The Boltzmann-Grad Limit
for the Periodic Lorentz Gas

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In 1905, Lorentz proposed to describe the motion of electrons in metals by the methods of kinetic theory

- Gas of electrons described by its phase-space density \( f \equiv f(t, x, v) \)

(At time \( t \), there are \( f(t, x, v)dxdv \) electrons in a phase-space volume \( dxdv \) centered at \( (x, v) \), \( x = \) position, \( v = \) velocity)

- Electron-electron collisions neglected (unlike in the kinetic theory of gases)

- Only the collisions between electrons and metallic atoms are considered

\[ \Rightarrow \text{LINEAR KINETIC EQUATION} \]

\( \neq \) the Boltzmann equation in the kinetic theory of gases is NONLINEAR
The Lorentz kinetic model

- Equation for the phase-space density of electrons $f \equiv f(t, x, v)$:

$$
(\partial_t + v \cdot \nabla_x + \frac{1}{m} F(t, x) \cdot \nabla_v) f(t, x, v) = N_{at} r_{at}^2 |v| C(f(t, x, \cdot))(v)
$$

where $C$ is the Lorentz collision integral

$$
C(\phi)(v) = \int_{|\omega|=1, \omega \cdot v > 0} (\phi(\mathcal{R}_\omega v) - \phi(v)) \cos(v, \omega) d\omega
$$

with $\mathcal{R}_\omega$ denoting the specular reflection: $\mathcal{R}_\omega(v) = v - 2(v \cdot \omega)\omega$

Notation: $m =$ mass of the electron;

- $F \equiv F(t, x)$ is the electric force (known);

- $N_{at}, r_{at}$ density, radius of metallic atoms.
It is a **mesoscopic** model (between **microscopic** and **macroscopic**);

- it is a *single-particle* phase-space equation; but

- a *statistical* description and **not a first principle**

**Probabilistic interpretation**

Direction of each particle jumps at *exponentially distributed times*, so that

- jump times, and jumps in direction are independent;

  \[ \Rightarrow \text{the Lorentz collision integral } C \]

- between two jumps, each particle is driven by the electric force field \( F \)

  \[ \Rightarrow \text{the streaming operator } \partial_t + v \cdot \nabla_x + \frac{1}{m} F \cdot \nabla v \]
The microscopic model (Lorentz gas)

- Periodic configuration of spherical obstacles

\[ Z_r = \{ x \in \mathbb{R}^D \mid \text{dist}(x, Z^D) > r \} , \quad Y_r = Z_r/Z^D \]
Particles move freely between the obstacles

\[ \dot{x}(t) = v(t), \quad \dot{v}(t) = 0, \quad \text{if } x(t) \in Z_r \]

and are reflected upon impinging on the surface of the obstacles

\[ v(t^+) = R_{n_x(t)} v(t^-), \quad \text{whenever } x(t) \in \partial Z_r \]

(with \( n_x \) the inward unit normal at \( x \in \partial Z_r \)).
The prescription above define a (broken) flow
\[(x, v) \mapsto (X^r_t(x, v), V^r_t(x, v))\]

Define then a phase-space density (propagated by the flow above)
\[f_\epsilon(t, x, v) \equiv f_{\text{in}}(\epsilon X^r_t(x/\epsilon, v), V^r_t(x/\epsilon, v)), \quad \text{with } r = \epsilon^{\frac{1}{D-1}}\]

**Question**

Does \(f_\epsilon \to f\), the solution of the Lorentz kinetic equation, as \(\epsilon \to 0\)?

• Proved for a Poisson distribution of obstacles (Gallavotti 1972)

• See also Spohn (CMP 1978), Boldrighini-Bunimovich-Sinai (JSP 83)
**Distribution of free path lengths**

- **Free path length** (i.e. exit time)

  \[ \tau_r(x, v) = \inf \{ t > 0 \mid x + tv \in \partial Z_r \} \]

- For \((x, v)\) uniformly distributed on \(Z_r \times S^{D-1}\)

  \[ \Phi_r(t) = \text{Prob}\{(x, v) \mid \tau_r(x, v) > t\} \]
Theorem. There exists $0 < C_- < C_+$ such that, for all $t > 1/r^{D-1}$

$$\frac{C_-}{tr^{D-1}} \leq \Phi_r(t) \leq \frac{C_+}{tr^{D-1}}$$

Upper bound + lower bound for $D = 2$: Bourgain-G-Wennberg CMP 1998


$$\Rightarrow \langle \tau_r \rangle = \int_{Y_r \times S^{D-1}} \tau_r(x, v) \frac{dxdv}{|Y_r||S^{D-1}|} = +\infty \quad \text{infinite mean free path}$$
Log-log-plot of $\Phi_r(t)$ vs. $t$ for $r = 0.01$, $r = 0.03$ and $r = 0.001$
Plot of $tr\Phi_r(t)$ vs. $t$
**Theorem.** There exists $0 < C_- < C_+$ such that, for all $t > 1/r^{D-1}$

