American Option Pricing: a Variance Reduction Technique

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PLAN OF THE TALK

1. The price of an American option: Dynamic Programming and the Longstaff-Schwartz algorithm
2. Reducing Variance by changing the drift
3. Numerical results
PRICE OF AN AMERICAN OPTION

Two equivalent formulations

<table>
<thead>
<tr>
<th>Optimal Stopping Time Problem</th>
<th>Dynamic Programming Problem</th>
</tr>
</thead>
</table>

- compact and “clean” mathematical formulation
- easy to handle when studying theory (*)
- untreated from a practical point of view: “impossible” to have a straightforward numerical implementation.

\[ U_0 = \mathbb{E} f(\tau_0, X_{\tau_0}) \]

- no closed formula
- may reveal “uncomfortable” for explicit calculus
- easy numerical implementation (*)

\[
\begin{align*}
    U_M &= f(T, X_T) \\
    U_i &= \max\{f(t_i, X_{t_i}), \mathbb{E}(U_{i+1}|X_{t_i})\}
\end{align*}
\]
THE LONGSTAFF-SCHWARTZ ALGORITHM ('01)

\[ E(U_{j+1} | X_{t_j}) = \phi_j(X_{t_j}) \]

(Markov Property)

\[ \phi_j(\cdot) \approx \alpha_j \cdot e(\cdot) \]

\[ \alpha_j = \arg \min_{\alpha \in \mathbb{R}^m} E \left[ U_{j+1} - \alpha \cdot e(X_{t_j}) \right]^2 \]

Monte Carlo

\[ \alpha_i^N = \arg \min_{\alpha \in \mathbb{R}^m} \sum_{n=1}^{N} \frac{1}{N} \left[ U_{i+1}^{(n)} - \alpha \cdot e(X_{t_i}^{(n)}) \right]^2 \]

\[ \phi_j(x) = E(U_{j+1} | X_j = x) \]
THE LONGSTAFF-SCHWARTZ ALGORITHM ('01)

\[
\begin{aligned}
\begin{cases}
U_M &= f(T, X_T) \\
U_i &= \max\{f(t_i, X_{t_i}), \mathbb{E}(U_{i+1}|X_{t_i})\}
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
U_M^m &= f(T, X_T) \\
U_i^m &= \max\{f(t_i, X_{t_i}), \alpha_i \cdot e(X_{t_i})\}
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
U_M^{(m,N,n)} &= f(T, X_T^{(n)}) \\
U_i^{(m,N,n)} &= \max\{f(t_i, X_{t_i}^{(n)}), \alpha_i^N \cdot e(X_{t_i}^{(n)})\}
\end{cases}
\end{aligned}
\]

Convergence $L^2$

T.C.L.
$N \rightarrow +\infty$
REDUCING VARIANCE BY CHANGING THE DRIFT

Original Model

\[ \frac{dX_t}{X_t} = \mu dt + \sigma dW_t \]

Drifted model

\[ \frac{dX_t^\theta}{X_t^\theta} = \mu dt + \sigma \theta dt + \sigma dW_t \]
REDDUCING VARIANCE BY CHANGING THE DRIFT

Thanks to Girsanov’s Theorem, we have, $\forall \theta \in \mathbb{R}^D$

$$\mathbb{E} f(\tau_0, X_{\tau_0}) = \mathbb{E} e^{-\frac{1}{2}||\theta||^2_{\tau_0} - \theta \cdot W_{\tau_0}} f(\tau_0, X_{\tau_0}^\theta) \overset{\text{def}}{=} \mathbb{E} f^\theta(\tau_0, X_{\tau_0}^\theta)$$

that is, we dispose of a set of equivalent pricing problems.

We want to find the optimal drift i.e. the drift which speeds up the convergence of the L.S. algorithm

Central Limit Theorem

$$U_0(m,N) = \max\{f(t_0, X_{t_0}), \sum_{n=1}^{N} U_1^{(m,N,n,\theta)}/N\}$$

Remarks: - $\tau_0$ is a r.v. This is more than a change in integration measure!
- $\tau_0$ does NOT depend on $\theta!$
NUMERICAL RESULTS
Put (Basket) American Option, 1D

Ratios \( \frac{\text{Price}(\theta)}{\text{Price}(0)} \) and \( \sqrt{\frac{\text{Var}(\theta)}{\text{Var}(0)}} \)

- Variance is reduced up to 2.3% of its initial value!
- Prices are in very good accord with the Benchmark (Premia tree)
NUMERICAL RESULTS

