Coupling Index and Stocks

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Outline

- Handling both an Index and its composing stocks is still a challenging task.
- Standard approach : a model for the stocks (with smile) + a correlation matrix (historical estimation). Then, reconstruct the index local/implied vol. (Avellaneda *et al.* [1], Lee *et al.* [6], ...)
  - Difficulty to retrieve the good shape of the index smile (steeper than stock smile).
  - Dealing with the correlation matrix is tedious (keep it positive definite ? implied correlation matrix ?).

Our objective : a new modeling approach allowing for a good fit of both Index and stocks.

- Another viewpoint : a factor model (the index represents the market and influences the stocks).
- A new correlation structure. Correlation risk $\rightsquigarrow$ Index vol. risk.
- (In discrete time) Cizeau, Potters and Bouchaud [2001] show that it is possible to capture the essential features of stocks cross-correlations by a simple non-Gaussian one factor model, specially in extreme market conditions :

$$r_i(t) = \beta_i r_m(t) + \epsilon_i(t)$$

  where $r_i(t) = \frac{S_i(t)}{S_i(t-1)} - 1$ and r_m is the market daily return.
- Our model can be seen as an extension in continuous time.
- Calibration to both index and stocks is feasible.

Outline

Consider an Index composed of M underlyings :

$$I_t^M = \sum_{j=1}^{M} w_j S_t^{j,M}$$

In a risk-neutral world, we specify the following dynamics for the stocks :

$$\forall j \in \{1, \ldots, M\}, \quad \frac{dS_t^{j,M}}{S_t^{j,M}} = (r - \delta_j)dt + \beta_j \sigma(t, I_t^M)dB_t + \eta_j(t, S_t^{j,M})dW_t^j \quad (1)$$

- r is the short interest rate.
- $\delta_j \in [0, \infty[$ incorporates both repo cost and dividend yield of the stock j.
- β_j is the usual beta coefficient of the stock j.
- $(B_t)_{t\in[0,T]}, (W_t^1)_{t\in[0,T]}, \ldots, (W_t^M)_{t\in[0,T]}$ are independent BMs.
- The functions $\sigma, \eta_1, \ldots, \eta_M$ satisfy the usual Lipschitz and growth assumptions that ensure existence and strong uniqueness of the solutions (see for example Theorem 5.2.9 of Karatzas and Shreve [5])

- We have M coupled SDEs and $M+1$ noise sources.
- The dynamics of a given stock depends on all the other stocks composing the index through the volatility term $\sigma(t, I_t^M)$.
- The cross-correlations between stocks are not constant but stochastic :

$$\rho_{ij} = \frac{\beta_i \beta_j \sigma^2(t, I_t^M)}{\sqrt{\beta_i^2 \sigma^2(t, I_t^M) + \eta_i^2(t, S_t^{i,M})}\sqrt{\beta_j^2 \sigma^2(t, I_t^M) + \eta_j^2(t, S_t^{j,M})}}$$

Note that they depend not only on the stocks but also on the index.

The index I^M satisfies the following SDE

$$dI_t^M = rI_t^M dt - \left(\sum_{j=1}^M \delta_j w_j S_t^{j,M}\right) dt + \left(\sum_{j=1}^M \beta_j w_j S_t^{j,M}\right) \sigma(t, I_t^M) dB_t + \sum_{j=1}^M w_j S_t^{j,M} \eta_j(t, S_t^{j,M}) dW_t^j \qquad (2)$$

Our model is inline with Cizeau *et al.* [2] :

- The beta coefficients are narrowly distributed around 1 $\Rightarrow \sum_{j=1}^M \beta_j w_j S_t^{j,M} \simeq I_t^M$.
- For large M, we will show that the term $\sum_{j=1}^M w_j S_t^j \eta_j(t, S_t^j) dW_t^j$ can be neglected.

$$\Longrightarrow r_j = \beta_j r_{I^M} + \eta_j \Delta W^j + \text{drift}$$

where r_j (resp. r_{I^M}) is the log-return of the stock j (resp. the index).

The return of a stock is decomposed into a systemic part driven by the index, which represents the market, and a residual part.

