
Optimal regularity in the obstacle problem for path-dependent American Options

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Introduction

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● Model equation

● Asian Options

● Path-dependent volatility

● Path-dependent vol: PROs

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Kolmogorov Equation

Outline of the proof

- PDE approach to path-dependent American Options.
- Main results: Obstacle problem, regularity theory.
- Plans for the future: more regularity, numerics.

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- PDE approach to path-dependent American Options.
- Main results: Obstacle problem, regularity theory.
- Plans for the future: more regularity, numerics.

Agenda:

- Kolmogorov Equation: basic theory and functional setting.
- Examples: Asian options, Hobson-Rogers model.
- Main result: existence of a strong solution.
- Optimal regularity (in progress).

Model equation

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Kolmogorov Equation

Outline of the proof

[Kolmogorov] (1934), [Fokker] (1914), [Planck] (1917)

$$\partial_{vv} + v\partial_x - \partial_t, \quad (x, v, t) \in \mathbb{R}^3$$

Ornstein-Uhlenbeck

$$\begin{cases} dX_t = V_t dt \\ dV_t = \sigma dW_t, \end{cases}$$

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[Kolmogorov] (1934), [Fokker] (1914), [Planck] (1917)

$$\partial_{vv} + v\partial_x - \partial_t, \quad (x, v, t) \in \mathbb{R}^3$$

Ornstein-Uhlenbeck

$$\begin{cases} dX_t = V_t dt \\ dV_t = \sigma dW_t, \end{cases}$$

Applications to Kinetic theory: Boltzmann-Landau

$$a(\cdot, f)\partial_{vv}f + v\partial_x f - \partial_t f$$

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Black & Scholes setting

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t, \\ dB_t = r B_t dt, \end{cases}$$

[Barraquand-Pudet] (1996), [Barucci-P.-Vespri] (2001), [Jiang-Dai] (2004)

■ Arithmetic average: $A_t = \int_0^t S_\tau d\tau$, $dA_t = S_t dt$

$$\frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial A}$$

■ Geometric average: $A_t = \int_0^t \log(S_\tau) d\tau$, $dA_t = \log(S_t) dt$

$$\frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \log(S) \frac{\partial V}{\partial A}$$

Path-dependent volatility

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Outline of the proof

[Hobson, Rogers] (1998); [Di Francesco, Pascucci] (2004);
[Foschi, Pascucci] (2007)

$$dS_t = \mu S_t dt + \sigma(D_t) dW_t,$$

D_t = deviation from the normal trend

$$D_t = S_t - \lambda \int_0^{+\infty} e^{-\lambda\tau} S_{t-\tau} d\tau$$

Pricing operator:

$$\frac{\partial V}{\partial t} = \frac{1}{2} \sigma(S - A)^2 \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial A}$$

