Sigma^*)||$ is bounded
(iv)$-1 < \gamma_{ij}(\lambda, u) = \gamma_{ij}(u) \leq M$ for all $(\lambda, u) \in R^d \times R^n \setminus \{0\}$ where $M > 0$.

Regularity of the value function

Theorem

Under Assumption **(H3)**, *there exists a solution $\phi \in C^{1,2}([0,T) \times R^d) \cap C^0([0,T] \times R^d)$ to the semi-linear equation* (1) *with terminal condition* (2) *and linear growth condition in λ uniformly in t on the derivative $D\phi$.*

$$\begin{aligned}\frac{dS_t^i}{S_{t^-}^i} &= b_i dt + \nu_i(\Lambda_{it}) \sum_{j=1}^{i} \rho_{ij}(\rho_j dW_t^j + \sqrt{1-\rho_j^2} d\bar{W}_t^j) \\ &+ \sum_{j=1}^{n} \int_{R^n \setminus \{0\}} \gamma_{ij}(u) \tilde{\mu}_j(dt, du), \\ d\Lambda_{it} &= (a_i - \theta_i \Lambda_{it}) dt + dW_t^i, \text{ for all } i \in \{1, ..., n\}.\end{aligned}$$

$(\nu_i)_{1 \leq i \leq n}$ bounded C^1 functions with bounded derivatives and lower bounded by a positive constant $\epsilon_i > 0$ for all $i \in \{1, ..., n\}$ a_i and θ_i are constants, ρ_{ij} is the constant correlation between the two Brownian motions of S^i and S^j, ρ_i is the constant correlation between S^i and its volatility and $\gamma_{ij}(u) > -1$ and is bounded.

The optimal portfolio is given by:

$$
\begin{aligned}
\hat{\pi}(t,\Lambda_t) \in \arg\min_{\pi \in \triangle} \Bigg[& -\delta \sum_{i=1}^{n} \pi_i b_i \\
+ \quad & \frac{\delta(1-\delta)}{2} \sum_{i,j=1}^{n} \pi_i \pi_j \left((\nu_i(\Lambda_{it}))(\nu_j(\Lambda_{jt})) \sum_{k=1}^{\inf(i,j)} \rho_{ik}\rho_{jk} \right) \\
+ \quad & \delta \sum_{i=1}^{n} \sum_{j=1}^{i} \pi_i \rho_{ij} \rho_j (\nu_i(\Lambda_{it})) \frac{\partial \phi}{\partial \lambda_j}(t,\Lambda_t) \\
- \quad & \sum_{i=1}^{n} \int_{R^n \setminus \{0\}} \left((1+\pi^* \gamma_i(u))^\delta - 1 - \delta \pi^* \gamma_i(u) \right) q_i(du) \Bigg].
\end{aligned}
$$

We define two processes (Y, Z) by

$$Y_t := \phi(t, \Lambda_t),\ Z_t := D\phi(t, \Lambda_t) \text{ for all } t \in [0, T].$$

Proposition

Suppose that Assumptions **(H1)**, **(H2)** *and* **(H3)** *hold. We define the processes* $(\bar{Y}, \bar{Z})$ *as follows*

$$\bar{Y}_t := exp(Y_t),\ \bar{Z}_t := \bar{Y}_t Z_t,\ \text{for all } t \in [0, T],$$

then the couple $(\bar{Y}, \bar{Z})$ *is a solution of the following BSDE*

$$-d\bar{Y}_t = \bar{g}(\Lambda_t, \bar{Y}_t, \bar{Z}_t)dt - \bar{Z}_t dW_t,$$

with terminal condition

Proposition

$$\bar{Y}_T = 1$$

where

$$\begin{aligned} \bar{g}(\lambda, \bar{y}, \bar{z}) &= -\min_{\pi \in \Delta} \Big\{ -\delta(\pi^* b(\lambda)\bar{y} - \pi^* \sigma(\lambda)\bar{z}) \\ &+ \bar{y}\frac{\delta(1-\delta)}{2}|\Sigma(\lambda)^*\pi|^2 \\ &- \bar{y}\sum_{i=1}^{n}\int_{R^n\setminus\{0\}} \Big((1+\pi^*\gamma_i(u))^\delta - 1 - \delta\pi^*\gamma_i(u)\Big)\, q_i(du) \Big\}. \end{aligned}$$

Discretization and simulation of the decoupled FBSDE

Step 1. Problem discretization. $(t_k := kh = \frac{kT}{N})_{0 \leq k \leq N}$.