A simplified Model

We look at the asymptotics for a large number of underlying stocks.
Consider the limit candidate $(I_t)_{t\in[0,T]}$ solution of

$$dI_t/I_t = (r-\delta)dt + \beta\sigma(t,I_t)dB_t; \quad I_0 = I_0^M \tag{3}$$

Theorem 1

Let $p \in \mathbb{N}^*$. If

($\mathcal{H}$1) $\exists K_b$ s.t. $\forall (t,s)$, $|\sigma(t,s)| + |\eta_j(t,s)| \le K_b$

$\exists K_\sigma$ s.t. $\forall (t,s_1,s_2)$, $|s_1\sigma(t,s_1) - s_2\sigma(t,s_2)| \le K_\sigma |s_1 - s_2|$.

then, $\exists C_T$ a constant independent of M such that

$$\mathbb{E}\left(\sup_{0\le t\le T} |I_t^M - I_t|^{2p}\right) \le C_T\left(\left(\sum_{j=1}^M w_j^2\right)^p + \left(\sum_{j=1}^M w_j|\beta_j - \beta|\right)^{2p} + \left(\sum_{j=1}^M w_j|\delta_j - \delta|\right)^{2p}\right)$$

We control the error when we replace I^M by I in the dynamics of the stocks :

Theorem 2

Denote by $\bar{I}_t^M = \sum_{j=1}^M w_j S_t^j$. *Under the assumptions of Theorem 1 and if*
($\mathcal{H}$2) $\exists K_\eta$ *s.t.* $\forall (t, s_1, s_2)$, $|s_1 \eta(t, s_1) - s_2 \eta(t, s_2)| \leq K_\eta |s_1 - s_2|$
$\exists K_{Lip}$ *s.t.* $\forall (t, s_1, s_2)$, $|\sigma(t, s_1) - \sigma(t, s_2)| \leq K_{Lip} |s_1 - s_2|$
then, $\forall j \in \{1, \ldots, M\}$, $\exists \widetilde{C}_T^j$ *s.t.*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |S_t^{j,M} - S_t^j|^{2p}\right) \leq \widetilde{C}_T^j \Bigg(\left(\sum_{j=1}^M w_j^2\right)^p + \left(\sum_{j=1}^M w_j |\beta_j - \beta|\right)^{2p} + \left(\sum_{j=1}^M w_j |\delta_j - \delta|\right)^{2p}\Bigg)$$

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |I_t^M - \bar{I}_t^M|^{2p}\right) \leq \max_{1 \leq j \leq M} \widetilde{C}_T^j \left(\sum_{j=1}^M w_j\right)^{2p} \Bigg(\left(\sum_{j=1}^M w_j^2\right)^p + \left(\sum_{j=1}^M w_j |\beta_j - \beta|\right)^{2p} + \left(\sum_{j=1}^M w_j |\delta_j - \delta|\right)^{2p}\Bigg)$$

The limit $M \to +\infty$

Corollary

Under the additional assumptions

($\mathcal{H}$3) $\exists A$ *s.t.* $\max_{j\geq 1}\left((S_0^{j,M})^2 + (\beta_j^M)^2 + (\delta_j^M)^2\right) \leq A$,

($\mathcal{H}$4) $P_w^M = \sum_{j=1}^M (w_j^M)^2 \longrightarrow_{M\to\infty} 0$,

($\mathcal{H}$5) $P_\beta^M = \sum_{j=1}^M w_j^M |\beta_j^M - \beta| \longrightarrow_{M\to\infty} 0$,

($\mathcal{H}$6) $P_\delta^M = \sum_{j=1}^M w_j^M |\delta_j^M - \delta| \underset{M\to\infty}{\longrightarrow} 0$,

one has $\mathbb{E}\left(\sup_{0\leq t\leq T} |I_t^M - I_t|^2\right) \longrightarrow_{M\to\infty} 0$ *and* $\forall j \in \{1,\ldots,M\}$, $\quad \mathbb{E}\left(\sup_{0\leq t\leq T} |S_t^{j,M} - S_t^j|^2\right) \longrightarrow_{M\to\infty} 0$. *If, in addition,* $\sup_M \sum_{j=1}^M w_j^M < \infty$ *then* $\mathbb{E}\left(\sup_{0\leq t\leq T} |I_t^M - \bar{I}_t^M|^2\right) \longrightarrow_{M\to\infty} 0$.