Path-dependent vol: PROs

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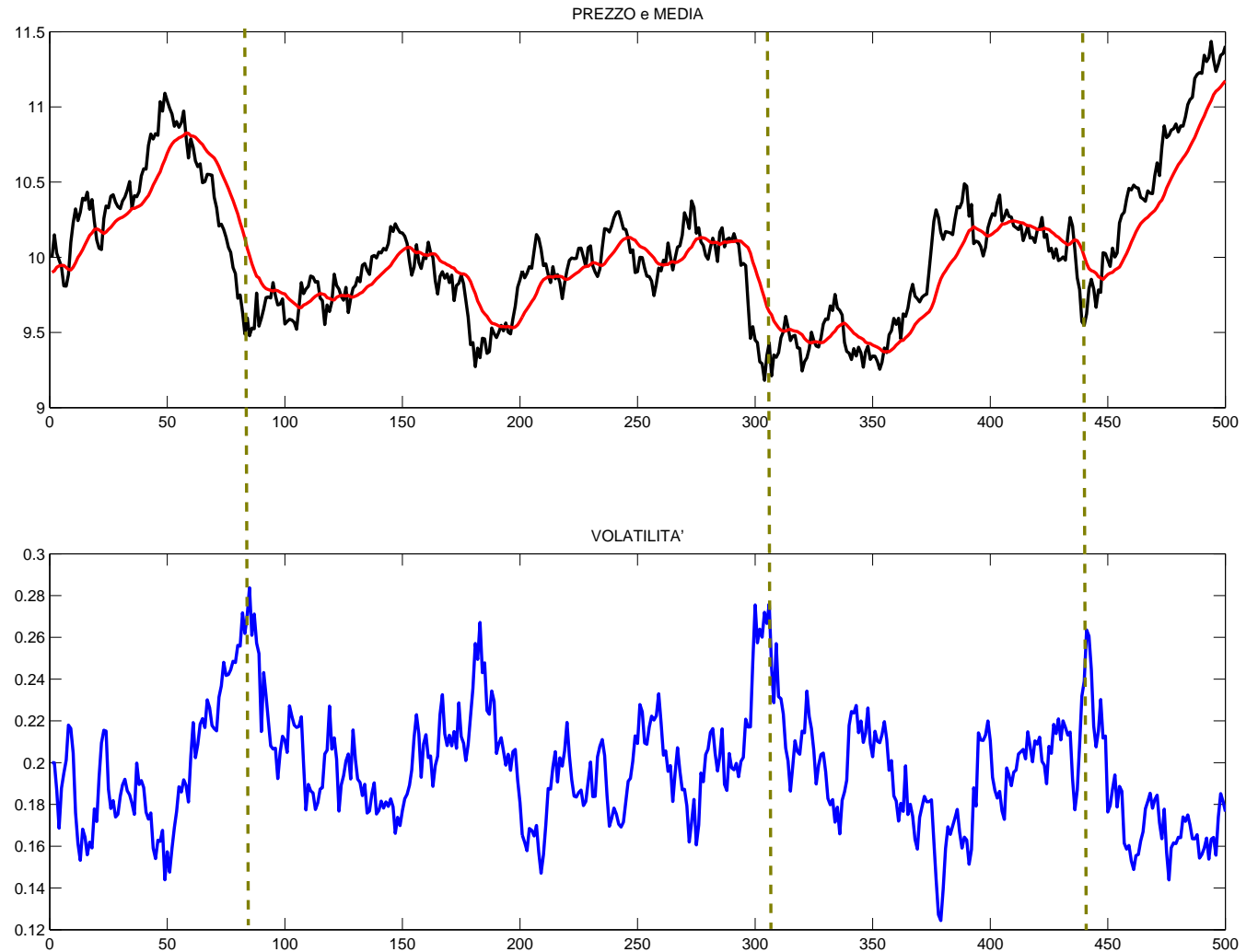
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■ realistic asset/volatility dynamics



Optimal stopping

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● Obstacle problem

● Uniformly parabolic PDEs

● Degenerate equation

● Our main results

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Outline of the proof

Price of the **American Option**:

$$u(t, x) = \sup_{\tau \in \mathcal{T}_{t, T}} \mathbb{E} [\varphi(\tau, X_{\tau}^{t, x})]$$

- $X^{t, x}$ diffusion starting from (x, t)
- φ payoff function
- $\mathcal{T}_{t, T}$ set of all stopping times with values in $[t, T]$.

