Asymptotics of the maximal radius of an L^r-optimal sequence of quantizers.

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Conference on Numerical Methods in Finance
April 15-17 2009

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Quantization problem

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ be a $r.v.$ with distribution $\mathbb{P}_X = P$ and $X \in L^r(\mathbb{P})$. Quantizing X (w.r.t the L^r norm, at level n) consists on finding the best approximation of X (w.r.t the L^r norm) by an $\mathbb{R}^d$ $r.v.$ $q(X)$ of cardinal at most n, with q a Borel function on $\mathbb{R}^d$. The associated L^r-mean quantization error is given by

$$e_{n,r}(X) = \inf\{\|X - q(X)\|_r, q : \mathbb{R}^d \stackrel{\text{Borel}}{\longrightarrow} \mathbb{R}^d,\ \mathrm{card}(q(\mathbb{R}^d)) \le n\}.$$

Let $q : \mathbb{R}^d \mapsto \alpha$ and set

$$\widehat{X}^\alpha = \sum_{a \in \alpha} a \mathbf{1}_{\{X \in C_a(\alpha)\}} \tag{1}$$

with

$$C_a(\alpha) \subset \{x \in \mathbb{R}^d : |x - a| = \min_{b \in \alpha} |x - b|\}. \tag{2}$$

Then the L^r mean quantization problem reads

$$e_{n,r}(X) \;=\; \inf\{\|X - \widehat{X}^\alpha\|_r, \alpha \subset \mathbb{R}^d,\ \mathrm{card}(\alpha) \le n\} \tag{3}$$

Zador Theorem

Definition

A sequence of quantizers $(\alpha_n)_{n\geq 1}$ *is* $L^r(\mathbb{P}_X)$*-optimal if for every* $n \geq 1$, α_n *realize the infimum (with is in fact a minimum) in* (3).

1. The L^r-mean quantization error $e_{n,r}(X) \searrow 0$ as $n \to +\infty$.
2. Let $P = P_a + P_s$. Let $X \in L^{r+\eta}(\mathbb{P})$ for some $\eta > 0$. Then

$$\lim_{n\to+\infty} n^{1/d} e_{n,r}(P) = Q_r(P)^{1/r} \tag{4}$$

with $Q_r(P) = J_{r,d} \left\| \frac{dP_a}{d\lambda_d} \right\|_{\frac{d}{d+r}}$.

Applications to Numerical probabilities

▷ We estimate $\mathbb{E}f(X)$ by $\mathbb{E}f(\widehat{X}) = \sum_{i=1}^{n} f(x_i^\star)\mathbb{P}(X \in C_i(x_i^\star))$ owing to the weighted empirical measure theorem: $\sum_{i=1}^{n} p_i^\star \delta_{x_i^\star} \Longrightarrow P.$

▷ In nonlinear filtering problem: if X is a signal process and Y the observed process we estimation $\mathbb{E}(f(X_n)|(Y_1, \cdots, Y_n) = y)$ by $\sum_{i=1}^{N_n} \widehat{\Pi}_y^i f(x_i)$.

• Bally, Pagès, Printems: A quantization tree method for pricing and hedging multidimensional American options.

• Pagès, Pham: Optimal quantization methods for nonlinear filtering with discrete-time observations.

• Pagès, Pham, Printems: An Optimal markovian quantization algorithm for multidimensional stochastic control problems.

• Callegaro, Sagna: Estimation of the default probability in partial information model by an hybrid MC-Quantization

• Pagès, Printems: Functional quantization for numerics with an application to option pricing.

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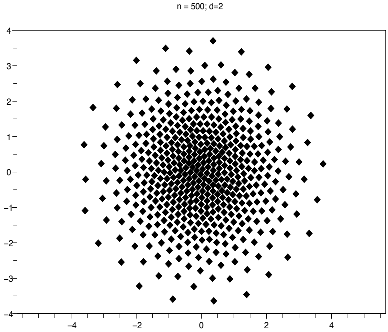
Definition of the maximal radius



In this talk we are interested to the asymptotics of the maximal radius sequence of an $L^r(P)$-optimal sequence of quantizers (α_n) defined for every $n \geq 1$ by

$$\rho(\alpha_n) := \rho_n := \max\{|a|,\ a \in \alpha_n\}. \tag{5}$$

limit of the maximal radius

Proposition

If $\mathrm{card}(\mathrm{supp}(P)) = +\infty$ *then for every* $L^r(P)$*-optimal sequence of quantizers* (α_n),

$$\lim_{n\to+\infty} \rho_n = \sup_{n\geq 1} \rho_n = \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{6}$$