- Assumption ($\mathcal{H}$**4**) prevents concentration in the weights (for example uniform weights lead to $P_w^M = \sum_{j=1}^M \frac{1}{M^2} = \frac{1}{M} \longrightarrow_{M\to\infty} 0$).

- In order to remain coherent with the definition of the beta coefficients, we have to take $\beta = 1$.

P_w^M	β_{opt}	$(P_{\beta_{opt}}^M)^2$	$(P_{\beta=1}^M)^2$
0.026	0.975	0.0173	0.0174

TABLE: Computation of P_w^M, β_{opt} and $(P_{\beta_{opt}}^M)^2$ for the Eurostoxx index at December 21, 2007. The beta coefficients are estimated on a two year history.

To sum up, under mild assumptions, when the number of underlying stocks is large, our original model may be approximated by

$$\begin{aligned} \forall j \in \{1, \dots, M\}, \quad \frac{dS_t^j}{S_t^j} &= (r - \delta_j)dt + \beta_j \sigma(t, I_t)dB_t + \eta_j(t, S_t^j)dW_t^j \\ \frac{dI_t}{I_t} &= (r - \delta_I)dt + \sigma(t, I_t)dB_t. \end{aligned} \tag{4}$$

We end up with

- A local volatility model for the index
- A stochastic volatility model for each stock, decomposed into a systemic part driven by the index level and an intrinsic part.

Careful ! Our simplified model is not valid for options written on the index together with all its composing stocks since the index is no longer an exact, but an approximate, weighted sum of the stocks. Instead, one should consider the reconstructed index $\bar{I}_t^M = \sum_{j=1}^M w_j S_t^j$ or use the original model.

Outline

Calibration of the simplified model

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\delta)dt + \beta\,\sigma(t,I_t)dB_t + \eta(t,S_t)dW_t \\ \frac{dI_t}{I_t} &= (r-\delta_I)dt + \sigma(t,I_t)dB_t. \end{aligned} \tag{5}$$

- Fitting the index smile boils down to the calibration of a local volatility model.
- Fitting an individual stock smile is more tedious.

$\Rightarrow$ Our model gives an advantage to the fit of index option prices (index options are usually more liquid than individual stock options).

Non-parametric estimation of η

We have a relation between the local volatility and the stochastic volatility (see Gyöngi [4] or Dupire [3]) :

$$v_{loc}(t, K) = \eta^2(t, K) + \beta^2 \mathbb{E}\left(\sigma^2(t, I_t) \,|\, S_t = K\right)$$

So,

$$\eta(t, K) = \sqrt{v_{loc}(t, K) - \beta^2 \mathbb{E}\left(\sigma^2(t, I_t) \,|\, S_t = K\right)}. \tag{6}$$

- v_{loc} can be calibrated with the best-fit of a parametric form to the stock market smile.
- Estimating the conditional expectation is more challenging (it depends implicitly on η as it is the case for (S_t, I_t)).

We investigate a simulation based approach yielding a non-parametric estimation of η.

If we plug the formula (6) in the dynamics of the stock we obtain

$$\frac{dS_t}{S_t} = (r-\delta)dt + \beta\,\sigma(t, I_t)dB_t + \sqrt{v_{loc}(t, S_t) - \beta^2 \mathbb{E}\left(\sigma^2(t, I_t)\,|\,S_t\right)}dW_t$$

$$\frac{dI_t}{I_t} = (r-\delta_I)dt + \sigma(t, I_t)dB_t$$

- This SDE is non-linear in the sense of McKean.
- Kernel estimators of the Nadaraya-Watson type :

$$\mathbb{E}\left(\sigma^2(t, I_t)\,|\,S_t = s\right) \simeq \frac{\sum_{i=1}^{N} \sigma^2(t, I_t^i) K\left(\frac{s - S_t^i}{h_N}\right)}{\sum_{i=1}^{N} K\left(\frac{s - S_t^i}{h_N}\right)}$$

where K is a non-negative kernel s.t. $\int_{\mathbb{R}} K(x)dx = 1$ and $\lim_{N\to\infty} h_N = 0$.