Obstacle problem

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K Kolmogorov operator

$$\begin{cases} \max\{Ku, \varphi - u\} = 0, & \text{in }]0, T[\times \mathbb{R}^N \\ u(T, \cdot) = \varphi(T, \cdot), & \text{in } \mathbb{R}^N \end{cases}$$

Obstacle problem

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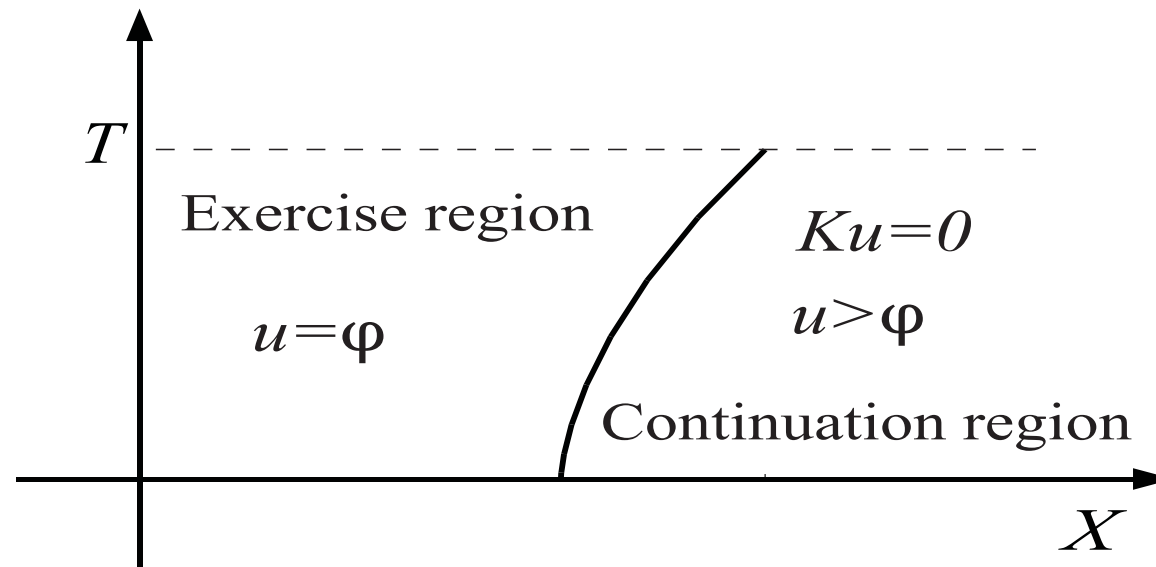
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Outline of the proof

K Kolmogorov operator

$$\begin{cases} \max\{Ku, \varphi - u\} = 0, & \text{in }]0, T[\times \mathbb{R}^N \\ u(T, \cdot) = \varphi(T, \cdot), & \text{in } \mathbb{R}^N \end{cases}$$



Uniformly parabolic PDEs

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- Variational solutions:
Bensoussan & Lions (1978)
Kinderlehrer & Stampacchia (1980)
- Obstacle and optimal stopping:
van Moerbeke (1975)
Bensoussan & Lions (1978)
- American options:
Bensoussan (1984), Karatzas (1988)
Jaillet & Lamberton & Lapeyre (1990)
- Viscosity solutions:
Fleming & Soner (2006), Barles (1997)
- Strong solutions:
Friedman (1975)

Degenerate equation

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- Barraquand and Pudet (1996)
- Barles (1997)
- Parrott and Clarke (1998)
- Wu, Yu and Kwok (1999)
- Hansen and Jorgensen (2000)
- Ben-Ameur, Breton and L'Ecuyer (2002)
- Barone-Adesi, Bermudez and Hatgioannides (2003)
- Marcozzi (2003)
- Fu and Wu (2003)
- Jiang and Dai (2004)
- d'Halluin, Forsyth and Labahn (2005)
- Huang and Thulasiram (2005)
- Bermudez, Nogueiras and Vazquez (2006)
- Barbu and Marinelli (2008)

Our main results

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Outline of the proof

[1] Di Francesco, Pascucci, P. (2008) Existence of a **strong solution** to the obstacle problem for non-uniformly parabolic pricing PDEs for Asian options

[2] Pascucci (2008) Solution to the optimal stopping and pricing problems for Asian options

[3] Frentz, Nyström, Pascucci, P. (2008) Optimal interior regularity for solutions to the obstacle problem

Kolmogorov Equations

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- Properties of u
- Optimal regularity

Outline of the proof

Consider the Stochastic Differential Equation in \mathbb{R}^N

$$dX_t = BX_t dt + \sigma dW_t, \quad X_0 = x,$$

B : $N \times N$ constant matrix,

σ : $N \times m$ constant matrix,

W_t m -dimensional Wiener process, $1 \leq m \leq N$.

