Jump-adapted discretization schemes for Lévy-driven SDEs

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Weak approximation of Lévy-driven SDEs

We are interested in numerical evaluation of

$$E[f(X_1)], \quad \text{where} \quad X_t = X_0 + \int_0^t h(X_{s-})dZ_s, \quad X \in \mathbb{R}^n$$

where $Z \in \mathbb{R}^d$ is a pure-jump Lévy process:

$$Z_t = \gamma t + \int_0^t \int_{|y|\leq 1} y\widehat{N}(dy, ds) + \int_0^t \int_{|y|>1} yN(dy, ds)$$

Discretization with constant time step

The Euler scheme with constant time step

$$\hat{X}^n_{\frac{i+1}{n}} = \hat{X}^n_{\frac{i}{n}} + h(\hat{X}^n_{\frac{i}{n}})(Z_{\frac{i+1}{n}} - Z_{\frac{i}{n}})$$

has the convergence rate (Protter and Talay '97) $|E[f(X_1)] - E[f(\hat{X}^n_1)]| \leq \frac{C}{n}$ but suffers from two difficulties

- The increments of Z cannot usually be simulated in closed form;
- A large jump in Z between two discretization dates may lead to a large discretization error.

Jump-adapted discretization

A natural idea to solve both problems, due to Rubenthaler '03, is

- Approximate Z with a compound Poisson process

$$Z_t^\varepsilon := \gamma_\varepsilon t + \int_0^t \int_{|y|>\varepsilon} yN(dy, ds), \quad \gamma_\varepsilon = \gamma - \int_{\varepsilon<|y|\leq 1} y\nu(dy).$$

- Apply the Euler scheme at every jump time of Z^ε.

The convergence rate may be computed in terms of *expected* number of discretization dates, proportional to $\lambda_\varepsilon = \int_{|y|\geq\varepsilon} \nu(dy)$.

This rate may range from very good to very bad (for Z of infinite variation) because

- The variance of small jumps may go to zero very slowly;
- The drift γ_ε may explode as $\varepsilon \to 0$.

Taking into account the structure of Z

- View the Lévy process between the jumps of Z^ε as a deterministic ODE perturbed by noise (small jumps).
- Solve the deterministic ODE explicitly, or with a higher-order scheme, which is easy to construct.
- Approximate small jumps by Brownian motion (Asmussen and Rosinski '01):

$$\left(\frac{1}{\sigma_\varepsilon^2}\int_0^t \int_{|z|\leq\varepsilon} y\hat{N}(dy,ds)\right)_{0\leq t\leq 1} \xrightarrow[\varepsilon\to 0]{d} (W_t)_{0\leq t\leq 1}$$

  where

$$\sigma_\varepsilon^2 = \int_{|y|\leq\varepsilon} y^2\nu(dy).$$

- Construct an approximation to the SDE around the explicit solution of the ODE

Constructing the approximating process

In the 1-d case, the explicit solution of

$$dX_t = h(X_t)dt, \quad X_0 = x,$$

can always be written as

$$X_t := \theta(t,x) = F^{-1}(t + F(x)), \qquad F \text{ is the primitive of } \frac{1}{h(x)}.$$

We denote the jump times of Z^ε by $(T_i^\varepsilon)_{i\geq 1}$ and set $\eta_t := \sup\{T_i^\varepsilon : T_i^\varepsilon \leq t\}$.

Constructing the approximating process

We define inductively $\hat{X}(0) = X_0$ and for $i \geq 1$,

$$\hat{X}(T_i^\varepsilon) = \hat{X}(T_i^\varepsilon -) + h(\hat{X}(T_i^\varepsilon -))\Delta Z(T_i^\varepsilon) \quad \text{(Euler at jump times)}$$

$$\hat{X}(T_{i+1}^\varepsilon -) = \theta\Big(\gamma_\varepsilon(T_{i+1}^\varepsilon - T_i^\varepsilon) + \sigma_\varepsilon(W(T_{i+1}^\varepsilon) - W(T_i^\varepsilon)) - \frac{1}{2}h'(X(T_i^\varepsilon))\sigma_\varepsilon^2(T_{i+1}^\varepsilon - T_i^\varepsilon), \hat{X}(T_i^\varepsilon)\Big)$$

(approx. solution of continuous SDE between jumps)

The approximating process $\hat{X}$ satisfies the SDE

$$d\hat{X}_t = h(\hat{X}_{t-})\left\{dZ_t^\varepsilon + \sigma_\varepsilon dW_t + \gamma_\varepsilon dt + \frac{1}{2}(h'(\hat{X}_t) - h'(\hat{X}_{\eta(t)}))\sigma_\varepsilon^2 dt\right\}.$$

- Explicitly solvable approximation which is exact for deterministic Z and/or linear h.

