The newsvendor problem

Michel De Lara

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1 The newsvendor problem (integer formulation)

- Each morning, the newsvendor must decide how many copies $u \in \{1, 2, \dots, u^\sharp\}$ of the day's paper to order. The variable u is called *control*.
- During the day, the newsvendor will meet an unknown demand $w \in \{1, 2, \dots, w^\sharp\}$. The variable w is called *uncertainty*.
- The newsvendor faces an economic tradeoff:
  - he pays the unitary purchasing cost c per copy, when he orders stock;
  - he sells a copy at price p;
  - if he remains with an unsold copy, it is worthless (perishable good).
- Therefore, the newsvendor's costs are (where $w \in \{1, 2, \dots, w^\sharp\}$ is a possible value of the demand)

$$j(u, w) = c \underbrace{u}_{\substack{\text{quantity}\\\text{purchased}}} - p \underbrace{\min\{u, w\}}_{\text{quantity sold}} \ . \tag{1}$$

  The newsvendor's *payoff* is $-j(u, w)$.

Now, we introduce a *random variable* $\mathbf{W}$, where $\mathbf{W} : \Omega \to \{1, 2, \dots, w^\sharp\}$. Here, Ω is an underlying probability space, equipped with a probability $\mathbb{P}$. We suppose that the newsvendor knows the probability distribution $\mathbb{P}_{\mathbf{W}}$ of the demand $\mathbf{W}$.

Thus equipped, we consider the stochastic optimization problem of expected costs minimization:

$$\min_{u \in \{1,2,\dots,u^\sharp\}} J(u) = \mathbb{E}_{\mathbb{P}}[j(u, \mathbf{W})] \ . \tag{2}$$

```
// demand
wsharp=100;// no larger, else the Poisson distribution cannot be computed
wflat=1;
demand=[wflat:wsharp];

// control
control=[demand,1+demand($)];

// Criterion / costs
cc=1;pp=10*cc;
// cc=1 ; pp=1.1*cc ;
// avoid that cc/pp is the inverse of an integer
// when the distribution of demand is uniform
costs=cc*control'*ones(demand)-pp*mini(ones(control')*demand,control'*ones(demand));
// one row by control, one column by demand
```

We will consider different demand distributions.

1.1 The demand distribution is uniform

First, we suppose that the demand distribution $\mathbb{P}_{\mathbf{W}}$ is uniform as follows.

```
probab=ones(demand);
probab=probab/sum(probab);
```

Question 1

a) ***[1]*** *Draw a histogram of the random demand* $\mathbf{W}$.

b) ***[1+1]*** *In the scilab code above, what does the matrix* `costs` *represent? (What do you find at the intersection of a row and of a column?) Explain in detail why we have that* `criterion = probab*costs'` *is a row vector made of the values of* $J(u) = \mathbb{E}_{\mathbb{P}}[j(u, \mathbf{W})]$ *for* $u \in \{1, 2, \dots, u^\sharp\}$*?*

c) ***[1+1]*** *Draw the mapping* $u \in \{1, 2, \dots, u^\sharp\} \mapsto J(u)$*. Thanks to the scicoslab macro* `mini` *(that provides the minimum and the argmin index of a vector), give the numerical value of the decision* $u^\star$ *(optimal order) that minimizes* $u \mapsto J(u)$*.*

d) ***[1+1]*** *What does the vector* `decumprobab=1-cumsum(probab)` *represent? Explain your answer. Check that, in agreement with the theory, we numerically have that*

$$\mathbb{P}(W > u^\star - 1) \geq \frac{c}{p} \geq \mathbb{P}(W > u^\star) \ .$$

e) ***[2+1]*** *For a given value of u, explain why the random variable $j(u, \mathbf{W})$ can at most take the values $\{j(u,1), \ldots, j(u,u-1), j(u,u)\}$. Give, for each of the u elements of this list, the corresponding probability that $j(u, \mathbf{W})$ takes this value in the list. In the end, you will provide an expression of the probability distribution of $j(u, \mathbf{W})$, using* `probab` *and* `decumprobab`*.*