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B : $N \times N$ constant matrix,

σ : $N \times m$ constant matrix,

W_t m -dimensional Wiener process, $1 \leq m \leq N$.

- Solution: $X_t = e^{tB} \left(x + \int_0^t e^{-sB} \sigma dW_s \right)$
- X_t is Gaussian. Mean: $E(X_t) = e^{tB} x$,
- Covariance matrix: $C(t) = \int_0^t e^{-sB} \sigma (e^{-sB} \sigma)^T ds$

Transition density

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Outline of the proof

$$dX_t = BX_t dt + \sigma dW_t$$

$\mathcal{C}(t)$ is **positive definite** \Leftrightarrow Hörmander condition

- X has a transition density
- Gaussian fundamental solution of the Kolmogorov op.

$$K = \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t, \quad (t, x) \in \mathbb{R}^{N+1}$$

where

$$A = \frac{1}{2} \sigma \sigma^T \approx \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$$

Model equation

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Outline of the proof

$$\partial_x^2 u + x \partial_y u = \partial_t u \quad (x, y, t) \in \mathbb{R}^3$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad C(t) = \begin{pmatrix} t & -\frac{t^2}{2} \\ -\frac{t^2}{2} & \frac{t^3}{3} \end{pmatrix}$$

$$\Gamma(x, y, t) = \frac{1}{12\pi t^2} \exp\left(-\frac{x^2}{t} - 3\frac{xy}{t^2} - 3\frac{y^2}{t^3}\right).$$

Non Euclidean structure

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Outline of the proof

$$K = \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t, \quad (x, t) \in \mathbb{R}^{N+1}$$

Translations:

$$(\xi, \tau) \circ (x, t) := (x + e^{-tB}\xi, t + \tau), \quad (\xi, \tau), (x, t) \in \mathbb{R}^{N+1}$$

Invariance: Let $v(x, t) := u(x + e^{-tB}\xi, t + \tau)$. Then

$$(Ku)((\xi, \tau) \circ (x, t)) = Kv(x, t)$$

Homogeneous norm

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Outline of the proof

Dilations: Let $v(x_1, x_2, t) = u(\lambda x_1, \lambda^3 x_2, \lambda^2 t)$. Then

$$u_{x_1 x_1} + x_1 u_{x_2} = u_t \quad \Leftrightarrow \quad v_{x_1 x_1} + x_1 v_{x_2} = v_t$$

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$$u_{x_1 x_1} + x_1 u_{x_2} = u_t \quad \Leftrightarrow \quad v_{x_1 x_1} + x_1 v_{x_2} = v_t$$

Homogeneous norm

$$|(x_1, x_2)|_K := |x_1| + |x_2|^{1/3} \quad \|(x, t)\|_K := |x|_K + |t|^{1/2}$$

Hölder continuity in x

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, t)| \leq C(|x_1 - \xi_1| + |x_2 - \xi_2|^{1/3})^\alpha$$

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$$u_{x_1 x_1} + x_1 u_{x_2} = u_t \quad \Leftrightarrow \quad v_{x_1 x_1} + x_1 v_{x_2} = v_t$$

Homogeneous norm

$$|(x_1, x_2)|_K := |x_1| + |x_2|^{1/3} \quad \|(x, t)\|_K := |x|_K + |t|^{1/2}$$

Hölder continuity in x

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, t)| \leq C (|x_1 - \xi_1| + |x_2 - \xi_2|^{1/3})^\alpha$$

Hölder continuity in (x, t)

