Path-Dependent Dividends and the American Put Option

M.H. Vellekoop[1] J.W. Nieuwenhuis[2]

[1] Department of Applied Mathematics
University of Twente

[2] Department of Economics and Business Administration
University of Groningen

Numerical Methods in Finance, Paris 2009

Outline

1. Problem Formulation
   - Standard Black-Scholes-Merton Model
   - Including Dividends
   - Assumptions
2. Main Theorem
   - Early Exercise Premium with Dividends
   - Calculating the Optimal Exercise Boundary
3. Numerical Results
   - Knock Out Dividend Model
   - Proportional Cash Dividend Model
   - Fixed Cash Dividend Model

Standard Equity Model

Consider the standard Black-Scholes-Merton model for stock and bond prices

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ dB_t &= rB_t dt \end{aligned}$$

for time period $t \in [0, T]$ with

- S_0, B_0, r, σ given strictly positive constants and
- W a one-dimensional Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$.

American and European Put Options

Let AP and EP denote the American and European Put price processes for the maturity T and a strike $K > 0$:

$$
\begin{aligned}
EP_t &= \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T}(K - S_T)^+ \mid \mathcal{F}_t\right] \stackrel{def}{=} E(t, S_t) \\
AP_t &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_\tau}(K - S_\tau)^+ \mid \mathcal{F}_t\right] \stackrel{def}{=} A(t, S_t)
\end{aligned}
$$

where $\mathcal{T}_{[t,T]}$ is the set of all stopping times with values in $[t, T]$.
Optimal stopping problem for American Put has the solution

$$
\tau^* = \inf\{u \geq t : A(u, S_u) = (K - S_u)^+\} = \inf\{u \geq t : S_u \leq S_u^*\}
$$

where S^* denotes what is called the optimal exercise boundary.

Early Exercise Premium

The difference between the American and European option price is known as the Early Exercise Premium

$$EE(t,s) = A(t,s) - E(t,s)$$

and it can be characterized as follows:

Early Exercise Representation
(Carr et al '92), (Jacka '93), (Kim '90)

$$EE(t,s) = rK\,\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-r(u-t)}\mathbf{1}_{\{S_u \le S_u^*\}}du \mid S_t = s\Big]$$

Dividend Models

We would like to extend these results to the case where dividends are included, i.e.

$$dS_t = rS_t dt + \sigma S_t dW_t - dD_t^S$$

with, for example, continuous, proportional or fixed cash dividends

$$\begin{aligned} dD_t^S &= qS_t dt \\ dD_t^S &= (1-\alpha)S_{t-} d\mathbf{1}_{\{t \geq t_D\}} \\ dD_t^S &= \min\{d, S_{t-}\} d\mathbf{1}_{\{t \geq t_D\}}. \end{aligned}$$

for a given $t_D \in]0, T[$ and $d > 0$, $q > 0$, $\delta, \alpha \in]0, 1[$.

Path-Dependent Dividend

Here, we focus on 'knock-out' version of proportional dividends:

$$dD_t^S = (1-\alpha)S_{t-}\mathbf{1}_{\{\min_{u\in[0,t_d]} S_u \geq \delta S_0\}}\, d\mathbf{1}_{\{t\geq t_d\}}$$

This models the fact that the company which issued the stock will only pay dividends if the stock price has not fallen below the level δS_0 before the dividend date.

This obviously makes the dividend path-dependent.

Path-Dependent Dividend

European Put option with knockout dividends can be priced in closed form:

Lemma

For general $\alpha \in]0,1]$ and $m_t \geq \delta S_0$ we have that the European put option price $E(t, S_t)$ with strike K and maturity T equals, for all $t \in [0, T]$,

$$P_{K,T}(t, S_t) + \mathbf{1}_{\{t<t_d\}} K e^{-r(T-t)} \left(\hat{h}_{t_d,T}(\delta S_0, K) - L_t^{\beta} \hat{h}_{t_d,T}\left(\frac{S_t}{L_t}, \frac{K}{L_t^2}\right) \right)$$
$$-\mathbf{1}_{\{t<t_d\}} e^{-r(T-t)} \left(\hat{H}_{t_d,T}(\delta S_0, K) - L_t^{2+\beta} \hat{H}_{t_d,T}\left(\frac{S_t}{L_t}, \frac{K}{L_t^2}\right) \right)$$

with $L_t = \frac{\delta S_0}{S_t}$, $\beta = 2r\sigma^{-2} - 1$ and

$$\begin{aligned} \hat{H}_{t_1,t_2}(x_1, x_2) &= \alpha \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t_2} \mathbf{1}_{\{\tilde{S}_{t_2} \leq \frac{x_2}{\alpha}, \tilde{S}_{t_1} > x_1\}} \mid \mathcal{F}_t] \\ &\quad - \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t_2} \mathbf{1}_{\{\tilde{S}_{t_2} \leq x_2, \tilde{S}_{t_1} > x_1\}} \mid \mathcal{F}_t] \\ \hat{h}_{t_1,t_2}(x_1, x_2) &= \mathbb{Q}(\tilde{S}_{t_1} > x_1,\ x_2 < \tilde{S}_{t_2} \leq \frac{x_2}{\alpha} \mid \mathcal{F}_t) \end{aligned}$$

where $\tilde{S}$ is the asset process with zero dividends.