Put Basket American Option, 2D, uncorrelated symmetric asset

Very good accord with reference price!
NUMERICAL RESULTS

Put Basket American Option, 2D, uncorrelated symmetric asset

Ratios $\sqrt{\text{Var}(\theta)/\text{Var}(0)}$

Min=0.15 (Min Var < 2.3%)
NUMERICAL RESULTS

Put Basket American Option, 5D, uncorrelated asset

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>Price($\theta$)/Price(0)</th>
<th>$\sqrt{\text{Var}(\theta)/\text{Var}(0)}$.</th>
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</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2</td>
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<td>0.897768</td>
<td>1.08737</td>
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<tr>
<td>-1</td>
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<td>0.443951</td>
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<td>-1</td>
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<td>-0.5</td>
<td>1.01677</td>
<td>0.345993</td>
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<tr>
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<td>1.0</td>
<td>-1.0</td>
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<td>-1.5</td>
<td>0.956187</td>
<td>0.337243</td>
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<tr>
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<tr>
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<td>-2</td>
<td>-0.5</td>
<td>-2</td>
<td>0.920403</td>
<td>0.626063</td>
</tr>
</tbody>
</table>
NUMERICAL RESULTS

Put Basket American Option, 2D, correlated asset

Ratio $\sqrt{\text{Var}(\theta)}/\text{Var}(0)$

Min 0.2 (MinVar 4%)
NUMERICAL RESULTS

**Put Min** American Option, 2D, correlated asset

Ratio $\sqrt{\text{Var}(\theta)/\text{Var}(0)}$

Min 0.37 (MinVar 14%)
NUMERICAL RESULTS

Best of Call American Option, 2D, correlated asset

\[ \text{Ratio } \sqrt{\text{Var}(\theta)/\text{Var}(0)} \]

Min 0.11 (Min Var 1.2%)
HOW COULD WE ESTIMATE THE MINIMUM?

Let us compare our problem to the corresponding European one:

• are the minima “quite near”?

• what about setting $\theta_{\text{USA}}^{\text{min}} \approx \theta_{\text{EU}}^{\text{min}}$

• whenever worthwhile, could we estimate $\theta_{\text{EU}}^{\text{min}}$
HOW COULD WE ESTIMATE THE MINIMUM?

<table>
<thead>
<tr>
<th>Model</th>
<th>$\sqrt{\text{Var}(\theta_{\text{min}}^{\text{US}})/\text{Var}(0)}$</th>
<th>$\sqrt{\text{Var}(\theta_{\text{min}}^{\text{EU}})/\text{Var}(0)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PB 1D, $K = 80$, $X_0 = 100$</td>
<td>0.14</td>
<td>0.15</td>
</tr>
<tr>
<td>PB 2D, $K = 90$, $X_0 = (100, 100)$</td>
<td>0.15</td>
<td>0.18</td>
</tr>
<tr>
<td>PB 2D, $K = 90$, $X_0 = (105, 70)$</td>
<td>0.23</td>
<td>0.37</td>
</tr>
<tr>
<td>PB 2D, $K = 90$, $X_0 = (100, 100), \rho = 0.8$</td>
<td>0.20</td>
<td>0.22</td>
</tr>
<tr>
<td>BC 2D, $K = 90$, $X_0 = (110, 90), \rho = 0.7$</td>
<td>0.11</td>
<td>0.30</td>
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<tr>
<td>PM 2D, $K = 90$, $X_0 = (100, 100), \rho = 0.6$</td>
<td>0.37</td>
<td>0.43</td>
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<tr>
<td>PB 3D, $K = 95$, $X_0 = (100, 100, 100)$</td>
<td>0.20</td>
<td>0.31</td>
</tr>
<tr>
<td>PB 5D, $K = 100, X_0 = (100, 100, 100, 100, 100)$</td>
<td>0.21</td>
<td>0.31</td>
</tr>
</tbody>
</table>

1. $\theta_{\text{min}}^{\text{EU}}$ gives good *sub-optimal* results

2. $\theta_{\text{min}}^{\text{EU}}$ can be estimated by fast and precise stochastic algorithms as the Robbins-Monro ones
CONCLUSIONS

Changing diffusion drift $\rightarrow$ Monte Carlo Variance Reduction for the L.-S. Algorithm

Main features of our method:

1. **Generality:** it works for ALL
   - $L^{2+\xi}$ payoff function
   - Diffusion Markov process $\rightarrow$ extendable to stochastic volatility models (price to pay = supplementary dimensions)

2. **Efficiency:** reduction of variance up to 2%

3. **Versatility:** we have some further *sub-optimal* approximations