A system of interacting particles

Replacing the conditional expectation by its non-parametric estimator yield the following system : $\forall 1 \leq i \leq N$,

$$\frac{dS_t^{i,N}}{S_t^{i,N}} = (r-\delta)dt + \beta\,\sigma(t, I_t^i)dB_t^i + \sqrt{v_{loc}(t, S_t^{i,N}) - \beta^2 \frac{\sum_{j=1}^N \sigma^2(t, I_t^j) K\left(\frac{S_t^{i,N} - S_t^{j,N}}{h_N}\right)}{\sum_{j=1}^N K\left(\frac{S_t^{i,N} - S_t^{j,N}}{h_N}\right)}}\, dW_t^i$$

$$\frac{dI_t^i}{I_t^i} = (r - \delta_I)dt + \sigma(t, I_t^i)dB_t^i$$

$(B^i, W^i)_{i \geq 1}$ is a sequence of independent two-dimensional Brownian motions.

$\Rightarrow$ This $2N$-dimensional linear SDE may be discretized using a simple Euler scheme !

Calibration of the original model

$$\forall j \in \{1,\dots,M\}, \quad \frac{dS_t^{j,M}}{S_t^{j,M}} = (r-\delta_j)dt + \beta_j\,\sigma(t, I_t^M)dB_t + \eta_j(t, S_t^{j,M})dW_t^j$$

$$I_t^M = \sum_{i=1}^{M} w_i S_t^{i,M}$$

- A perfect calibration of both the index and the individual stocks is complicated... but we can
  - take for σ the calibrated local vol of the index and then calibrate the η_j coefficients in order to fit all the individual stock smiles $\Rightarrow$ the index is not perfectly calibrated but the error should be small (Theorem 1).
  - take for σ and η_j the calibrated coefficients in the simplified model $\Rightarrow$ the index and the stocks are not perfectly calibrated but the error should be small (Theorems 1 and 2).
- We allow for a slight error in the calibration but the additivity constraint is observed.

Outline

An acceleration technique

- The simulation of the particle system is time consuming : a global complexity of order $O(nN^2)$ where n is the number of time steps in the Euler scheme.
- A possible acceleration technique : neglect particles which are far away from each other.
- How ? Sort the particles and stop the estimation of the conditional expectation whenever the contribution of a particle is lower than some fixed threshold.
- We lose in precision but we gain much more in computation time.

Data :

- Local volatilities of the Eurostoxx index and of Carrefour at December 21, 2007.
- Beta coefficient estimated on a two years history ($\beta = 0.7$).
- Short interest rate and dividend yields as of December 21, 2007.
- Maturity $T = 1$.
- Threshold for the accelerated technique : $\frac{1}{N}$.
- Smoothing parameter : $h_N = N^{-\frac{1}{10}}$.
- Number of time steps for the Euler scheme : $n = 20$.

Moneyness ($\frac{K}{S_0}$)	0.5	0.7	0.9	1	1.1	1.2	1.5	2
Error : $\lvert\widehat{\sigma}_{simul} - \widehat{\sigma}_{exact}\rvert$	36	8	2	1	2	9	32	56

TABLE: Error (in bp) on the implied volatility with $N = 200000$ particles.

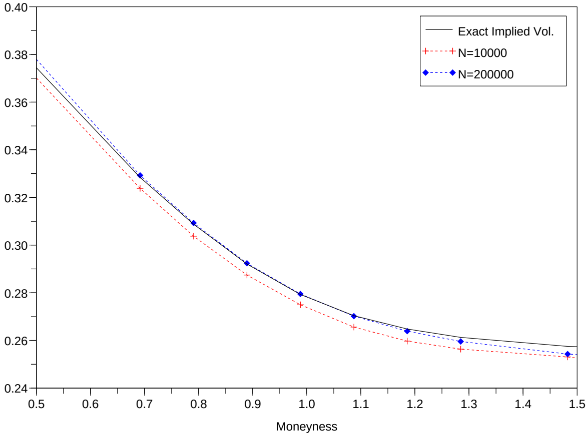

FIGURE: Convergence of the implied volatility obtained with non-parametric estimation.