The convergence rate

$(\mathbf{H_n})$ $f \in C^n$, $h \in C^n$ $f^{(k)}$ and $h^{(k)}$ are bounded for $1 \leq k \leq n$ and $\int z^{2n}\nu(dz) < \infty$.

$(\mathbf{H'_n})$ $f \in C^n$, $h \in C^n$, $h^{(k)}$ are bounded for $1 \leq k \leq n$, $f^{(k)}$ have at most polynomial growth for $1 \leq k \leq n$ and $\int |z|^k \nu(dz) < \infty$ for all $k \geq 1$.

- Assume $(\mathbf{H_3})$ or $(\mathbf{H'_3})$. Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C\left(\frac{\sigma_\varepsilon^2}{\lambda_\varepsilon}(\sigma_\varepsilon^2 + |\gamma_\varepsilon|) + \int_{|y|\leq\varepsilon} |y|^3 \nu(dy)\right)$$

- Assume $(\mathbf{H_4})$ or $(\mathbf{H'_4})$ and $\nu(dy) = (1+\xi(y))\nu_0(dy)$, where ν_0 is a symmetric measure and $\xi(y) = O(y)$. Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C\left(\frac{\sigma_\varepsilon^2}{\lambda_\varepsilon} + \int_{|y|\leq\varepsilon} |y|^4 \nu(dy)\right).$$

Worst-case bounds

For general Lévy measures,

$$|E[f(\hat{X}_1) - f(X_1)]| \leq o(\lambda_\varepsilon^{-\frac{1}{2}}),$$

and in the locally symmetric case,

$$|E[f(\hat{X}_1) - f(X_1)]| \leq o(\lambda_\varepsilon^{-1}).$$

- For all known examples, the convergence rates are better.

Stable-like behavior and other examples

- Assume that ν has a density $\nu(z) = \frac{f(z)}{|z|^{1+\alpha}}$ with $0 < f(0-), f(0+) < \infty$. Then

$$|E[f(\hat{X}_1) - f(X_1)]| \leq O(\lambda_\varepsilon^{1-\frac{3}{\alpha}}),$$

and if the Lévy measure is symmetric near zero (CGMY),

$$|E[f(\hat{X}_1) - f(X_1)]| \leq O(\lambda_\varepsilon^{1-\frac{4}{\alpha}}).$$

- The NIG process has a symmetric stable-like Lévy measure with $\alpha = 1$

$$\Rightarrow \quad |E[f(\hat{X}_1) - f(X_1)]| \leq O(1/\lambda_\varepsilon^3)$$

- In the VG model, the convergence is exponential:

$$|E[f(\hat{X}_1) - f(X_1)]| \leq C\frac{e^{-2\lambda_\varepsilon}}{\lambda_\varepsilon}.$$

Constructing the approximating process

- Start by replacing the small jumps of Z with a Brownian motion:

$$d\bar{X}_t = h(\bar{X}_{t-})\{\gamma_\varepsilon dt + dW_t^\varepsilon + dZ_t^\varepsilon\},$$

where W^ε is a d-dimensional Brownian motion with covariance matrix Σ^ε. This process can also be written as

$$\bar{X}(t) = \bar{X}(\eta_t) + \int_{\eta_t}^t h\left(\bar{X}(s)\right) dW^\varepsilon(s) + \int_{\eta_t}^t h\left(\bar{X}(s)\right) \gamma_\varepsilon ds,$$

$$\bar{X}(T_i^\varepsilon) = \bar{X}(T_i^\varepsilon -) + h(\bar{X}(T_i^\varepsilon -))\Delta Z(T_i^\varepsilon),$$

- Expand the solution of the SDE between the jumps of Z^ε around the solution of ODE, treating the stochastic term as a small random perturbation in the spirit of Friedlin and Wentzell '79.