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, \tau)| \leq C \left(|x_1 - \xi_1| + |x_2 - \xi_2 - (\tau - t)\xi_1|^{1/3} \right)^\alpha + |t - \tau|^{\alpha/2}$$

Hölder spaces

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Outline of the proof

$$K = \sum_{i,j=1}^m a_{ij}(x,t) \partial_{x_i x_j} + \sum_{i=1}^m a_i(x,t) \partial_{x_i} + a(x,t) + \underbrace{\langle Bx, \nabla \rangle - \partial_t}_Y$$

Hölder spaces

$$\begin{aligned} \|u\|_{C^{2,\alpha}(\Omega)} &= \|u\|_{C^\alpha(\Omega)} + \sum_{i=1}^m \|\partial_{x_i} u\|_{C^\alpha(\Omega)} \\ &+ \sum_{i,j=1}^m \|\partial_{x_i x_j} u\|_{C^\alpha(\Omega)} + \|Yu\|_{C^\alpha(\Omega)} \end{aligned}$$

Sobolev spaces

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Outline of the proof

$$K = \sum_{i,j=1}^m a_{ij}(x,t) \partial_{x_i x_j} + \sum_{i=1}^m a_i(x,t) \partial_{x_i} + a(x,t) + \underbrace{\langle Bx, \nabla \rangle - \partial_t}_Y$$

Sobolev spaces

$$\begin{aligned} \|u\|_{\mathcal{S}^p(\Omega)} &= \|u\|_{L^p(\Omega)} + \sum_{i=1}^m \|\partial_{x_i} u\|_{L^p(\Omega)} \\ &+ \sum_{i,j=1}^m \|\partial_{x_i x_j} u\|_{L^p(\Omega)} + \|Y u\|_{L^p(\Omega)} \end{aligned}$$

A priori estimates

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Outline of the proof

■ Embedding theorem

$$\|u\|_{C^{1,\alpha}(O)} \leq c \|u\|_{S^p(\Omega)}$$

for

$$p \geq Q + 2, \quad \alpha = 1 - \frac{Q + 2}{p}$$

where $O \subset\subset \Omega$ and $c = c(K, O, \Omega, p)$
 Q is the homogeneous dimension

A priori estimates

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Outline of the proof

■ Embedding theorem

$$\|u\|_{C^{1,\alpha}(O)} \leq c \|u\|_{S^p(\Omega)}$$

for

$$p \geq Q + 2, \quad \alpha = 1 - \frac{Q + 2}{p}$$

where $O \subset\subset \Omega$ and $c = c(K, O, \Omega, p)$
 Q is the homogeneous dimension

■ Schauder and Sobolev type a-priori estimates

Bramanti, Cerutti, Manfredini (1996)

Di Francesco, P. (2006)

Di Francesco, Pascucci, P. (2007)

Strong solution

$$\max\{Ku, \varphi - u\} = 0, \quad \text{a.e.}$$

where

$$K = \sum_{i,j=1}^m a_{ij}(x,t) \partial_{x_i x_j} + \sum_{i=1}^m a_i(x,t) \partial_{x_i} + a(x,t) + \underbrace{\langle Bx, \nabla \rangle - \partial_t}_Y$$

$u \in \mathcal{S}_{loc}^1$ that is

$$u, \partial_{x_i} u, \partial_{x_i x_j} u, Yu \in L_{loc}^1$$

for $i, j = 1, \dots, m$

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Obstacle-Payoff function

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Outline of the proof

$\varphi \in \text{Lip}_{\text{loc}}(\mathbb{R}^N \times]0, T[)$.

For every compact set $H \subset \mathbb{R}^N \times]0, T[$ there exists a constant $c_H \in \mathbb{R}$ (possibly $c_H < 0$), such that

$$\sum_{i,j=1}^m \xi_i \xi_j \partial_{x_i x_j} \varphi \geq c_H |\xi|^2 \quad \text{in } H, \quad \xi \in \mathbb{R}^m$$

in the distributional sense.