f) ***[1+2]*** *Draw histograms of the probability distribution of the random payoff (the opposite of the costs) $-j(u, \mathbf{W})$ for $u = u^\star$ (the optimal decision) and for $u = \mathbb{E}_{\mathbb{P}}[\mathbf{W}]$ (the naive deterministic solution consisting in ordering the mean demand $\mathbb{E}_{\mathbb{P}}[\mathbf{W}]$). Draw the two histograms on the same picture, so that they have the same scale. Comment on the differences between the two histograms.*

g) ***[1+1+1+2]*** *The vector* `grand(365,"markov",ones(probab')*probab,1)` *represents a sequence of realizations of 365 i.i.d. random variables having the same distribution than the demand $\mathbf{W}$. Simulate and draw the trajectory of the cumulated payoffs of the newsvendor during one year if, every day, he orders the optimal quantity $u = u^\star$. Do the same for $u = \mathbb{E}_{\mathbb{P}}[\mathbf{W}]$ and draw it on the same picture. In what sense does the the optimal decision $u = u^\star$ does better than $u = \mathbb{E}_{\mathbb{P}}[\mathbf{W}]$? Justify in detail why the two trajectories are approximately straight lines; to what correspond the slopes?*

h) *Now, we study if the results are robusts to changes in the the ratio between the unitary purchasing cost c and the selling price p.*

   1) ***[2]*** *Take $c < p$ with $c \approx p$. Find the optimal decision $u^\star$. Draw histograms of the probability distribution of the random payoff $-j(u, \mathbf{W})$ for $u = u^\star$ and for $u = \mathbb{E}_{\mathbb{P}}[\mathbf{W}]$. Simulate and draw trajectories of the corresponding cumulated payoffs.*

   2) ***[2]*** *Same question with $c << p$.*

   3) ***[2]*** *Discuss. In particular, how does the optimal solution $u^\star$ vary with the ratio c/p?*

```
//exec('newsvendor_data.sce');exec('newsvendor_uniform.sce');exec('newsvendor_main.sce

xset("window",1);clf();plot2d2(demand,probab)
xtitle("Histogram of the demand")

// Criterion / expected costs
criterion=probab*costs';
```

```
// a row vector
xset("window",3);clf();plot2d2(control,criterion)
xtitle("The expected costs as function of the decision")

// Optimal decision
[lhs,optcont]=mini(criterion);
disp("The optimal decision is "+string(optcont))

// Naive deterministic solution
meandemand=probab*demand';
disp("The expected demand is "+string(round(meandemand)))

// Check that the optimal decision satisfies the optimality condition
cumprobab=cumsum(probab);
decumprobab=1-cumprobab;

xset("window",4);clf();plot2d2(demand,decumprobab,rect = [demand(1)-1,0,demand($),1]);
plot2d(demand,cc/pp*ones(decumprobab),style = 5);
xtitle("The decumulative distribution of the demand")

disp("Is it true that "+string(cc/pp)+" lies between "+string(decumprobab(optcont))+ ...
     " and "+string(decumprobab(optcont-1))+"?")

NS=365;
// simulated demands
DD=grand(NS,'markov',ones(probab')*probab,1);

time=[1:NS];

xset("window",8);clf();
plot2d2(time,cumsum(-costs(round(meandemand),DD)),style = 3);
plot2d2(time,cumsum(-costs(optcont,DD)),style = 5);
legends(["optimal solution "+string(optcont);
         "naive deterministic solution "+string(round(meandemand))],[5,3],"lr");
xtitle("The cumulated payoffs as function of the number of days","time","payoff")


xset("window",10);clf();
uu=optcont;
ss=5;
plot2d2([costs(uu,[wflat:(uu-1)]),costs(uu,uu)], ...
       [probab([wflat:(uu-1)]),decumprobab(uu-1)],style = ss, ...
```

```
        rect = [mini(costs),0,maxi(costs),1]);
//
uu=round(meandemand);
ss=3;
plot2d2([costs(uu,[wflat:(uu-1)]),costs(uu,uu)], ...
        [probab([wflat:(uu-1)]),decumprobab(uu-1)],style = ss, ...
        rect = [mini(costs),0,maxi(costs),1]);
//
legends(["optimal solution "+string(optcont);
         "naive deterministic solution "+string(round(meandemand))],[5,3],"ur");
xtitle("Histograms of the costs")
```

