The Decoupling Approach to Binomial Pricing of Multi-Asset Options

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Outline

The Continuous-Time Setting

We consider an m-dimensional Black-Scholes model with time horizon T where the stock price dynamics under the risk-neutral measure Q are given by

$$dS_i(t) = S_i(t)(r\mathrm{d}t + \sigma_i \mathrm{d}W_t^i), \quad S_i(0) = s_{0,i} \qquad \text{for } i = 1, \dots, m$$

for Brownian motions W^i and W^j with correlation ρ_{ij} for $i \neq j$. The variance-covariance matrix Σ is positive-definite.

Then,

$$\text{Corr}\left[\frac{\mathrm{d}S_i(t)}{S_i(t)}, \frac{\mathrm{d}S_j(t)}{S_j(t)}\right] = \rho_{ij}\,\mathrm{d}t$$

Standard Multi-Dimensional Binomial Trees

Ansatz

Approximation of the joint evolution process (Boyle (1988), Boyle & Evnine & Gibbs (1989), Kamrad & Ritchken (1991), ...).

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Set $N =$ number of periods and $\Delta t = T/N$.
For period $k \leq N$, the set of possible up-down-scenarios is

$$\mathcal{E}_k := \{\, \underline{\omega}_k = (\omega_{k,1}, \ldots, \omega_{k,m}) \mid \omega_{k,i} \in \{-1, 1\} \quad \forall\, i = 1, \ldots, m\}$$

$\Rightarrow$ The up-down behaviour of a path is described by

$$\omega = (\omega_{1,1}, \ldots, \omega_{1,m}, \ldots, \omega_{N,1}, \ldots, \omega_{N,m}) \in \mathcal{E}_1 \times \ldots \times \mathcal{E}_N =: \mathcal{E}^{(N)}$$

Standard Multi-Dimensional Binomial Trees (2)

For $1 \leq i \leq m$, starting from $S_i^{(N)}(0) = s_{0,i}$ we define

$$S_i^{(N)}(k\Delta t) := S_i^{(N)}((k-1)\Delta t)\, e^{\alpha_i \Delta t + \beta_i \sqrt{\Delta t}\, Z_{k,i}},$$

where $Z_{k,i} : \mathcal{E}^{(N)} \to \{1, -1\}$ is the coordinate map $\omega \mapsto \omega_{k,i}$.

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The constants α_i and $\beta_i > 0$ and the sequence of probability measures $(P^{(N)})_{N \in \boldsymbol{N}}$ on $(\mathcal{E}^{(N)}, \mathcal{P}(\mathcal{E}^{(N)}))_{N \in \boldsymbol{N}}$ are such that

- for fixed N, $(Z_{k,1}, \ldots, Z_{k,m})$, $k = 1, \ldots, N$, are i.i.d.
- the first two moments of the continuous-time log-returns are asymptotically matched

$\Rightarrow$ Weak convergence to the continuous-time price process

Standard Multi-Dimensional Binomial Trees (3)

In addition to the 1D setting, the correlation structure of the continuous model has to be asymptotically matched.

$\Rightarrow$ It holds that as $N \to \infty$,

$$\beta_i \beta_j \operatorname{Cov}_{P(N)}\left(Z_{k,i}, Z_{k,j}\right) \to \rho_{i,j}\sigma_i\sigma_j \qquad \text{for } i = 1, \ldots, m;\ j < i$$

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Drawbacks

- Construction is tedious
- Application is restricted by the model parameters
- Research on 1D trees is not directly applicable

Standard Multi-Dimensional Binomial Trees (4)

Example: Negative "transition probabilities"

Let S_1, S_2, S_3 with $\rho_{12} = -0.8$, $\rho_{23} = -0.6$ and $\rho_{13} = 0.2$.
Then for the tree suggested by Boyle, Evnine and Gibbs (1989),

$$P^{(N)}\left(\omega_{k1} = \omega_{k2} = \omega_{k3} = -1\right) = \tfrac{1}{8}\Big(-0.2 - \sqrt{\Delta t}\sum_{i=1}^{3}\tfrac{r-\frac{1}{2}\sigma_i^2}{\sigma_i}\Big),$$

which is negative for all $\Delta t > 0$ if $(r - 1/2\sigma_i^2) > 0$ for all assets i.

Here the problem cannot be fixed by choosing a sufficiently large number of periods!

Non-monotone Convergence to the Option Price

Example

Cash-or-nothing option with an up-and-in barrier B_1 on stock 1 and a down-and-out barrier B_2 on stock 2; i.e.