Obstacle-Payoff function

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For every compact set $H \subset \mathbb{R}^N \times]0, T[$ there exists a constant $c_H \in \mathbb{R}$ (possibly $c_H < 0$), such that

$$\sum_{i,j=1}^m \xi_i \xi_j \partial_{x_i x_j} \varphi \geq c_H |\xi|^2 \quad \text{in } H, \quad \xi \in \mathbb{R}^m$$

in the distributional sense.

Examples:

- C^2 functions
- Lipschitz continuous functions, *convex w.r.t.* x_1, \dots, x_m
- call and put options

$$\varphi(x) = (E - x)^+ \quad \Longrightarrow \quad \varphi'' = \delta_E \geq 0$$

The existence result

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Outline of the proof

$$\begin{cases} \max\{Lu - f, \varphi - u\} = 0, & \text{in } \mathbb{R}^N \times]0, T[, \\ u(\cdot, 0) = g, & \text{in } \mathbb{R}^N, \end{cases}$$

Let $\varphi : \mathbb{R}^N \times]0, T[\rightarrow \mathbb{R}$ **locally** Lipschitz and such that

$$\sum_{i,j=1}^m a_{ij}(x, t) \xi_i \xi_j \partial_{x_i x_j} \varphi \geq c |\xi|^2 \quad (\text{locally})$$

Theorem [Di Francesco, Pascucci, P.] (2008) If there exists a super-solution \bar{u} of the obstacle problem, then there exists a unique strong solution u .

Regularity: $u \in S_{\text{loc}}^p$ for every $p \in [1, \infty[$,

$u \in C_{\text{loc}}^{1,\alpha}$ for every $\alpha \in]0, 1[$.

Properties of u

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Outline of the proof

$$\begin{cases} \max\{Lu - f, \varphi - u\} = 0, & \text{in } \mathbb{R}^N \times]0, T[, \\ u(\cdot, 0) = g, & \text{in } \mathbb{R}^N, \end{cases}$$

Let $\varphi : \mathbb{R}^N \times]0, T[\rightarrow \mathbb{R}$ **locally** Lipschitz and such that

$$\sum_{i,j=1}^m a_{ij}(x, t) \xi_i \xi_j \partial_{x_i x_j} \varphi \geq c |\xi|^2 \quad (\text{locally})$$

Theorem [Di Francesco, Pascucci, P.] (2008) u is a viscosity solution.

Theorem [Pascucci] (2008) $u(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} [\varphi(\tau, X_{\tau}^{t,x})]$.

$\mathcal{T}_{t,T}$ = set of all stopping times with values in $[t, T]$.

Optimal regularity

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Outline of the proof

$$\max\{Lu - f, \varphi - u\} = 0, \quad \text{in } \Omega \subset \mathbb{R}^N \times]0, T[.$$

Theorem [Frentz, Nyström, Pascucci, P.] (2008) Assume that the coefficients a_{ij} of K are Hölder continuous. Then:

- If $\varphi \in S_{\text{loc}}^p$, then $u \in S_{\text{loc}}^p$, $1 \leq p \leq \infty$
- If $\varphi \in C_{\text{loc}}^\alpha$, then $u \in C_{\text{loc}}^\alpha$, $0 < \alpha \leq 1$
- If $\varphi \in C_{\text{loc}}^{1+\alpha}$, then $u \in C_{\text{loc}}^{1+\alpha}$, $0 < \alpha \leq 1$

Proof: Blow up argument (see [Petrosyan, Shahgholian] (2007)).

