A HYPERVISCOSITY SPECTRAL LARGE EDDY SIMULATION MODEL FOR TURBULENT FLOWS
Overview

Part I: Basic Facts about the Navier-Stokes Equations

Part II: Proposition of a Paradigm for LES

Part III: The Hyperviscosity Model

Part IV: Spectral Hyperviscosity Approximation

THE NAVIER-STOKES EQUATIONS
BASIC FACTS ABOUT THE NAVIER-STOKES EQUATIONS
The Navier-Stokes Equations

\[ \begin{align*}
\frac{\partial}{\partial t} u &+ \nabla \cdot u = f \\
\nabla \cdot u &= 0 \\
\n u(0, x) &= u_0
\end{align*} \]

\( u \) is the fluid domain •
\( d \) is chosen equal to unity.
\( f \) is a source term.
\( u_0 \) is the initial data •

\[ \begin{align*}
\n\frac{\partial}{\partial t} n &= 0 = \nabla \cdot n \\
\n0 &= n \cdot \Delta \\
\n(\Delta n)_\Omega &= n \cdot \Delta - \nabla \cdot \nabla + n \Delta \cdot n + n \Delta \cdot \nabla
\end{align*} \]

\( \Omega \) is the fluid domain •
\( n \) is velocity •
\( p \) is pressure •
The question of the existence of classical solutions for long times is open. This question is linked to the question of uniqueness of turbulent solutions.

Existence.

J. Leray (1934) introduces the notion of turbulent solution.

Existence and uniqueness.

Clay Institute 1M$ prize.

J. Leray uses mollification to prove existence:

\[ J = \frac{3}{2}n \Delta n - \frac{3}{2}d \Delta + \frac{3}{2}n \Delta \cdot (\frac{3}{2}n \phi) + \frac{3}{2}n^3 \theta \]

\[ \phi_T^3 = x \phi_T^1, \quad 0 \leq \phi_L \leq \phi, \quad \phi \in \mathbb{R}^3 \Delta \subset \phi \]

E. Hopf (1951) et al. uses the Galerkin technique to prove:

\[ ((U)_T^3, L, 0) \in T \cup ((U)_H^1, L, 0) \subset T \]

A turbulent solution is a weak solution in

The question of the existence of classical solutions for long times is open. This question is linked to the question of uniqueness of turbulent solutions.

Existence and uniqueness.
Suitable Weak Solution


Best partial regularity theorem to date.

Solutions in $[0, +\infty) \times \mathbb{R}$ is less than 1.

"Dimension" of the set of singular points of suitable weak
dissipative (or

DEFINITION: A NS weak solution is said to be suitable (or

V. Sheffer (1976) introduces suitable weak solutions:

\begin{align*}
0 \geq n \cdot 1 - \zeta (n \Delta) \nu + (\zeta n^\frac{\nu}{1})_v \Delta \nu - ((d + \zeta n^{\frac{\nu}{1}}) \nu) \cdot \Delta + (\zeta n^{\frac{\nu}{1}}) \theta
\end{align*}
\[ 0 = (S)^1 \iff \\
\{ \Lambda \ni (t,x) : \Lambda \lambda \epsilon (\Lambda)_{\infty | T \not\ni n} \}^0 \cup \{ x \in (t,x) \} = S \]

Singular set: \( S \)

\[ \{ g > \frac{1}{4} \epsilon (t,x) \partial \cap X \} \cup \{ \inf \lim_{\epsilon \to 0^+} = (X)^1 \} \]
THEOREM: (Duchon-Robert (2000))

Leray-regularized NS solutions are dissipative. The question is open for Galerkin weak NS solutions.

Proof: $\phi^3 n f - \phi_\varepsilon (\varepsilon n \Delta) \mathbf{a} + \phi \Delta (\varepsilon n^2 d + \varepsilon n^2 n \ast \phi) - (\phi \varepsilon \Delta \mathbf{a} + \phi^3 \mathbf{\theta}) \varepsilon n_\varepsilon = 0$.