$$\begin{aligned} \Lambda^N_{l,t_0} &= \Lambda_{l,0} \\ \Lambda^N_{l,t_{k+1}} &= \Lambda^N_{l,t_k} + \eta(\Lambda^N_{l,t_k})h + \triangle W_{l,k}, \end{aligned}$$

$\triangle W_{l,k} = W_{l,t_{k+1}} - W_{l,t_k}$, $1 \leq l \leq d$.

$$\begin{aligned} \bar{Y}^N_{t_N} &= 1 \\ \bar{Z}^N_{l,t_k} &= \frac{1}{h} E_k(\bar{Y}^N_{t_{k+1}} \triangle W_{l,k}), \\ \bar{Y}^N_{t_k} &= E_k(\bar{Y}^N_{t_{k+1}}) + hE_k(\bar{g}(\Lambda^N_{t_k}, \bar{Y}^N_{t_{k+1}}, \bar{Z}^N_{t_k})), \end{aligned}$$

where $E_k(.) = E(.|\mathcal{F}_{t_k})$.

Error induced the discretization in time

Remark

Since $Y^N_{t_N}$ is a constant and Λ^N is a Markov Chain, it is easy to see that $\bar{Y}^N_{t_k} = \bar{y}^N_k(\Lambda^N_{t_k})$ and $\bar{Z}^N_{t_k} = \bar{z}^N_k(\Lambda^N_{t_k})$ where $\bar{y}^N_k$ and $\bar{z}^N_k$ are unknown regression functions defined in a backward manner.

Zhang (2004), Bouchard and Touzi (2004), Gobet et al. (2005) proved For h small enough

Theorem

$$\max_{0 \le k \le N} E|\bar{Y}_{t_k} - \bar{Y}^N_{t_k}|^2 + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} E|\bar{Z}_t - \bar{Z}^N_{t_k}|^2 dt \le C\left(1 + |\Lambda_0|^2\right) h.$$

Discretization and simulation of the decoupled FBSDE

Step 2. Localization. Localization of the Brownian increments and the function $\bar{g}$

- $[\triangle W_{l,k}]_w = -R_0\sqrt{(}h) \vee \triangle W_{l,k} \wedge R_0\sqrt{(}h)$
- $\bar{g}^R(\lambda, \bar{y}, \bar{z}) = \bar{g}^R(-R_1 \vee \lambda_1 \wedge R_1, ..., -R_d \vee \lambda_d \wedge R_d, \bar{y}, \bar{z})$

The localization modifies the numerical scheme. We define $(\bar{Y}^{N,R}, \bar{Z}^{N,R})$ by

$$
\begin{aligned}
\bar{Y}^{N,R}_{t_N} &= 1 \\
\bar{Z}^{N,R}_{l,t_k} &= \frac{1}{h} E_k(\bar{Y}^{N,R}_{t_{k+1}}[\triangle W_{l,k}]_w), \\
\bar{Y}^{N,R}_{t_k} &= E_k(\bar{Y}^{N,R}_{t_{k+1}}) + hE_k(\bar{g}^R(\Lambda^N_{t_k}, \bar{Y}^{N,R}_{t_{k+1}}, \bar{Z}^{N,R}_{t_k})),
\end{aligned}
$$

Discretization and simulation of the decoupled FBSDE

Remark

The main interest of the localization is to provide bounded regression functions $\bar{y}_k^{N,R}$ and $\bar{z}_k^{N,R}$. One has $||\bar{y}_k^{N,R}||_\infty \leq C_y(R)$ and $||\bar{z}_k^{N,R}||_\infty \leq C_z(R)$(See Proposition 1 (Lemor Gobet and Warin (2006))

Step 3. Function bases.

- We approximate $\bar{Y}_{t_k}^{N,R}$ and $\bar{Z}_{l,t_k}^{N,R}$ for all $1 \leq l \leq d$ by a projection on a finite-dimensional function bases $p_{0,k}(\Lambda_{t_k}^{N,R})$ and $p_{l,k}(\Lambda_{t_k}^{N,R})$ for all $1 \leq l \leq d$.
- We set $\alpha_{l,k}$ for all $0 \leq l \leq d$ the projection coefficients of $\bar{Y}_{t_k}^{N,R}$ and $\bar{Z}_{l,t_k}^{N,R}$, $1 \leq l \leq d$ on the function bases $(p_{l,k})_{0 \leq l \leq d}$. We denote by $K_{l,k}$ the size of $p_{l,k}$.

Discretization and simulation of the decoupled FBSDE

Step 4. Monte-Carlo Simulations.