Penalization technique

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● Penalization technique

● Limit: $\varepsilon \rightarrow 0$

● Proof of the key estimate

● Finite difference

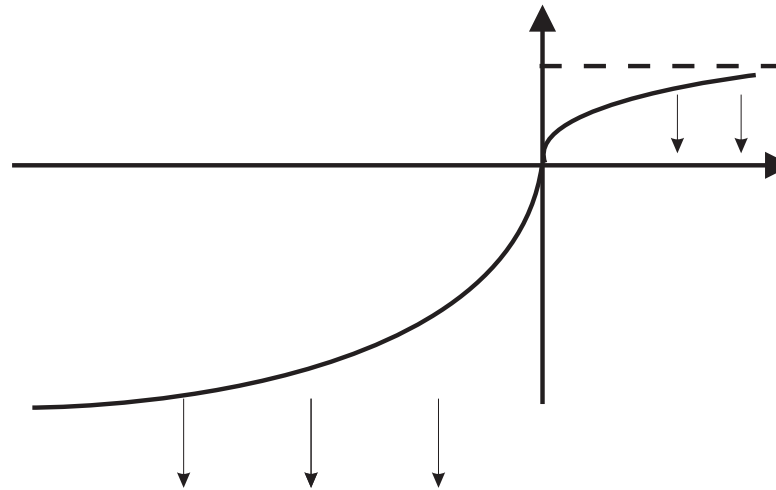
Let H be a cylinder

$$\begin{cases} Ku = \beta_\varepsilon(u - \varphi), & \text{in } H \\ u|_{\partial_P H} = g \end{cases}$$

where $(\beta_\varepsilon)_{\varepsilon>0}$ are in $C^\infty(\mathbb{R})$, increasing, $\beta_\varepsilon(0) = 0$ and

■ if $u > \varphi$ then $\beta_\varepsilon(u - \varphi) \leq \varepsilon$

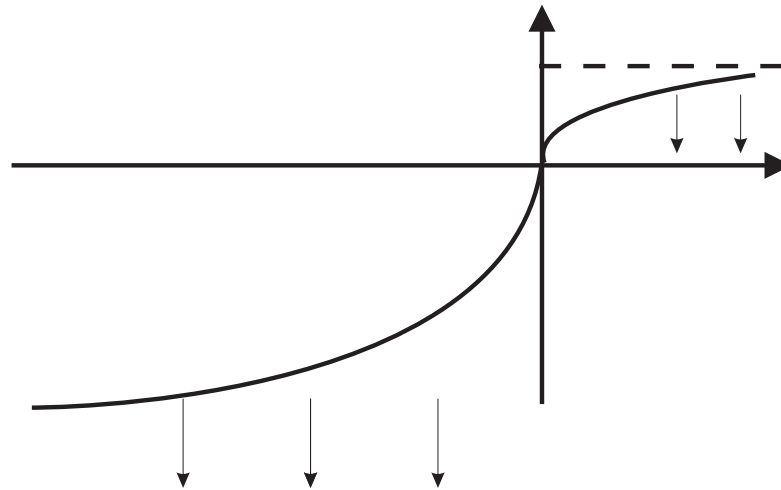
■ if $u < \varphi$ then $\beta_\varepsilon(u - \varphi) \rightarrow -\infty$ as $\varepsilon \rightarrow 0$



Limit: $\varepsilon \rightarrow 0$

$$\begin{cases} Ku^\varepsilon = \beta_\varepsilon(u^\varepsilon - \varphi), & \text{in } H(T) \\ u^\varepsilon|_{\partial_P H(T)} = g \end{cases}$$