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**Theorem. (Caglioti-G CMP 2003)** For $D = 2$,

$$\lim_{r \to 0^+} \frac{1}{|\ln \epsilon|} \int_{\epsilon}^{1/4} \Phi_r(t/r) \frac{dr}{r} = \frac{2}{\pi^2 t} + O(1/t^2)$$
Non convergence to the Lorentz kinetic equation

For $n \geq 1$ and $r = n^{D-1}$, and for an initial phase space density $\rho^{in} \equiv \rho^{in}(x) \geq 0$ independent of $v$, set

$$f_n(t, x, v) = \rho^{in}(\frac{1}{n}X^{r}_{nt}(nx, v))$$

**Theorem.** For some $\rho^{in} \in L^{\infty}(T^D)$, neither $f_n$ nor any subsequence thereof converges in $L^{\infty}$ weak-* to the solution of

$$(\partial_t + v \cdot \nabla_x) f = C(f) \text{ on } \mathbb{R}^D \times S^{D-1}, \quad f \big|_{t=0} = \rho^{in}$$

where $C$ is the Lorentz collision integral

$$C(\phi)(v) = \int_{|\omega|=1, \omega \cdot v > 0} \left( \phi(\mathcal{R} \omega v) - \phi(v) \right) \cos(v, \omega) d\omega$$
The same is true if the Lorentz collision integral is replaced with any operator of the form

$$C(f) = \sigma \int_{S^D - 1} p(v, v')(f(v') - f(v))dv'$$

where $\sigma > 0$ and the function $p$ is the kernel of a compact operator on $L^2(S^{D-1})$ that satisfies

$$p(v, v') = p(v', v) \geq 0, \quad \int_{S^D - 1} p(v, v')dv' = 1$$
Method of proof

- **Spectral theory of transport operators** (Ukai-Ghidouche-Point JMPA 1977)
  \[ \| f(t) - \langle \rho^{in} \rangle_{L^2(T^D \times S^{D-1})} \|_{L^2(T^D)} \leq c e^{-\gamma t} \| \rho^{in} \|_{L^2(T^D)} \]

- **Pointwise inequality**
  \[ f_n(t, x, v) \geq \rho^{in}(x - tv) 1_{\tau_r(nx, v) \geq nt} \]

- If some subsequence \( f_n' \rightharpoonup f \) in \( L^\infty \) weak-*,
  \[ \| f(t, \cdot, \cdot) \|_{L^2_{x,\omega}} \geq \| \rho^{in} \|_{L^2_x} \Phi_r(nt) \geq \frac{C_- \| \rho^{in} \|_{L^2_x}}{ntr^{D-1}} = C_- \| \rho^{in} \|_{L^2_x} \]

- **Concentration argument:** pick \( \rho^{in}_\delta \) such that
  \[ \| \rho^{in}_\delta \|_{L^2(T^D)} = 1 \text{ while } \langle \rho^{in}_\delta \rangle \to 0 \text{ as } \delta \to 0^+ \]

  \[ \Rightarrow \text{Contradiction with the spectral bound} \]
Case of absorbing obstacles, $D = 2$

**Pbm:** to find the limit as $r \to 0^+$ of $g_r$ s.t.

\[
(\partial_t + v \cdot \nabla_x) g_r = 0, \quad x \in rZ_r, \quad v \in S^{D-1},
\]

\[
g_r(t, x, v) = 0, \quad x \in \partial(rZ_r), \quad v \cdot n_x > 0,
\]

\[
g_r \big|_{t=0} = g^{in} \big|_{rZ_r}.
\]
Theorem. (Caglioti-G CMP 2003) Let \( g^{in} \geq 0 \) be in \( C^1_c(\mathbb{R}^2 \times S^1) \). For each \( \chi \in C^1_c(\mathbb{R}^2 \times S^1) \),

\[
\lim_{r \to 0^+} \frac{1}{|\ln r|} \int_r^{1/4} g_r(t) \chi \frac{dr}{r} = \langle g(t)\chi \rangle + O(1/t^2)
\]

where

\[
g(t,x,v) = \frac{2g^{in}(x - tv,v)}{\pi^2 t}.
\]

• This suggests that the limiting equation for the above model should be

\[
\partial_t g + v \cdot \nabla x g + \frac{1}{t} g = 0, \quad t > 0, \quad x \in \mathbb{R}^2, \quad |v| = 1.
\]

This, however, holds \text{for large} \( t \) only.
Method of proof

Idea no.1 Given a linear flow with irrational slope on a 2-torus with a disk removed, what is the longest orbit of this flow? (question raised by R. Thom in 1989).

