SPDEs and Level Sets

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Plan

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   1. Overview
   2. Main advantages
   3. Applications to Computer Vision

2. Adding Stochastic Perturbations to LSM-based Shape Optimization Algorithms
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Level Sets Method

$\Gamma(t) \subset \mathbb{R}^n$ interface

$\Omega(t) \subset \mathbb{R}^n$ such that $\Gamma(t) = \partial \Omega(t)$

Consider $u : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$\Gamma(t) = \{x \in \mathbb{R}^n : u(t, x) = 0\}$

$u(t, x) < 0 \quad \forall x \in \Omega(t)$

$u(t, x) > 0 \quad \forall x \notin \overline{\Omega(t)}$

$\frac{\partial \Gamma}{\partial t} = \beta n \quad \Leftrightarrow \quad du = \beta |Du| \quad \beta$ intrinsic
Example of Level Sets Evolution

\[ \frac{\partial \Gamma}{\partial t} = kn \quad du = |Du| \text{div} \left( \frac{Du}{|Du|} \right) \]
Main advantages

- Handles automatically topological changes
- Robust mathematical theory behind – viscosity solutions
- Stable numerical schemes
- Natural extensions to higher dimensions
Applications to Computer Vision

• Mean Curvature Motion
• Shape Optimisation
  – Active Contours
  – Active Regions
  – Adaptative Active Regions
Mean Curvature Motion

Isotropic smoothing of a curve in the Euclidian plane

Interface Evolution

\[ \frac{\partial \Gamma}{\partial t} = \frac{\partial^2 \Gamma}{\partial v^2} n = k n \]

Corresponding implicit function evolution

\[ du = |Du| \text{div} \left( \frac{Du}{|Du|} \right) \]
Active Contours

Energy to minimize
\[
\int_{\Gamma(t)} g(|D I(p)|) \, dp
\]

Euler-Lagrange equation
\[
\frac{\partial \Gamma}{\partial t} = g(I) k n - (D g \cdot n) n
\]

Corresponding evolution of the implicit function
\[
du = g(I) |Du| k + Dg(I) \cdot Du
\]
Active Regions

Paragios Deriche 1999

Feature extraction step – supervised

Energy to minimize

\[
(1 - \alpha) \int_{\Gamma(t)} g(p(I(m))) \, dm \\
+ \alpha \left[ \int_{\Omega(t)} \log(p_A(I(m))) \, dm \\
+ \int_{\Omega^C(t)} \log(p_B(I(m))) \, dm \right]
\]
Unsupervised Active Regions

Rousson Brox Deriche 2003

Segment an image in 2 regions, called generically the interior and the exterior, based on a single Gaussian distribution assumption both of the inside and the outside.

\[
E(\Gamma, \mu_1, \Sigma_1, \mu_2, \Sigma_2) = \int_{\Gamma} e_1(x) + \int_{D \setminus \Gamma} e_2(x) + \nu \text{length}(\partial \Gamma)
\]

\[
e_i(x) = -\log p_{\mu_i, \Sigma_i}(I(x))
\]

\[
p_{\mu_i, \Sigma_i} = C|\Sigma_i|^{-\frac{1}{2}} e^{-(I(x) - \mu_i)^T \Sigma_i^{-1} (I(x) - \mu_i) / 2}
\]
Evolution sometimes gets stuck in local minima
Adding Stochastic Perturbations to Shape Optimization Algorithms

- SAC (Stochastic Active Contours)
  - Single Gaussian Model
  - Gaussian Mixtures
- Mathematical Elements
  - General Theory
  - Noise
  - Implementation
Shape Optimization problems through Simulated Annealing

Stochastic Active Contours (SAC)

Drawbacks of classical Active Contours/Regions methods
- Sometimes get stuck in local minima;
- Euler-Lagrance equations do not always provide explicit gradients.
Single Gaussian Model SAC

Adaptative Segmentation [Rousson Deriche 2002] +
Simulated Annealing through
Stochastic Mean Curvature Motion (SMCM)

Segment an image in 2 regions, called generically the interior and the exterior, based on a single Gaussian distribution assumption both of the inside and the outside.

\begin{align*}
\text{Interior} & \quad \mu_1, \Sigma_1 \quad \Gamma \\
\text{Exterior} & \quad \mu_2, \Sigma_2 \quad D \setminus \Gamma \\
E(\Gamma, \mu_1, \Sigma_1, \mu_2, \Sigma_2) &= \int_{\Gamma} e_1(x) + \int_{D \setminus \Gamma} e_2(x) + \nu \text{ length}(\partial \Gamma) \\
e_i(x) &= -\log p_{\mu_i, \Sigma_i}(I(x)) \\
p_{\mu_i, \Sigma_i} &= C|\Sigma_i|^{-\frac{1}{2}}e^{-(I(x)-\mu_i)^T\Sigma_i^{-1}(I(x)-\mu_i)/2}
\end{align*}
Euler-Lagrange simplifies to [Rousson Deriche]

\[ du = \left( e_2(x) - e_1(x) + \nu \text{div} \left( \frac{Du}{|Du|} \right) \right) |Du|dt + \text{noise} \]

Standard approach sometimes gets stuck in local minima, while SMCM does not!