Dividend Model Assumptions

In more general setup, we assume

1. S is an adapted càdlàg semimartingale, Markov, and distribution of S_t has a density for all t
2. D^S is an adapted, increasing càdlàg semimartingale, continuous in all but countable number of time points
3. $S/B + \int dD^S/B$ is a $\mathbb{Q}$-martingale, and the functions

$$x \to \mathbb{E}^{\mathbb{Q}}[\Phi(S_u) \mid S_t = x], \qquad x \to \mathbb{E}^{\mathbb{Q}}[\int_t^u dD_v^S/B_v \mid S_t = x]$$

are increasing for all $0 \leq t < u \leq T$ and for all increasing functions $\Phi : \mathbb{R} \to \mathbb{R}$.

Dividend Model Assumptions

These assumptions include the dividend models mentioned earlier, but also stock-dependent volatility models (Babilua et al. '07).

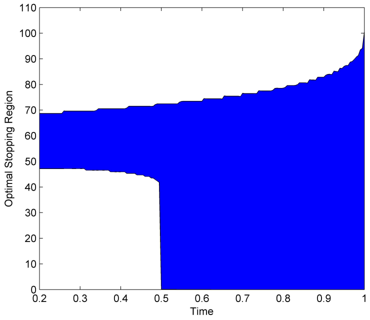
They guarantee that if we define

$$S_t^* = \inf\{s > 0 : A(t,s) > K - s\}$$

then

$$s > S_t^* \quad \Leftrightarrow \quad A(t,s) > K - s$$

i.e. it excludes possibility that there are several boundaries which separate different continuation and stopping regions.



$$dD_t^S = \min\{S_{t_D-}, \max\{40 - S_{t_D-}, 0\}\} d\mathbf{1}_{\{t \geq t_D\}}$$

Early Exercise Premium with Dividends

For stock price models with dividends, the early exercise premium can be characterized as follows:

Theorem (Early Exercise Representation including Dividends)

Under the assumptions stated above, and the additional assumption that the optimal exercise boundary S^* is continuous apart from at most a countable number of points, the American and European Put price processes satisfy

$$AP_t - EP_t = -B_t \, \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \mathbf{1}_{\{S_{u-} \leq S_u^* \wedge \Delta S_u = 0\}} \big(d\big(\tfrac{K}{B_u}\big) + \tfrac{dD_u^S}{B_u}\big) \mid \mathcal{F}_t \Big].$$

Continuity of optimal exercise boundary is subject of ongoing research, in collaboration with CERMICS.

Part I

- First prove that $s \to A(t, s)$ is Lipschitz continuous, uniformly over t
- Apply Meyer-Ito formula for convex mappings to the process $X = (AP - K + S)/B$.
- Use the fact that AP/B, as Snell envelope of upper semicontinuous process, is a continuous positive supermartingale, and
- that S_t^* is continuous apart from a countable number of points,
- to show that the local time of X in zero equals zero.

Part II

- Since $S/B + \int dD^S/B$ is a $\mathbb{Q}$-martingale, it turns out that the remaining technical point is to prove that

$$\int_0^t \mathbf{1}_{\{S_{u-} > S_u^*\}} d\left(\frac{AP_u}{B_u}\right)$$

is also a martingale under $\mathbb{Q}$.

- We do this by extending earlier defined methods (El Karoui '79, Karatzas & Shreve '88, Jacka '93) for optimal stopping problems, exploiting the fact that we may show that we should never stop in points where S is discontinuous.