Expanding the solution

- Consider a family of processes $(Y^\alpha)_{0\leq\alpha\leq 1}$ defined by

$$Y^\alpha(t) = \bar{X}(\eta_t) + \alpha \int_{\eta_t}^t h\left(Y^\alpha(s)\right) dW^\varepsilon(s) + \int_{\eta_t}^t h\left(Y^\alpha(s)\right) \gamma_\varepsilon ds$$

- Our idea is to replace the process $\bar{X} := Y^1$ with its first-order Taylor approximation:

$$\bar{X}(t) \approx Y^0(t) + \frac{\partial}{\partial\alpha} Y^\alpha(t)|_{\alpha=0}.$$

- The 1st order approximation yields exactly the same rates as the 1d scheme, and 2nd order approximation does not yield an improvement because the error from replacing X by $\bar{X}$ dominates.

The approximation

The new approximation $\tilde{X}$ is defined by

$$\tilde{X}(t) = Y_0(t) + Y_1(t), \quad t > \eta_t,$$
$$\tilde{X}(T_i^\varepsilon) = \tilde{X}(T_i^\varepsilon-) + h(\tilde{X}(T_i^\varepsilon-))\Delta Z(T_i^\varepsilon),$$
$$Y_0(t) = \tilde{X}(\eta_t) + \int_{\eta_t}^t h(Y_0(t))\gamma_\varepsilon ds$$
$$Y_1(t) = \int_{\eta_t}^t \frac{\partial h}{\partial x_i}(Y_0(s))\, Y_1^i(s)\gamma_\varepsilon ds + \int_{\eta_t}^t h(Y_0(s))\, dW^\varepsilon(s)$$

where we used the Einstein convention for summation over repeated indices.

Computing Y_0 and Y_1

- Y_0 is the solution of an ODE and can be computed, e.g., by a 4th order Runge-Kutta scheme.
- Conditionally on the jump times $(T_i^\varepsilon)_{i\geq 1}$, the random vector $Y_1(t)$ is Gaussian with mean zero, and we only need its terminal covariance.
- Its covariance matrix $\Omega(t)$ satisfies the (matrix) linear equation

$$\Omega(t) = \int_{\eta_t}^{t} (\Omega(s)M(s) + M^{\perp}(s)\Omega^{\perp}(s) + N(s))ds$$

where $M^{\perp}$ denotes the transpose of the matrix M and

$$M_{ij}(t) = \frac{\partial h_{ik}(Y_0(t))}{\partial x_j}\gamma_\varepsilon^k \quad \text{and} \quad N(t) = h(Y_0(t))\Sigma^\varepsilon h^{\perp}(Y_0(t)).$$

A multidimensional example: Libor market model

- The Libor market model (general case of BGM model) describes joint arbitrage-free dynamics of a set of forward interest rates.
- Libor market models with jumps were considered by Jamshidian '99, Glasserman and Kou '03, Eberlein and Özkan '05 and others.
- Let $T_i = T_1 + (i-1)\delta$, $i = 1, \dots, n+1$ be a set of dates called *tenor* dates. The *Libor* rate L_t^i is the forward rate defined at t for the period $[T_i, T_{i+1}]$:

$$L_t^i = \frac{1}{\delta}\left(\frac{B_t(T_i)}{B_t(T_{i+1})} - 1\right),$$

where $B_t(T)$ is the price at t of a zero-coupon bond with maturity T.

A multidimensional example: Libor market model

Following Jamshidian '99, an arbitrage-free dynamics of n forward Libors $L^1_t, \dots, L^n_t$ can be constructed via the multi-dimensional SDE

$$\frac{dL^i_t}{L^i_{t-}} = \sigma^i(t)dZ_t - \int_{\mathbb{R}^d} \sigma^i(t)z \left[\prod_{j=i+1}^{n} \left(1 + \frac{\delta L^j_t \sigma^j(t)z}{1+\delta L^j_t}\right) - 1 \right] \nu(dz)dt,$$

where Z is a d-dimensional martingale pure jump Lévy process with Lévy measure ν under the terminal measure Q and $\sigma^i(t)$ are deterministic volatility functions.