- We simulate M independent Monte Carlo simulations of $(\Lambda^N_{t_k})_{0\le k\le N}$ and $(\triangle W_k)_{0\le k\le N-1}$. We denote $(\Lambda^{N,m}_{t_k})_{1\le m\le M, 0\le k\le N}$ and $(\triangle W^m_k)_{1\le m\le M, 0\le k\le N-1}$ these simulations.
- We write $p_{l,k}(\Lambda^{N,m}_{t_k}) = p^m_{l,k}$ and $B^M_{l,k}$ the matrix of size $M \times K_{l,k}$ which rows are $(p^m_{l,k})^*$. we denote by $K^M_{l,k}$ the rank of $B^M_{l,k}$.

Step 5. Truncations. We force our approximation of $\bar{Y}^{N,R}$ and $\bar{Z}^{N,R}$ to be bounded by $C_y(R)$ and $C_z(R)$. For a function ψ, we define the truncations by $[\psi]_y(x) = -C_y(R) \vee \psi(x) \wedge C_y(R)$ and $[\psi]_z(x) = -C_z(R) \vee \psi(x) \wedge C_z(R)$

The algorithm

$\rightarrow \bar{Y}_{t_N}^{N,R,M} = 1$

$\rightarrow$ iteration: for $k = N-1, ..., 0$, assume that $\bar{y}^{N,R,M}(\Lambda_{t_{k+1}}^N)$ is known, then we solve the $d+1$ least-squares problems:

$$\alpha_{l,k}^M = \arg\inf_{\alpha} \frac{1}{M} \sum_{m=1}^{M} |\bar{y}_{k+1}^{N,R,M}(\Lambda_{t_{k+1}}^{N,m}) \frac{[\triangle W_{l,k}^m]_w}{h} - \alpha.p_{l,k}^m|^2$$

$$\alpha_{0,k}^M = \arg\inf_{\alpha} \frac{1}{M} \sum_{m=1}^{M} |\bar{y}^{N,R,M}(\Lambda_{t_{k+1}}^{N,m}) + h\bar{g}^R(\Lambda_{t_k}^{N,m}, \bar{y}_{k+1}^{N,R,M}(\Lambda_{t_{k+1}}^{N,m}), [\alpha_{l,k}^M.p_{l,k}^m]_z) - \alpha.p_{0,k}^m|^2$$

$\rightarrow \bar{y}_k^{N,R,M}(.) = [\alpha_{0,k}^M.p_{0,k}]_y(.)$ and $\bar{z}_k^{N,R,M}(.) = [\alpha_{l,k}^M.p_{l,k}]_z(.)$.

The algorithm

Assumption (H4) $(p_{l,k})_{0\leq l\leq d, 0\leq k\leq N}$ is a complete orthonormal system with respect to the empirical scalar product $< .,. >_{k,M}$

$$< \psi_1, \psi_2. >_{k,M} = \frac{1}{M}\sum_{m=1}^{M} \psi_1(\Lambda_{t_k}^{N,m})\psi_2(\Lambda_{t_k}^{N,m})$$

$$\rightarrow \frac{(B_{l,k}^M)^*(B_{l,k}^M)}{M} = Id$$

$$\alpha_{l,k}^M = \frac{1}{M}\sum_{m=1}^{M} p_{l,k}^m \bar{y}_{k+1}^{N,R,M}(\Lambda_{t_{k+1}}^{N,m})\frac{[\triangle W_{l,k}^m]_w}{h}$$

$$\alpha_{0,k}^M = \frac{1}{M}\sum_{m=1}^{M} p_{0,k}^m \Big(\bar{y}_{k+1}^{N,R,M}(\Lambda_{t_{k+1}}^{N,m}) + h\bar{g}^R(\Lambda_{t_k}^{N,m}, \bar{y}_{k+1}^{N,R,M}(\Lambda_{t_{k+1}}^{N,m}), [\alpha_{l,k}^M . p_{l,k}^m]_z)\Big)$$

The algorithm

- For the convergence, we refer to Lemor, Gobet Warin (2006)
- The error induced by the localization has a contribution of order h
- We choose the hypercube basis. We choose a domain D and we partition it into small hypercubes of edge κ
- To get a global (squared) error of order h^{β}, $\beta \in (0,1]$, we have to choose $\kappa \approx h^{\frac{\beta+1}{2}}$ (i.e. $K \approx Ch^{-\frac{d(\beta+1)}{2}}$)and $M \approx Ch^{-d(\beta+1)-(\beta+2)} log(h^{-\frac{d(\beta+1)}{4}-\frac{\beta+1}{2}})$

Numerical Approach

We simulate the d-dimensional stochastic factor $\Lambda = (\Lambda_t)_{t \in [0,T]}$ by using Monte-Carlo method. We denote by $\Lambda_{t_k}^{N,m} \in R^d$ the m-th approximation of Λ_t. We determine $d_{i,k}^{max} = \max_{1 \leq m \leq M} \Lambda_{i,t_k}^{N,m}$ and $d_{i,k}^{min} = \min_{1 \leq m \leq M} \Lambda_{i,t_k}^{N,m}$. We consider

$$D_k = \left\{ \lambda \in R^d, \lambda_i \in [d_{i,k}^{min}, d_{i,k}^{max}], \text{ s.t. for all } 1 \leq i \leq d \right\}.$$

We partition into small hypercubes of edge κ the domain D_k.