Integral equation for Optimal Exercise Boundary

Inserting $s = S_t^*$ as initial condition at time t gives

Corollary

Under the assumptions of the previous theorem, the optimal exercise boundary satisfies

$$K - S_t^* = E(t, S_t^*) - B_t \, \mathbb{E}^{\mathbb{Q}}[\int_t^T \mathbf{1}_{\{S_{u-} \leq S_u^* \wedge \Delta S_u = 0\}} (d(\frac{K}{B_u}) + \frac{dD_u^S}{B_u}) \mid S_t = S_t^*].$$

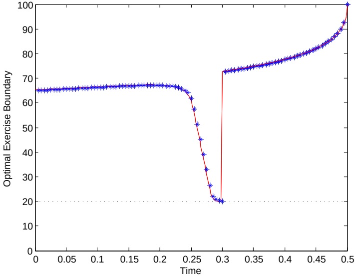
Knock Out Dividend Model

Equity model

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t - dD_t^S \\ dD_t^S &= (1-\alpha) S_{t-} \mathbf{1}_{\{\min_{u \in [0,t_d]} S_u \geq \delta S_0\}} d\mathbf{1}_{\{t \geq t_d\}}. \end{aligned}$$

Parameter values

$S_0 = K = 100,\ \sigma = 0.50,\ r = 0.125,\ T = 0.50,\ t_D = 0.30,\ \alpha = .99,\ \delta = .20.$



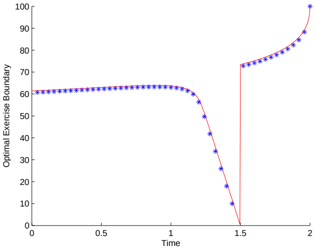
Proportional Cash Dividend Model

Equity model

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t - dD_t^S \\ dD_t^S &= (1-\alpha) S_{t-} d\mathbf{1}_{\{t \geq t_D\}}. \end{aligned}$$

Parameter values

$$S_0 = 100,\ \sigma = 0.30,\ r = 0.04,\ K = 100,\ T = 2.00,\ t_D = 1.50,\ \alpha = 0.98$$



Fixed Cash Dividend Model

Equity model

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t - dD_t^S \\
dD_t^S &= \min\{D, S_{t-}\} d\mathbf{1}_{\{t \geq t_D\}}.
\end{aligned}
$$

results in following integral equation

$$
\begin{aligned}
EEP(t,s) &= rK \int_t^{t_d} e^{-r(u-t)} N\left(\frac{\ln(S_u^*/s) - \tilde{r}(u-t)}{\sigma\sqrt{u-t}}\right) du \\
&+ rKe^{-r(t_d-t)} \int_{t_d}^{T} \int_z^{\infty} N\left(\frac{\ln(se^{\tilde{r}(t_d-t)+x\sigma\sqrt{t_d-t}} - D) - \ln S_u^* + \tilde{r}(u-t_d)}{\sigma\sqrt{u-t_d}}\right) dN(x) du
\end{aligned}
$$

where

$$
z = \frac{\ln(D/s) - \tilde{r}(t_d - t)}{\sigma\sqrt{t_d - t}}.
$$

and where N is the cumulative standard normal distribution function.

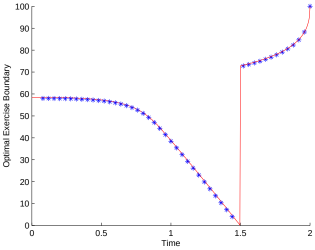
Fixed Cash Dividend Model

Equity model

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t - dD_t^S \\ dD_t^S &= \min\{d, S_{t-}\} d\mathbf{1}_{\{t \geq t_D\}}. \end{aligned}$$

Parameter values

$$S_0 = 100,\ \sigma = 0.30,\ r = 0.04,\ K = 100,\ T = 2.00,\ t_D = 1.50,\ d = 5.$$



Conclusions

- We extended the early exercise representation formula to a rather general model for stocks with 'knock-out' dividends.
- However, we need to assume that the optimal exercise boundary is continuous apart from a countable number of points in time.
- Establishing how to prove this a priori is known to be hard, and is the subject of ongoing research.

References I

P. Babilua, I. Bokuchava, B. Dochviri, and M. Shashiashvili.
The American put option in a one-dimensional diffusion model with level-dependent volatility.
Stochastics, 79:5–25, 2007.

P. Carr, R. Jarrow, and R. Myneni.
Alternative characterizations of American puts.
Mathematical Finance, 2:87–106, 1992.

N. El Karoui.
Les aspects probabilistes du contrôle stochastique, Lecture Notes in Mathematics 876.
Springer-Verlag, Berlin, 1981.

S.D. Jacka.
Local times, optimal stopping and semimartingales.
Annals of Probability, 21:329–339, 1993.

References II

S. Kallast and A. Kivinukk.
Pricing and hedging American options using approximations by Kim integral equations.
European Finance Review, 7:361–383, 2003.

I.J. Kim.
The analytical valuation of American options.
Review of Financial Studies, 3:547–472, 1990.

G. Peskir.
On the American option problem.
Mathematical Finance, 15:169–181, 2005.