The norm $|\cdot|$ *is Euclidean.*

▷ A Idea of the proof: One shows that

- $$\liminf_{n\to+\infty} \rho_n \geq \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{7}$$
- For every n such that $\mathrm{card}(\mathrm{supp}(P)) \geq n$,

$$\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))} \quad \text{and} \quad \rho_n \leq \sup\{|x|,\ x \in \mathrm{supp}(P)\}. \tag{8}$$

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Main result

We will make the following hypothesis on P in the general settings

$$\textbf{(H)} \equiv \qquad P(dx) \geq \varepsilon_0 \, \mathbf{1}_{\{x \in \bar{B}(x_0, r_0)\}} \lambda_d(dx), \quad \varepsilon_0,\ r_0 > 0,\ x_0 \in \mathbb{R}^d.$$

Theorem

Suppose X has an unbounded support and **(H)** *holds. Let $(\alpha_n)_{n \geq 1}$ be $L^r(P)$-optimal. Then, for every $r > 0$, for every $d \geq 1$,*

$$\lim_{\varepsilon \searrow 0} \, \underline{\lim}_{n} \left(n^{1+\frac{r}{d}} \mathbb{E}\left(|X|^r \mathbf{1}_{\{|X| > \frac{\rho_n}{2+\varepsilon}\}} \right) \right) \geq C_{r,d}. \qquad (9)$$

Rem. If $d = 1, r \geq 1$, under Assumption $(\mathbf{G_r})$: $P = f \cdot \lambda_d$ where f is non-increasing toward 0 on $[A, +\infty)$ for some $A > 0$ and

$$\lim_{y \to +\infty} \int_1^{+\infty} (u-1)^{r-1} \frac{f(uy)}{f(y)} du = 0$$

then (9) holds with "$1 + \varepsilon$" instead of "$2 + \varepsilon$".

toward the proof of the theorem

- ▷ Let $M(\alpha_n) = \{a \in \alpha_n,\ |a| = \rho_n\}$. Then, $\forall\, \varepsilon > 0,\ \exists\, n_\varepsilon$ t.q. $\forall n \geq n_\varepsilon$,

$$\forall\ a \in M(\alpha_n),\ \forall\ \xi \in C_a(\alpha_n), \qquad |\xi| \geq \frac{\rho_n}{2+\varepsilon}. \tag{10}$$

  with "$1+\varepsilon$" in place of "$2+\varepsilon$" in (10) when $d = 1,\ r \geq 1$ and if $(\mathbf{G_r})$ holds.

- ▷ Let $(\alpha_n)_{n\geq 1}$ be $L^r(P)$-optimal. Suppose that **(H)** holds. Then for large enough n,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq C_{r,d}\ n^{-\frac{r+d}{d}}. \tag{11}$$

From now on, set for every $r > 0$,

$$\bar{F}_r(x) = \mathbb{E}\left(|X|^r \mathbf{1}_{\{|X|>x\}}\right) \text{ and } f^{\leftarrow}(x) = \inf\{t > 0,\ f(t) \geq x\}.$$

Asympt. from the upper bound of the asympt. inverse of $-log\,\bar{F}_r(\cdot)$

Suppose that X has an unbounded support and that (**H**) is fulfilled.
h is an asymptotic inverse function of g if $h(g(x)) = x + o(x)$

Proposition

(a) *If $\psi_r \in R_\delta$ is measurable nondecreasing, and $\psi_r(-\log \bar{F}_r(x)) \geq x + o(x)$ as $x \to +\infty$, then*

$$\overline{\lim_n} \frac{\rho_n}{\psi_r(\log(n))} \leq c_{r,d} \left(1 + \frac{r}{d}\right)^{\delta}. \tag{12}$$

with $c_{r,d} = 1$ if $d = 1$, $r \geq 1$ and $(\mathbf{G_r})$ holds; $c_{r,d} = 2$ otherwise.
(b) *If $\psi_r \in R_\delta$ is measurable nondecreasing, and $\psi_r(-\log \bar{F}_r(e^x)) \geq x + o(x)$ when $x \to +\infty$, then*

$$\overline{\lim_n} \frac{\log \rho_n}{\psi_r(\log(n))} \leq \left(1 + \frac{r}{d}\right)^{\delta}. \tag{13}$$

Choice of ψ_r for distributions with expon. and polyn. tails:

Criterion

(a) *Let* $\theta^\star = \sup\left\{\theta > 0,\ \limsup_{x\to+\infty}\ e^{\theta x^\kappa}\bar{F}_r(x) < +\infty\right\}$,

for some κ*, then* $(\psi_r(x) = (\frac{x}{\theta})^{1/\kappa},\ \forall\theta \in (0,\theta^\star)\ )$

$$\overline{\lim_n}\ \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \le \frac{c_{r,d}}{(\theta^\star)^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa}. \tag{14}$$

(b) *Let* $\zeta^\star = \sup\left\{\zeta > 0,\ \limsup_{x\to+\infty}\ x^{\zeta-r}\bar{F}_r(x) < +\infty\right\}$.

Then $(\psi_r(x) = \frac{x}{\zeta-r},\ \forall\zeta \in (0,\zeta^\star)\ )$

$$\overline{\lim_n}\ \frac{\log\rho_n}{\log(n)} \le \frac{1}{\zeta^\star - r}\frac{r+d}{d}. \tag{15}$$

Asymp. for a specified density

Corollary

(a) *If* $f(x) \sim |x|^c \, e^{-\vartheta |x|^\kappa}$ *as* $|x| \to +\infty$; $\vartheta, \kappa > 0$; $c > -d$

then $\theta^\star = \vartheta$ *and*

$$\overline{\lim_n} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \frac{c_{r,d}}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa}. \tag{16}$$

(b) *If* $f(x) \sim \dfrac{(\log |x|)^\beta}{|x|^c}$ *as* $|x| \to +\infty$; $\beta \in \mathbb{R}$, $c > r + d$

then $\zeta^\star = c - d$ *and*

$$\overline{\lim_n} \, \frac{\log \rho_n}{\log(n)} \leq \frac{1}{c - d - r} \frac{r + d}{d}. \tag{17}$$

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Preliminaries

Definition and limit of the maximal radius

convergence rate

- Limit superior
- Limit inferior

Bibliography

Distribution of type (r, s)

Any L^r-optimal sequence $(\alpha_n)_{n\geq 1}$, $s \leq r$, is L^s-**rate-optimal** *i.e.*

$$\overline{\lim_n}\ n^{1/d}\|X - \widehat{X}^{\alpha_n}\|_s < +\infty. \tag{18}$$

If $s > r$ (and $X \in L^s(\mathbb{P})$) this rate optimality usually fails. So is always the case then $s > r + d$ and $\lambda_d(f > 0) = \infty$.

Definition

*Let $s, r > 0, s > r$. A r.v. $X \in L^s(\mathbb{P})$ has an (r, s)-**distribution** if any L^r-optimal sequence of quantizers $(\alpha_n)_{n\geq 1}$ is L^s-rate-optimal.*

Suppose that X has an (r, s)-distribution for some $\nu > 0$ and set

$$\nu_X^\star := \sup\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\}$$

main result

Theorem

Suppose X has an $(r, r+\nu)$-distribution for some $\nu > 0$. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers. Then,

$$\overline{\lim_{n}} \sup_{u>1} \left(\left(1 - 1/u\right)^{r+\nu} \; n^{\frac{r+\nu}{d}} \bar{F}_{r+\nu}\left(u\rho_n\right) \right) < +\infty. \qquad (19)$$

How to choose $\psi_{r,\nu}$?

Criterion

i) *Let* $\kappa > 0$ *and set*

$$\theta_\star = \inf\left\{\theta > 0,\ \liminf_{x\to+\infty} e^{\theta x^\kappa}\mathbb{P}(|X| > x) > 0\right\}. \qquad (20)$$

Then

$$\lim_n \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \geq \frac{1}{(\theta_\star)^{1/\kappa}}\left(\frac{r+\nu^\star}{d}\right)^{1/\kappa}. \qquad (21)$$

ii) *Let*

$$\zeta_\star = \inf\left\{\zeta > 0,\ \forall \nu \in (0,\nu^\star),\ \liminf_{x\to+\infty} x^{\zeta-r-\nu}\bar{F}_{r+\nu}(x) > 0\right\}, \qquad (22)$$

then

$$\lim_n \frac{\log \rho_n}{\log(n)} \geq \frac{1}{\zeta_\star - r - \nu^\star}\,\frac{r+\nu^\star}{d}. \qquad (23)$$