1.2 The demand distribution is a mixture of two truncated Poisson distributions

Second, we suppose that the demand distribution $\mathbb{P}_{\mathbf{W}}$ is a mixture of two truncated Poisson distributions as follows:

$$\mathbb{P}_{\mathbf{W}}(w) = \mathbb{P}(\mathbf{W} = w) = \frac{1}{2} \times k^{\flat} \frac{(\lambda^{\flat})^w}{w!} + \frac{1}{2} \times k^{\sharp} \frac{(\lambda^{\sharp})^w}{w!} \ , \ \ \forall w \in \{1, 2, \dots, w^{\sharp}\} \ . \tag{3}$$

Question 2 *[6] Same questions as in Question* **??** *when the demand* $\mathbf{W}$ *follows a mixture of two truncated Poisson distributions.*

```
// Poisson
lambdaflat=wsharp/4;lambdasharp=3*wsharp/4;
//
probabflat=(lambdaflat .^demand) ./(factorial(demand));
probabflat=probabflat/sum(probabflat);
//
probabsharp=(lambdasharp .^demand) ./(factorial(demand));
probabsharp=probabsharp/sum(probabsharp);
//
probab=0.5*probabflat+0.5*probabsharp;
```

1.3 The demand distribution is triangular

Question 3 *[6] Same questions as in Question 1 when the demand* $\mathbf{W}$ *follows a triangular distribution over* $\{1, 2, \dots, w^{\sharp}\}$.

```
// Triangular distribution
lambda=floor(wflat+0.3*(wsharp-wflat));
//
probab=[cumsum(ones([wflat:lambda]))/sum(ones([wflat:lambda])), ...
        1-cumsum(ones([(lambda+1):wsharp]))/sum(ones([(lambda+1):wsharp]))];
probab=probab/sum(probab);
```

2 The newsvendor problem (continuous formulation)

Here, we consider that the decision can take continuous values: $u \in [0, u^\sharp]$.

We also adopt new notations. We suppose that the demand $\mathbf{W}$ can take a finite number S of possible values D_s, where s denotes a *scenario* in the finite set $\mathbb{S}$ (S=card($\mathbb{S}$)). We denote π_s the probability of scenario $s \in \mathbb{S}$, with

$$\sum_{s \in \mathbb{S}} \pi_s = 1 \quad \text{and} \quad \pi_s > 0 \ , \ \ \forall s \in \mathbb{S} \ . \tag{4}$$

Notice that we do not consider scenarios with zero probability.

We consider the stochastic optimization problem

$$\min_{u \in [0, u^\sharp]} J(u) = \mathbb{E}_{\mathbb{P}}[j(u, \mathbf{W})] \ . \tag{5}$$

We now show how to rewrite this problem as a linear program. First, we write:

$$j(u, w) = cu - p \min\{u, w\} \tag{6a}$$

$$= \max\{cu - pu, cu - pw\} \tag{6b}$$

$$= \min_{v \in \mathbb{R}} \{v \mid v \geq cu - pu \ , \ \ v \geq cu - pw\} \ . \tag{6c}$$