**Empirical evidence shows that SMCM is more robust wrt to interface initialization**
Test Image:

2 regions modeled by 2 unknown Gaussian distributions with

• Same mean
• Different variances

Test Image with Initial Contour
Standard algorithm  SAC
Deterministic Contour Evolution

SAC Evolution
Gaussian Mixtures SAC

• Extend the previous algorithm for the case when region statistics are modeled by a mixture of Gaussian distributions with parameters

\[ \Theta_i = \left( \pi_i^1, \mu_i^1, \Sigma_i^1 \ldots \pi_i^{n_i}, \mu_i^{n_i}, \Sigma_i^{n_i} \right) \]

\[ \sum_j \pi_i^j = 1 \]

• The model dynamically calculates the optimal number of Gaussian distributions and then tries to fit the weights of those distributions using some algorithm (e.g. k-means).

• In this case, the k-means algorithm acts like a black box, due to the complex dependency \( \Gamma \rightarrow \Theta_i(\Gamma) \)

• Cannot obtain an explicit form of the EL equation, but only the derivative of the energy wrt the shape at constant parameters.
Gaussian Mixtures – SAC

Deterministic Evolution with Approximated Gradient

SAC Evolution with Approximated Gradient
Why do we need maths now that we have results?

- Well posedness …
- Geometric properties of stochastic evol.
Mathematical Theory

- Stochastic Mean Curvature Motion
- Viscosity Solutions for SPDEs
- Numerical Scheme used (Ito and Stratonovitch)
- Geometric properties
- Open Questions
**Stochastic Mean Curvature Motion - SMCM**

Notation

- **Domain** \( \Omega \in \mathbb{R}^2 \)
- **Curve** \( \Gamma = \partial \Omega \)

**Stochastic Mean Curvature Motion**

\[
\frac{\partial \Gamma}{\partial t} = \kappa n + W(dt, x)n
\]

**White Noise**

\( W(t, x) \)
Intrinsic property

\[
\frac{\partial \Gamma}{\partial t} = (\kappa + W(dt, x)) n
\]

\[
du = |Du| \text{div} \left( \frac{Du}{|Du|} \right) dt + |Du|W(dt, x)
\]

The curve evolution should be invariant wrt the choice of the implicit function.

Simplified equation

\[
du = |Du|dW(t) \quad \text{(EQ)}
\]
\[ du = |Du| dW(t) \quad (EQ) \]

\[ \alpha : \mathbb{R} \to \mathbb{R} \text{ smooth strictly increasing function} \]

If \( u \) is solution of \((EQ)\), then \( \alpha(u) \) should be a solution of the same equation \((EQ)\)

\[ d \left[ \alpha(u) \right] = \alpha'(u) |Du| dW + \frac{1}{2} \alpha''(u) |Du|^2 dt \]

\[ = |D[\alpha(u)]| dW + \frac{1}{2} \alpha''(u) |Du|^2 dt \text{ Not intrinsic!} \]

The Itô form of the level sets SPDE is not intrinsic!

• Level Sets (Stratonovich) \( du = |Du| \circ dW(t) \)

\( \alpha \) - same as before

\[ d[\alpha(u)] = \alpha'(u) \circ du = \alpha'(u) |Du| \circ dW(t) \]

\[ = |D[\alpha(u)]| \circ dW(t) \]

The Stratonovich form of the SPDE satisfies the intrinsic property!
Well Posedness for Space-Independent Stochastic Hamiltonians

- Based on a series of articles of P.L. Lions and Souganidis

\[ du = F(D^2 u, Du, x, t) dt + \sum_i H_i(Du) \circ dW_i(t) \quad (SPDE) \]

**Theorem** The equation (SPDE) admits an a.s. unique stochastic viscosity solution.

\[ u^\epsilon_t = F(D^2 u^\epsilon, Du^\epsilon) + \sum_i H_i(Du)^\epsilon \xi^\epsilon_i(t) \]

\[ \xi^\epsilon \to W \text{ uniformly on } (0, T) \text{ and a.s.} \]

**Theorem** The solutions of the approximated PDE converge a.s. locally uniformly on \( \mathbb{R}^n \times [0, T] \) to the solution of (SPDE).
Noise \( W(t, x) \)

- Theoretical difficulties when working with white noise in space.
- Colored Noise in space: distribute noise on a discrete grid \( x_i \) at each moment in time

\[
W(t, x) = \sum_{i=1}^{m} \phi_i(x)W_i(t)
\]

Noise – Scale defined by the distance between the \( x_i \)’s
Implementation

- Explicit scheme for the Ito evolution

\[ u(t + \Delta t) = u(t) + |Du|(t) \text{ div} \left( \frac{Du}{|Du|} \right)(t) \Delta t + |Du| \mathcal{N}(0, 1) \sqrt{\Delta t} \]

- Narrow Band method
- The theory applies without problems in 3D
Implementation Details

Stratonovitch Drift

\[ du = F(D^2u, Du)dt + H(x, Du) \circ dW(t) \]

\[ d\langle H(x, Du), W \rangle_t = \left[ \left(D^2u \frac{\partial H}{\partial p}\right) \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial x} \right] dt \]

Adding the above drift to the scheme before yields convergence towards the Stratonovitch equation
Geometric Properties

Page under construction!...
Open Questions

• Do not have a theorem on the time-convergence of the scheme (Ito or Stratonovitch) when the stochastic Hamiltonian depends on x

• Presence of artifacts in the evolution due to the presence of noise (when not colored enough)? (implementation dependent)
Example of artifacts
Artifacts : implementation details

- Narrow Band Method
- Implicit function re-initialization
- Distance-function preserving schemes