Corollary

If

$$f(x) \sim |x|^c \, e^{-\vartheta |x|^\kappa}, \; as \; |x| \to +\infty; \; \vartheta, \kappa > 0; \; c > -d \tag{24}$$

then

$$\nu^\star = d \quad and \quad \theta^\star = \theta_\star = \vartheta,$$

and for every $r > 0$, for every $d \geq 1$,

$$\frac{1}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa} \leq \varliminf_n \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \varlimsup_n \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \frac{2}{\vartheta^{1/\kappa}} \left(1 + \frac{r}{d}\right)^{1/\kappa} \tag{25}$$

If $d = 1$, $r \geq 1$ we have

$$\lim_{n \to +\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = \left(\frac{r+1}{\vartheta}\right)^{1/\kappa}. \tag{26}$$

Corollary

(b) *If*

$$f(x) \sim \frac{(\log|x|)^{\beta}}{|x|^{c}} \quad as \quad |x| \to +\infty,\ \beta \in \mathbb{R},\ c > r + d \tag{27}$$

then

$$\nu^{\star} = d\left(1 - \frac{r+d}{c}\right) \in (0, d) \quad and \quad \zeta_{\star} = \zeta^{\star} = c - d$$

and for every $r > 0$, *for every* $d \geq 1$,

$$\lim_{n \to +\infty} \frac{\log \rho_n}{\log(n)} = \frac{1}{c - r - d} \frac{r + d}{d}. \tag{28}$$

Examples

(1) • If $X \sim \mathcal{N}(0, I_d)$, $\forall r > 0$, $\forall d \geq 1$,

$$\sqrt{\frac{2(r+d)}{d}} \leq \underline{\lim}_n \frac{\rho_n}{\sqrt{\log(n)}} \leq \overline{\lim}_n \frac{\rho_n}{\sqrt{\log(n)}} \leq 2\sqrt{\frac{2(r+d)}{d}}. \quad (29)$$

If $d = 1$ and $r \geq 1$ we have

$$\lim_{n \to +\infty} \frac{\rho_n}{\sqrt{\log(n)}} = \sqrt{2(r+1)}. \quad (30)$$

• If $X \sim \Gamma(a, \lambda)$, $a > 0, \lambda > 0$ or if X has a double gamma distribution then for every $\forall r \geq 1$,

$$\lim_{n \to +\infty} \frac{\rho_n}{\log(n)} = \frac{r+1}{\lambda}; \quad (31)$$

and $\forall r \in (0, 1)$,

$$\frac{r+1}{\lambda} \leq \underline{\lim}_n \frac{\rho_n}{\log(n)} \leq \overline{\lim}_n \frac{\rho_n}{\log(n)} \leq \frac{2(r+1)}{\lambda}. \quad (32)$$

Examples

• The logistic distribution has the same asymptotics as the exponential distribution with parameter $\lambda = 1$ since $f(x) \sim e^{-x}$.
• For the Weibull distribution with parameter $\kappa > 0$: $\forall r \geq 1$,

$$\lim_{n\to+\infty} \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} = (r+1)^{1/\kappa}.$$

When $r \in (0,1)$,

$$(r+1)^{1/\kappa} \leq \underline{\lim}_n \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq \overline{\lim}_n \frac{\rho_n}{\big(\log(n)\big)^{1/\kappa}} \leq 2(r+1)^{1/\kappa}.$$

(2) Suppose X has a Pareto distribution with $\gamma > r$:
$f(x) = \gamma x^{-(\gamma+1)}\mathbf{1}_{\{x>1\}}$.

$$\lim_{n\to+\infty} \frac{\log \rho_n}{\log(n)} = \frac{r+1}{\gamma - r}. \tag{33}$$

An alternative approach

Theorem

Let $X \sim P$ with $P_a \neq 0$. Assume $(\alpha_n)_{n\geq 1}$ is an $L^r(P)$-optimal sequence of n-quantizers. Let $(X_k)_{k\geq 1}$ be an i.i.d sequence of $\mathbb{R}^d$-valued random variables with probability distribution P. Then for every $\nu \in (0, \nu_X^\star)$,

$$\liminf_{n\to+\infty} \left(\rho_n - \mathbb{E}\big(\max_{k\leq [n^{(r+\nu)/d}]} |X_k|\big)\right) \geq -\ C_\nu \tag{34}$$

where C_ν is a positive real constant.