A multidimensional example: Libor market model

- Terminal measure: martingale measure for which the last zero-coupon bond $B_t(T_{n+1})$ is the *numéraire*.
- The price of any asset divided by $B_t(T_{n+1})$ is a martingale and in particular the price of an option which pays $H = h(L^1_{T_1}, \ldots, L^n_{T_1})$ at time T_1 (e.g. swaption) is given by

$$\pi_t(H) = B_t(T_{n+1}) E\left[\left. \frac{h(L^1_{T_1}, \ldots, L^n_{T_1})}{B_{T_1}(T_{n+1})} \right| \mathcal{F}_t \right]$$

$$= B_t(T_{n+1}) E\left[\left. h(L^1_{T_1}, \ldots, L^n_{T_1}) \prod_{i=1}^{n} (1 + \delta L^i_{T_1}) \right| \mathcal{F}_t \right].$$

- The price of any such option can therefore be computed by Monte Carlo using the Libor dynamics.

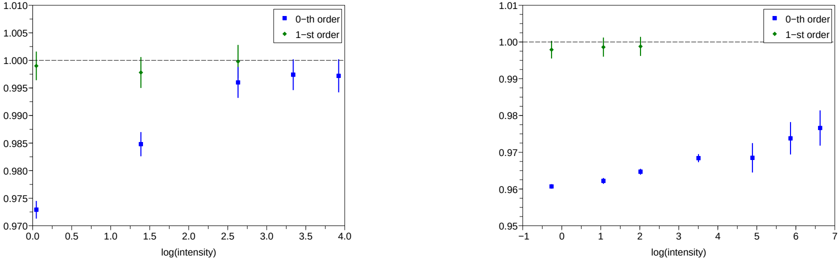
Numerical implementation

- We consider a Libor market model with tenor dates $\{5, 6, 7, 8, 9, 10\}$, a one-dimensional driving Lévy process and constant volatilities of all Libors ($\sigma^i(t) \equiv 1$).
- The initial values are fixed to 15% to emphasize the non-linear effects.
- The differential equations for $Y_0(t)$ and $\Omega(t)$ are solved simultaneously by fourth order Runge Kutta with $h = 1$.
- The Lévy measure is

$$C \frac{e^{-\lambda_+ x} 1_{x>0} + e^{-\lambda_- |x|} 1_{x<0}}{|x|^{1+\alpha}}$$

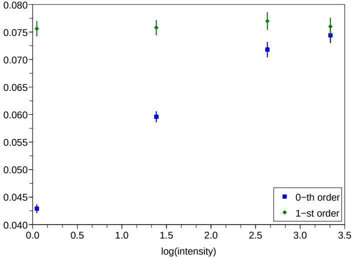
with $\lambda_+ = 10$, $\lambda_- = 20$ and $\alpha = 0.5$, $C = 1.5$ (Case 1) or $\alpha = 1.8$, $C = 0.01$ (Case 2). Both cases correspond to annualized standard deviation of about 24%.

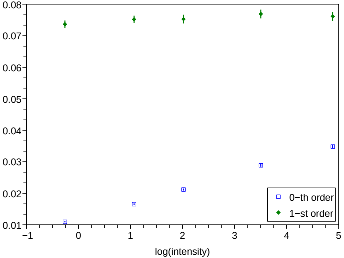
Sanity check: pricing a zero-coupon



Ratio of estimated to theoretical zero coupon bond price in Case 1 (left) and Case 2 (right). The theoretical convergence rate is λ^{-3} in Case 1 and $\lambda^{-0.11}$ in Case 2 for 0-th order approximation and λ^{-7} and $\lambda^{-1.22}$ for 1-st order approximation.

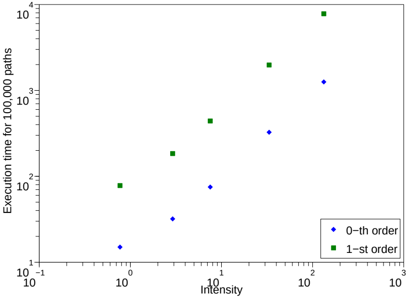
Pricing ATM receiver swaption





Estimated price of an ATM receiver swaption with maturity 5 years in Case 1 (left) and Case 2 (right).

Execution times



Execution times for the swaption example on a PIII PC without any code optimization. Even in Case 1 it is faster to use the 1st order scheme.