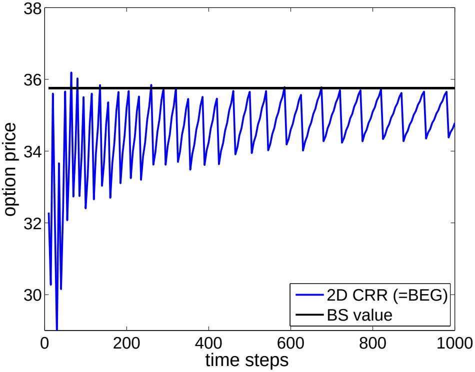
$$g(S_1, S_2) = G\,1_{\{S_1(t_0)\geq B_1 \text{ for some } t_0\in[0,T]\ ,\ S_2(t)\geq B_2\ \forall\, t\in[0,T]\}}.$$

Input parameters: $S_1(0) = 20.0$, $S_2(0) = 30.0$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $T = 1.0$ $r = 0.1$, $B_1 = 33.0$, $B_2 = 15.0$, $G = 100$ and **correlation** $\boldsymbol{\rho}$= **0.5**.

$\Rightarrow$ The BS value can be calculated analytically (He et al. (1998)).

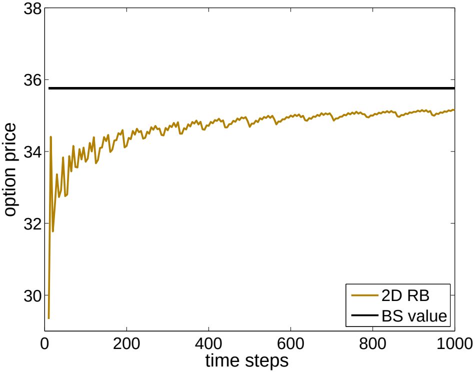
Non-monotone Convergence to the Option Price (2)

Convergence Pattern for the BEG tree



Non-monotone Convergence to the Option Price (3)

Convergence Pattern for the 2D Rendleman-Bartter tree (Amin (1991), Korn and M. (2009))



The Decoupling Approach to Multi-Dimensional Trees (1)

The basic idea

Transform the m-dimensional continuous stock price into a new process with independent components before setting up a discrete model (Hull and White (1990), Clewlow and Strickland (1998)).

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A general decoupling rule (Korn and M. (2009)):
Decompose Σ as

$$\Sigma = GDG^T$$

with $G \in \mathbb{R}^{m \times m}$ and $D \in \mathbb{R}^{m \times m}$, D diagonal. Define a new process Y by the transformation

$$S_t \quad \mapsto \quad Y_t := G^{-1} \left(\ln(S_t^1), \ldots, \ln(S_t^m) \right)^T \quad \forall\, t \in [0, T]$$

The Decoupling Approach to Multi-Dimensional Trees (2)

The process Y has dynamics

$$\begin{aligned} \mathrm{d}Y_j(t) &= \mu_j \mathrm{dt} + \sqrt{d_{jj}}\mathrm{d}\bar{W}_t^j \qquad \forall j = 1, \ldots, m \\ Y(0) &= G^{-1}(\ln(s_{0,1}), \ldots, \ln(s_{0,m})) \end{aligned}$$

where $(\bar{W}_t^1, \ldots, \bar{W}_t^m)^T$ is an m-dimensional standard BM and

$$\underline{\mu} = G^{-1}\left(r\underline{1} - 1/2\underline{\sigma}^2\right) \qquad \text{with } \underline{\sigma}^2 := (\sigma_1^2, \ldots, \sigma_m^2)^T.$$

$\Rightarrow$ The moment matching condition on the correlation structure becomes superfluous.

$\Rightarrow$ Moment matching has to be done componentwise only.

The Decoupling Approach to Multi-Dimensional Trees (3)

Advantages

- It offers an easy way to construct m-dimensional trees.
  $\Rightarrow$ The m-dimensional tree is obtained as the product of m 1D trees.
- It can be combined with any 1D discretization scheme for the individual stocks. In particular, we can choose different schemes for different components.
- It leads to well-defined transition probabilities.

Decoupled RB Tree

We choose a (log-) RB discretization for all 1D factorial trees:

Define $Y_0^{(N)} = Y_0$ and

$$Y_{k+1}^{(N)} = \begin{pmatrix} Y_{k,1}^{(N)} + \mu_1 \Delta t + Z_{k+1,1}\sqrt{d_{11}}\sqrt{\Delta t} \\ \vdots \\ Y_{k,m}^{(N)} + \mu_m \Delta t + Z_{k+1,m}\sqrt{d_{mm}}\sqrt{\Delta t} \end{pmatrix} \quad \forall\, k \leq N-1$$

Define $P^{(N)}$ on the path space $(\mathcal{E}^{(N)}, \mathcal{P}(\mathcal{E}^{(N)}))$ as the product of the probability measures P_k, $k = 1, \dots, m$, where

$$P_k(\{\underline{\omega}_k\}) := 2^{-m} \qquad \forall \underline{\omega}_k \in \mathcal{E}_k$$

Decoupled RB Tree (2)