Illustration of theorems 1 and 2

1. The original model

$$\forall j \in \{1, \ldots, M\}, \quad \begin{array}{l} \frac{dS_t^{j,M}}{S_t^{j,M}} = rdt + \sigma(t, I_t^M)dB_t + \eta(t, S_t^{j,M})dW_t^j \\ I_t^M = \sum_{i=1}^M w_i S_t^{i,M}. \end{array}$$

2. The simplified model

$$\forall j \in \{1, \ldots, M\}, \quad \begin{array}{l} \frac{dS_t^j}{S_t^j} = rdt + \sigma(t, I_t)dB_t + \eta(t, S_t^j)dW_t^j \\ \frac{dI_t}{I_t} = rdt + \sigma(t, I_t)dB_t. \end{array}$$

Reconstructed index $\bar{I}_t^M = \sum_{i=1}^M w_i S_t^i$.

3. The constant-correlation model

$$\forall j \in \{1, \ldots, M\}, \quad \begin{array}{l} \frac{dS_t^j}{S_t^j} = rdt + \sqrt{v_{loc}(t, S_t^j)}d\widetilde{W}_t^j \\ \forall i \neq j, d < \widetilde{W}^i, \widetilde{W}^j >_t = \rho dt. \end{array}$$

- We take for σ the calibrated local vol of the Eurostoxx.
- We choose an arbitrary parametric form for the vol coefficient η.
- We evaluate v_{loc} s.t. the constant-correl model and the simplified one yield the same implied vol for individual stocks $(v_{loc}(t,S) = \eta^2(t,S) + \mathbb{E}(\sigma^2(t,I_t)|S_t = S))$.
- We fix the correlation coefficient ρ s.t. the constant-correl model and the simplified one yield the same ATM implied vol for the index.
- We take the same market data as before.

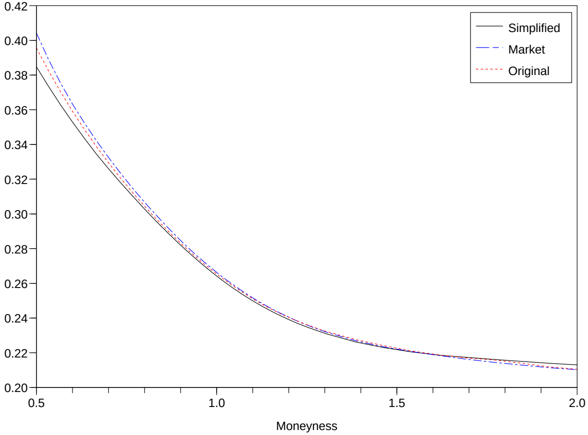

FIGURE: Implied volatility of an individual stock.

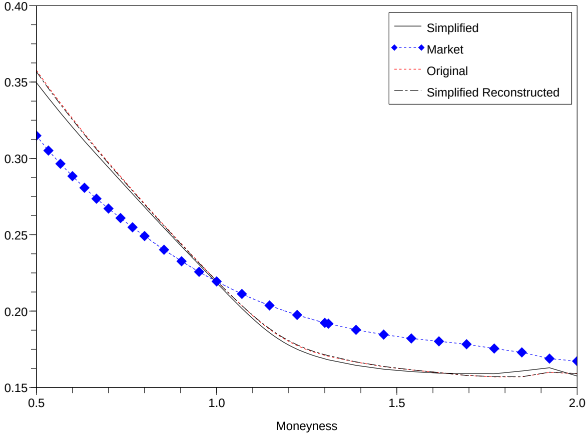

FIGURE: Implied volatility of the index.

Outline

- We have introduced a new model for describing the joint evolution of an index and its composing stocks.
- The index induces some feedback on the dynamics of its stocks.
- For large number of underlying stocks, the model reduces to a local vol model for the index and to a stochastic vol for each individual stock with volatility driven by the index.
- We favor the fit of the index smile.
- We have proposed a simulation based approach allowing to fit both the index and the stocks smiles.

Thank you !

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