Key estimate: $|\beta_\varepsilon(u^\varepsilon - \varphi)| \leq c$



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● Penalization technique

● Limit: $\varepsilon \rightarrow 0$

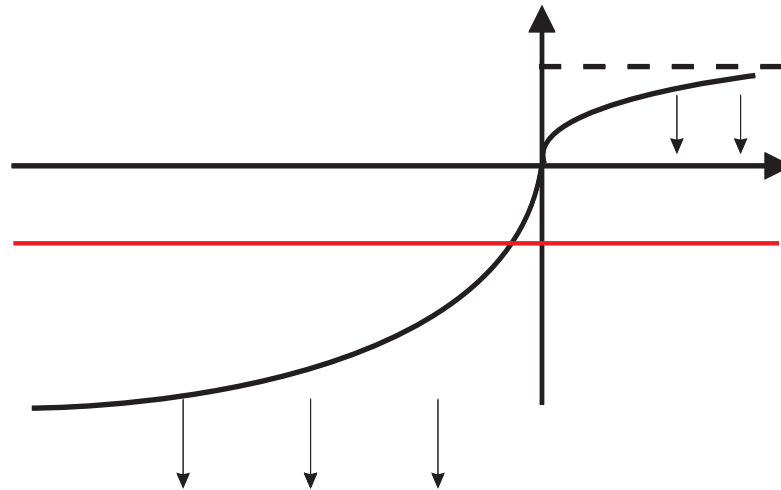
● Proof of the key estimate

● Finite difference

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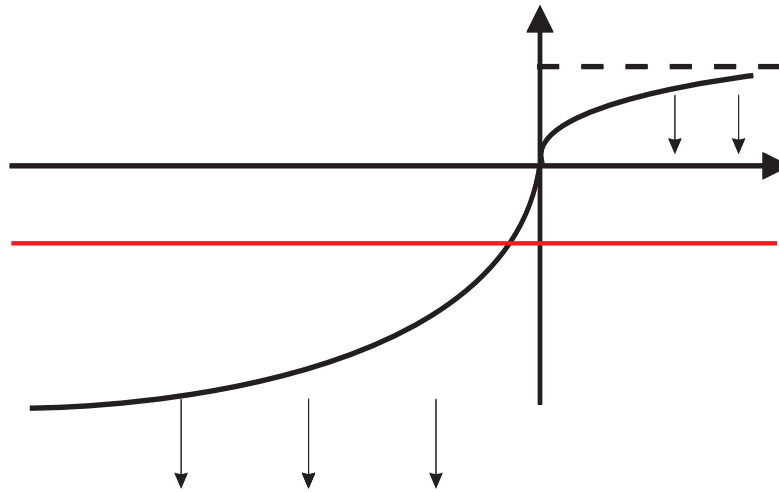
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Key estimate: $|\beta_\varepsilon(u^\varepsilon - \varphi)| \leq c$



Key estimate + barriers + interior estimates in \mathcal{S}^p

$$\|u^\varepsilon\|_{\mathcal{S}^p(O)} \leq c \left(\|u^\varepsilon\|_{L^p(\Omega)} + \|Ku^\varepsilon\|_{L^p(\Omega)} \right)$$

Proof of the key estimate

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$$|\beta_\varepsilon(u^\varepsilon - \varphi)| \leq c$$

Proof:

$$\zeta \text{ interior minimum} \implies K(u^\varepsilon - \varphi)(\zeta) \geq 0$$

$$\beta_\varepsilon(u^\varepsilon - \varphi)(\zeta) = Ku^\varepsilon(\zeta) \geq K\varphi(\zeta) \geq \tilde{C}$$

Finite difference

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$$\partial_x^2 u + x \partial_y u = \partial_t u \quad (x, y, t) \in \mathbb{R}^3$$

$$\partial_x^2 u(x, y, t) \approx \frac{u(x + \Delta_x, y, t) - 2u(x, y, t) + u(x - \Delta_x, y, t)}{\Delta_x^2}$$

$$x \partial_y u(x, y, t) - \partial_t u(x, y, t) \approx \frac{u(x, y + x \Delta_t, t - \Delta_t) - u(x, y, t)}{\Delta_t}$$

$$\text{Grid: } \left\{ (j \Delta_x, k \Delta_y, n \Delta_t) \mid j, k, n \in \mathbb{N} \right\} \quad \Delta_y = \Delta_x \Delta_t$$

Stability condition (only for the explicit method): $\Delta_t \leq \frac{1}{2} \Delta_x^2$