One risky asset model results

$$\frac{dS_t}{S_t} = bdt + \sigma(\Lambda_t)dW_t + \bar{\sigma}(\Lambda_t)d\bar{W}_t,$$
$$d\Lambda_t = (a - \theta\Lambda_t)dt + dW_t,\ \Lambda_0 = 0,$$

where $\sigma(\lambda) = \epsilon_1 + \frac{a_1}{\sqrt{\lambda^2+\beta_1^2}}$ and $\bar{\sigma}(\lambda) = \epsilon_2 + \frac{a_2}{\sqrt{\lambda^2+\beta_2^2}}$.
Table 1 gives the values of our model parameters.

Table: Values for the model's parameters

d	T	b	ϵ_1	ϵ_2	a_1
1	1	0.03	0.71×10^{-6}	0.704×10^{-6}	0.71
a_2	β_1	β_2	a	θ	δ
0.704	4	3	0.05	1	0.5

One risky asset model results

We determine $\bar{Y}_0$, then $v_1(0, x, \Lambda_0) := \frac{x^\delta}{\delta} \exp(-\bar{Y}_0)$. We compute $v_2(0, x, \Lambda_0) := \frac{1}{M} \sum_{m=1}^{M} U(X_T^m)$, where

$$X_T^m = x + \sum_{k=0}^{N-1} X_{t_k}^m \hat{\pi}_{t_k}^m \frac{S_{t_{k+1}}^m - S_{t_k}^m}{S_{t_k}^m}.$$

Table: Results for the value function

M	4000	6000	8000	10000
$v_1(0, 10, \Lambda_0)$	6.375	6.375	6.375	6.362
$v_2(0, 10, \Lambda_0)$	6.321	6.346	6.367	6.361

Two risky assets model results

$$\begin{aligned}
\frac{dS^1_t}{S^1_t} &= b_1 dt + \nu_1(\Lambda_{1t})(\rho_1 dW^1_t + \sqrt{1-\rho_1^2}\, d\bar{W}^1_t), \\
\frac{dS^2_t}{S^2_t} &= b_2 dt + \nu_2(\Lambda_{2t}) \left(\rho(\rho_1 dW^1_t + \sqrt{1-\rho_1^2}\, d\bar{W}^1_t) \right. \\
&+ \left. \sqrt{1-\rho^2}(\rho_2 dW^2_t + \sqrt{1-\rho_2^2}\, d\bar{W}^2_t) \right), \\
d\Lambda_{1t} &= (a_1 - \theta_1 \Lambda_{1t})dt + dW^1_t, \; \Lambda_{10} = 0, \\
d\Lambda_{2t} &= (a_2 - \theta_2 \Lambda_{2t})dt + dW^2_t, \; \Lambda_{20} = 0,
\end{aligned}$$

where $\nu_i(\lambda) = \epsilon_i + \frac{1}{\sqrt{\lambda^2+\beta_i^2}}$, $i \in \{1,2\}$. Our model parameters are given in Table 3.

Two risky assets model results

Table: Values for the model's parameters

d	T	h	b_1	b_2	a_1	θ_1	a_2	
2	0.4	0.04	0.05	0.03	0.75	1	0.5	
θ_2	ρ	ρ_1	ρ_2	δ	ϵ_1	ϵ_2	β_1	β_2
1	0.3	0.75	0.5	0.5	10^{-6}	10^{-6}	4	3

We determine $\bar{Y}_0$, then $v_1(0, x, \Lambda_0) := \frac{x^\delta}{\delta} \exp(-\bar{Y}_0)$.

Two risky assets model results

Table: Results for the value function

M	6000	8000	9000
$v_1(0, 10, \Lambda_0)$	6.376	6.372	6.372

The optimal investment strategy at $t = 0$ is given by $\hat{\pi}_0^* = (0, 0.91, 0.09)$.it is optimal to invest only in risky assets. It is better to invest more in the first risky asset since the drift of the second is higher and the level of the volatility of S^1 is lower $(\beta_1 > \beta_2)$.