▷ For the exponential distribution

$$\liminf_{n\to+\infty} \frac{\rho_n}{\log(n)} \geq \frac{r+1}{\lambda}$$

▷ For the Pareto distribution we have

$$\liminf_{n\to+\infty} \frac{\log \rho_n}{\log(n)} \geq \frac{r+1}{\gamma+1}.$$

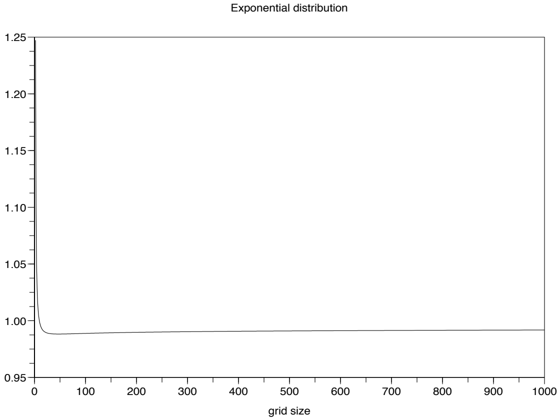

Figure: $\frac{\rho_n}{3\log(n)}$, $1 \leq n \leq 1\,000$ for the exponential distribution.

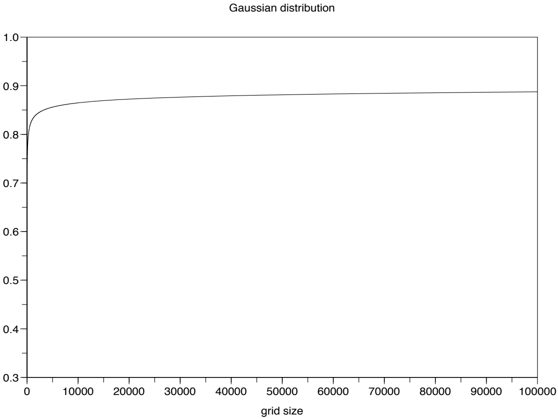

Figure: $\frac{\rho_n}{3\log(n)}$, $1 \leq n \leq 10\,000$ for the normal distribution.

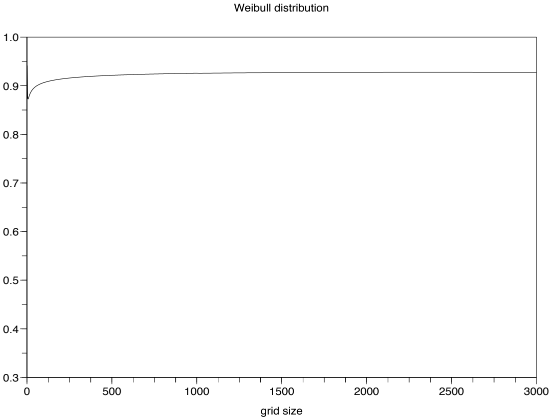

Figure: $\frac{\rho_n}{3\log(n)}$, $1 \leq n \leq 3\,000$ for the Weibull distribution.

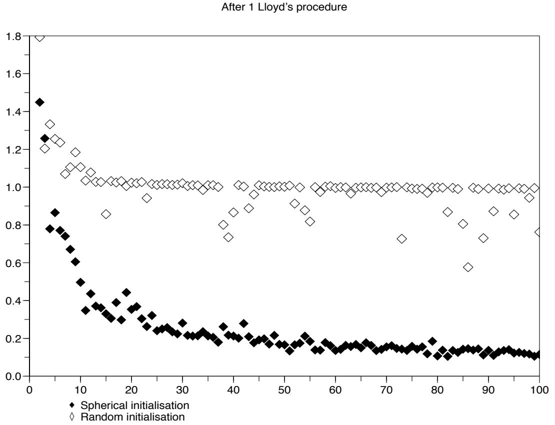

Figure: Comparaison of the distortions for different types of initializations after 1 Lloyd.

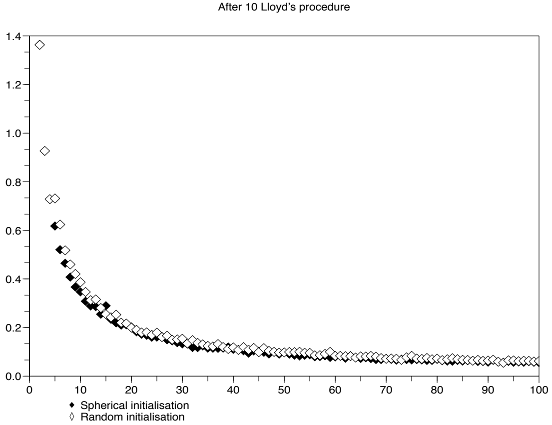

Figure: Comparaison of the distortions for different types of initializations after 10 Lloyd.

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