Second, we deduce

$$J(u) = \mathbb{E}_{\mathbb{P}}[j(u, \mathbf{W})] \tag{7a}$$

$$= \sum_{s \in \mathbb{S}} \pi_s j(u, D_s) \tag{7b}$$

$$= \sum_{s \in \mathbb{S}} \pi_s \min_{v_s \in \mathbb{R}} \{v_s \mid v_s \geq cu - pu \ , \ \ v_s \geq cu - pD_s\} \tag{7c}$$

$$= \min_{(v_s)_{s \in \mathbb{S}} \in \mathbb{R}^S} \sum_{s \in \mathbb{S}} \pi_s v_s \tag{7d}$$

$$\text{under the constraints} \quad \forall s \in \mathbb{S} \ , \ \ v_s \geq cu - pu \ , \ \ v_s \geq cu - pD_s \ . \tag{7e}$$

Third, we conclude

$$\min_{u \in [0, u^\sharp]} J(u) = \min_{u \in [0, u^\sharp], (v_s)_{s \in \mathbb{S}} \in \mathbb{R}^S} \sum_{s \in \mathbb{S}} \pi_s v_s \tag{8a}$$

$$\text{under the constraints} \quad \forall s \in \mathbb{S} \ , \ \ v_s \geq cu - pu \ , \ \ v_s \geq cu - pD_s \ . \tag{8b}$$

This is a linear program.

Question 4 *Suppose that* $\mathbf{W}$ *follows a uniform distribution over* $\{1, 2, \ldots, w^\sharp\}$.

a) *[2] Write the linear program (8) under the form adapted to the scicoslab macro* `linpro`.

b) ***[1]*** *Solve (8) with* ***`linpro`*** *and obtain the solution to the stochastic optimization problem (5).*

c) ***[1]*** *Compare with the optimal solution of Question 1.*

d) ***[2]*** *Increase the number $w^\sharp$ of values taken by the demand $\mathbf{W}$. When can you no longer solve numerically? Compare with the result of Question 1.*

3 The newsvendor problem with risk (continuous formulation)

Let $\lambda \in]0,1[$, that plays the role of a risk level. The Value at Risk of the cost $\mathbf{X}$ at level $\lambda \in]0,1[$ is

$$VaR_\lambda(\mathbf{X}) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(\mathbf{X} > x) < \lambda\}$$

The Tail Value at Risk of the cost $\mathbf{X}$ at level $\lambda \in]0,1[$ is

$$TVaR_\lambda(\mathbf{X}) = \frac{1}{1-\lambda}\int_\lambda^1 VaR_{\lambda'}(\mathbf{X})d\lambda'$$

We have the limit cases:

$$\begin{aligned} TVaR_0[\mathbf{X}] &= \mathbb{E}[\mathbf{X}] \\ TVaR_1[\mathbf{X}] &= \lim_{\lambda\to 1} TVaR_\lambda[\mathbf{X}] = \sup_{\omega\in\Omega} \mathbf{X}(\omega) \end{aligned}$$

The Rockafellar-Uryasev formula establishes that

$$TVaR_\lambda[\mathbf{X}] = \inf_{s\in\mathbb{R}} \left\{ \frac{\mathbb{E}[(\mathbf{X}-s)^+]}{1-\lambda} + s \right\} \ , \ \ \lambda \in [0,1[$$

We consider the risk averse stochastic optimization problem

$$\min_{u\in[0,u^\sharp]} J(u) = TVaR_\lambda[j(u,\mathbf{W})] \ . \tag{9}$$

We rewrite this problem as a linear program.

$$\min_{u\in[0,u^\sharp]} J(u) = \min_{u\in[0,u^\sharp], r\in\mathbb{R}, (v_s)_{s\in\mathbb{S}}\in\mathbb{R}^S} r + \frac{1}{1-\lambda}\sum_{s\in\mathbb{S}} \pi_s v_s \tag{10a}$$

$$\text{under the constraints} \quad \forall s \in \mathbb{S} \ , \ \ v_s \geq cu - pu \ , \ \ v_s \geq cu - pD_s \ . \tag{10b}$$