Backtransformation

Define $h : \mathbb{R}^m \to \mathbb{R}^m$ by

$$h(\underline{x}) := (e^{G_{1.}\underline{x}}, \ldots, e^{G_{m.}\underline{x}})^T$$

where $G_{i.} \in \mathbb{R}^{1\times m}$ is the i^{th} row of G $\quad\Rightarrow\quad S = \{h(Y_t)\}_{t\in[0,T]}$

Discrete asset price approximation

$$S_k^{(N)} := h\left(Y_k^{(N)}\right) \qquad \forall\, k = 0, \ldots, N$$

Decoupled RB Tree (3)

Justification of the method

Let us define the continuous process $\{S_t^{(c,N)}\}_{t\in[0,T]}$ by

$$S_i^{(c,N)}(t) := S_{k-1,i}^{(N)} + \frac{t-(k-1)\Delta t}{\Delta t}\left(S_{k,i}^{(N)} - S_{k-1,i}^{(N)}\right)$$

for $t \in [(k-1)\Delta t, k\Delta t]$, $i = 1, \ldots, m$. Then,

$$\boxed{S^{(c,N)} \Rightarrow_w S}$$

$\Rightarrow$ Under suitable conditions on the payoff $g : C[0,T]^m \to \mathbb{R}^+$,

$$E_{P_\infty}\left(e^{-rT} g\left(h\left(Y^{(c,N)}\right)\right)\right) \to e^{-rT} E_Q\left(g\left(S\right)\right)$$

Numerical Performance

Structural advantage: In the backtransformation h, the rectangular grid structure is destroyed.

$\Rightarrow$ The probability mass is smeared relative to barriers constant in the underlyings.

$\Rightarrow$ The fraction of nodes in the in-the-money region is more stable in N than under standard methods.

$\Rightarrow$ The convergence behavior is more regular.

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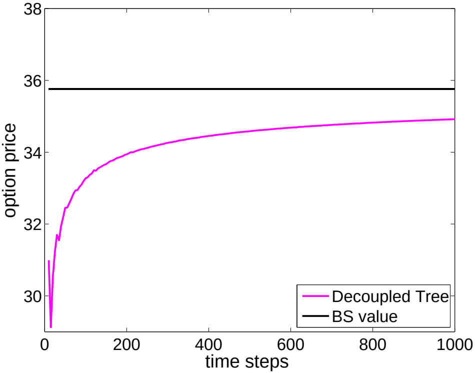
No free lunch: For path-dependent options, computational effort required for backtransformation contributes to the total effort's leading term.

$\Rightarrow$ Required CPU time increases.

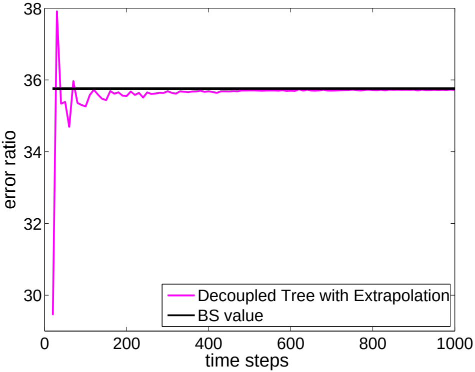
Numerical Performance (2)

Look at the convergence pattern for the valuation of the barrier option considered earlier with a decoupled RB tree, where decoupling is obtained with the Spectral decomposition:



Numerical Performance (3)

Result: Convergence is (approximately) monotone.
$\Rightarrow$ We can apply Richardson extrapolation.



Numerical Performance (Accuracy/Computing Time)

N	*BEG*	*2D RB*		*Decoupling*		*Decoupling extrapolation*	
100	32.40	33.71	83 %	33.28	250 %	35.26	283 %
250	34.88	34.55	73 %	34.12	227 %	35.65	253 %
500	35.65	34.69	88 %	34.59	302 %	35.71	316 %
750	35.04	35.08	89 %	34.79	293 %	35.73	311 %
1000	34.79	35.15	88 %	34.92	290 %	35.72	325 %
1250	35.09	35.18	86 %	35.01	287 %	35.74	324 %
1500	34.88	35.27	86 %	35.07	286 %	35.73	321 %
1750	35.41	35.29	88 %	35.12	295 %	35.74	330 %
2000	35.67	35.23	88 %	35.16	299 %	35.75	335 %
2750	35.34	35.38	88 %	35.25	299 %	35.75	335 %
BS	35.76						

Conclusion

- Decoupled trees are well-defined for any given correlation structure of a multi-asset Black-Scholes market.
- Decoupled trees exhibit a much more regular convergence behaviour than standard multi-dimensional tree methods.
- There are many tree modifications that are specially designed and optimized for the valuation of particular types of options. However, decoupled trees can be used as a universal recipe.
  $\Rightarrow$ Their application does not require specific knowledge of the problem under consideration.

Thank you for your attention!