Blank-Krikorian, IJM’93: On a 2-torus with a slit parallel to one of the coordinate axis, there are generically 3 classes of orbits (say $A$, $B$ and $C$). All orbits in a given class have the same length: $l(A)$, $l(B)$ and $l(C)$. Each such length is determined by the size of the slit and the continued fraction expansion of the slope.

- This defines a three-term partition of the 2-torus: each term of this partition is the union of all orbits of type $A$ (resp. $B$ and $C$).

- In each term of this partition, the distribution of exit times (from the 2-torus with the slit removed) knowing the direction $v$ is explicitly computed.
Gauss map \[ T : (0, 1) \to (0, 1) \text{ defined by } x \mapsto Tx = 1/x - [1/x] \]

Continued fractions For \( \alpha \in (0, 1) \setminus \mathbb{Q} \), one has
\[
\alpha = [a_1, a_2, a_3, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \quad \text{with } a_k = \left[ \frac{1}{T^{k-1}\alpha} \right]
\]

Convergents Truncated continued fractions give rational approximants
\[
\alpha \simeq [a_1, \ldots, a_{n-1}] =: \frac{p_n}{q_n}
\]
This defines recursively two sequences of integers \( p_n \) and \( q_n \)
\[
p_{n+1} = a_n p_n + p_{n-1}, \quad p_0 = 1, \quad p_1 = 0
\]
\[
q_{n+1} = a_n q_n + q_{n-1}, \quad q_0 = 0, \quad q_1 = 1
\]

Error \( d_n = (-1)^{n-1}(q_n\alpha - p_n) > 0 \) (signs of \( q_n\alpha - p_n \) alternate)
Let $\theta \in (0, \frac{\pi}{4})$, $\alpha = \tan \theta$ and $v = (\cos \theta, \sin \theta)$; set $R = 2r / \cos \theta$;
Let \( \frac{p_n}{q_n} = \text{n-th convergent} \) and \( d_n = \text{n-th error} \) in the continued fraction expansion of \( \alpha \).

**Partition of \((0, 1)\):** For \( I_{n,k} = \left[ \max(d_n, d_{n-1} - kd_n), d_{n-1} - (k - 1)d_n \right) \)

\[
(0, 1) = \bigcup_{n \geq 1} \bigcup_{1 \leq k \leq a_n} I_{n,k}
\]

- Assume \( R \in I_{n,k} \) — this defines a unique pair \((n, k)\) — and let \( t > 2 \).

- Let \( \psi_r(t, v) \) be the distribution of exit times for a particle moving at speed 1 in the direction \( v \) on a 2-torus punctured with a vertical slit of length \( R \).

Then

\[
\left| \psi_r(t, v) - \left(1 - \frac{R}{d_{n-1}} - t \frac{d_n}{R}\right) + \right| \leq \frac{4}{k} 1_{k \geq t-2}.
\]
If \( d_n < R < d_{n-1} \)

\[
k = -\frac{(d_{n-1} - R)}{d_n}
\]
Idea no.2 Given the direction \( v = (\cos \theta, \sin \theta) \), the distribution of exit times in a 2-torus with a disk of radius \( r \) removed is obtained by comparing \( 2r \) with the errors in the continued fraction expansion of \( \alpha = \tan \theta \).

• Observe that \( d_n(\alpha) = \alpha d_{n-1}(T\alpha) \).

• Renormalization replace the problem defined by a slope \( \alpha \) and a disk of radius \( r \) with the analogous problem with slope \( T\alpha \) and a disk of radius \( \alpha r \).

• This suggests seeking a fixed point of this transformation, by using some ergodic theorem where

\[
\text{TIME} = \ln(\text{DISK SIZE})
\]

\[\Rightarrow\] this explains why the Cesarò mean involves the invariant measure of the multiplicative group \( \mathbb{R}^*_+ \), i.e. \( \frac{dr}{r} \).
The Gauss map $T$ is ergodic on $(0, 1)$ with invariant measure

$$dg(\alpha) = \frac{1}{\ln 2} \frac{d\alpha}{1 + \alpha}$$

Define $N(\alpha, \epsilon) = \inf \{ n \in \mathbb{N} | d_n(\alpha) < \epsilon \}$; for $j = 0, 1$, we define

$$\Delta_j(\alpha, x) = -x - \ln d_{N(\alpha, e^{-x}) + 1 - j}(\alpha)$$

**Lemma.** Define $F(\theta) = \int_0^{\vert \ln \theta \vert} f(\vert \ln \theta \vert - y, -y) dy$; then, one has

$$\lim_{\epsilon \to 0^+} \frac{1}{\vert \ln \epsilon \vert} \int_{x^*}^{\vert \ln \epsilon \vert} f(\Delta_0(\alpha, x), \Delta_1(\alpha, x)) dx = \frac{12}{\pi^2} \int_0^1 F(\theta) d\theta$$

Conclude by applying the Lemma to $f(z_1, z_2) = (1 - e^{z_2} - te^{-z_1})_+$