Question 5 *Suppose that $\mathbf{W}$ follows a uniform distribution over $\{1,2,\ldots,w^\sharp\}$.*

a) ***[2]*** *Write the linear program (10) under the form adapted to the scicoslab macro* ***`linpro`****.*

b) ***[1]*** *Solve (10) with* ***`linpro`*** *and obtain the solution to the stochastic optimization problem (9).*

c) ***[1]*** *Compare with the optimal solution of Question 4.*

4 The newsvendor problem with backorder (continuous formulation)

Now, we suppose that the newsvendor

- pays the unitary purchasing cost c per copy, when he orders stock;
- sells a copy at price p; we have that $c > p$;
- can buy extra copies at the unitary backorder cost b, after he observes a demand w larger than the initial order u; we have that $b > c$
- pays the unitary holding cost h for each unsold copy, when the demand w is less than the initial order u.

Therefore, the newsvendor's costs are

$$j(u,w) = c \underbrace{u}_{\text{order}} + b \underbrace{[w-u]_+}_{\text{backorder}} + h \underbrace{[u-w]_+}_{\text{holding}} - p \underbrace{w}_{\text{sold}} , \tag{11}$$

where we recall that $x_+ = \max\{x, 0\}$.

```
cc=0.1;pp=10*cc;bb=10*cc;hh=10*cc;
```

Question 6 *We suppose that the demand* $\mathbf{W}$ *follows a uniform distribution over* $\{1, 2, \ldots, w^\sharp\}$.

a) ***[1]*** *Write a row vector* ***`criterion`*** *made of the values of* $J(u) = \mathbb{E}_{\mathbb{P}}[j(u, \mathbf{W})]$ *for* $u \in \{1, 2, \ldots, u^\sharp\}$.

b) ***[1+1]*** *Draw the mapping* $u \in \{1, 2, \ldots, u^\sharp\} \mapsto J(u)$*. Thanks to the scicoslab macro* ***`mini`*** *(that provides the minimum and the argmin index of a vector), give the numerical value of the decision* $u^\star$ *(optimal order) that minimizes* $u \mapsto J(u)$.

c) ***[1]*** *Check that, in agreement with the theory, we numerically have that*

$$\mathbb{P}(\mathbf{W} > u^\star - 1) \geq \frac{c+h}{b+h} \geq \mathbb{P}(\mathbf{W} > u^\star) .$$

d) ***[1+1]*** *For a given value of* u*, what is the set of values taken by the random variable* $j(u, \mathbf{W})$*? Give a scicoslab formula for the probability distribution of* $j(u, \mathbf{W})$*, using* `probab` *and* `decumprobab`*.*

e) ***[1+1]*** *Draw a histogram of the probability distribution of the random payoff (the opposite of the costs)* $-j(u, \mathbf{W})$ *for* $u = u^\star$ *(the optimal decision) and for* $u = \mathbb{E}_\mathbb{P}[\mathbf{W}]$ *(the naive deterministic solution consisting in ordering the mean demand* $\mathbb{E}_\mathbb{P}[\mathbf{W}]$*). Draw the two histograms on the same picture, so that they have the same scale. Comment on the differences between the two histograms.*

f) ***[1+1+1]*** *Simulate the trajectory of the cumulated payoffs of the newsvendor during one year if, every day, he orders the optimal quantity* $u = u^\star$*. Do the same for* $u = \mathbb{E}_\mathbb{P}[\mathbf{W}]$*. Compare.*

g) *[4]* *Multiply the unitary holding cost* h *by a factor 5. Comment on the changes that you observe.*

h) *[4]* *As in Section 2, write the new optimization problem as a linear program. Then, write this latter under the form adapted to the scicoslab macro* `linpro`*. Solve with* `linpro` *and compare with the optimal solution found above by the direct method.*

i) *[4]* *Increase the number* $w^\sharp$ *of values taken by the demand* $\mathbf{W}$*. When can you no longer solve numerically? Compare with the direct method.*