Test with $\varepsilon n$ since $\varepsilon n$ is smooth).

$0 = \frac{b - \varepsilon}{b \varepsilon} \geq d \geq 1$ (up to subsequences) and

Theorem: (Duchon-Robert (2000)) The limits (as $\varepsilon \to 0$) of $\varepsilon n$...
\[
\phi_{\zeta}(n \Delta) \int \preceq \phi_{\zeta}(\varepsilon n \Delta) \int 0 \to \varepsilon
\]
\[
\iff (n \Delta) \phi \int + n \Delta (n - \varepsilon n) \Delta \phi \int \gamma \preceq \phi_{\zeta}(\varepsilon n \Delta) \int
\]
\[
\left( T \right) \int n d \to \varepsilon n \varepsilon d \Leftarrow \left\{ \begin{array}{c}
\left( T \right) \nu T \int n \to n \\
\left( T \right) \frac{\nu}{\varepsilon} T \int d \to \varepsilon d
\end{array} \right.
\]
\[
\left( T \right) \frac{\nu}{\varepsilon} T \int d \to \varepsilon d \Leftarrow \nu T \varepsilon n \| c \preceq \nu T \varepsilon n \otimes \varepsilon n \| \varepsilon \| c \preceq \nu T \varepsilon d \|
\]
\[
\Leftarrow (\varepsilon n \otimes \varepsilon n \| \varepsilon \| \varepsilon p) \cdot \Delta \cdot \Delta = \varepsilon d \gamma \Delta
\]
\[
0 = \int \cdot \Delta \Delta
\]

\textbf{Proof (continued):}
LARGE EDDY SIMULATIONS
Almost no reasonable mathematical theory for LES.

Concept introduced by Leonard (1974)

The objective is to modify the NS model so that the new model is amenable to numerical simulations.

Objectives of LES
Let \((\cdot)\):
\[
\begin{align*}
t \otimes n - n \otimes n &= \mathbb{I} \\
\text{where we have introduced the so-called subgrid-scale tensor:}
\end{align*}
\]
\[
\begin{cases}
\mathbf{n} &= 0 = |\mathbf{n}|_0 \\
\text{or } \mathbf{n} \text{ is periodic}, \quad 0 &= \mathbb{I}|\mathbf{n}|_0 \\
0 &= \mathbf{n} \cdot \Delta \\
\mathbb{I} \cdot \Delta - \mathbb{I} &= n_z \Delta n - d \Delta + n \Delta \cdot n + n^2 \partial
\end{cases}
\]
Apply the filter operator to the Navier–Stokes equations:

Assume that the filter is linear and commutes with differential operators.

Let the filter be a regularizing operator acting on space-time-dependent functions (we call it a filter).

THE FILTERING-EVERYTHING PARADIGM
The closure problem: The goal of LES (Leonard, 1974) is to model in terms of $u$ only, without resorting to $n$. For approximating the NS solution as for approximating the filtered solution, one should expect to use the same number of degrees of freedom as for the solution set of NS is isomorphic to the solution set of the filtered equations. If the filter induces an isomorphism, the solution set of NS is isomorphic to the solution set of the filtered equations. Exact closure is possible, e.g., take

$$
\Lambda_{I}^{-1}(\Delta \cdot \nabla + I) =: \Lambda
$$

The filtering-everything paradigm is probably nonsensical.

Filtering the Navier–Stokes equations is an efficient approach only if inexact closure is performed.

Filtering and achieving exact closure may not reduce the number of degrees of freedom.

CONCLUSION:

THE FILTERING PARADOX
A sequence $(u^2; p^2) > 0$ is said to be a LES solution of NS if for all $(u^2; p^2) > 0$, $(u^2; p^2)$ is uniquely defined for all times (i.e. regularization technique (so-called DNS)).

\[ u^2 \rightarrow u, \quad p^2 \rightarrow p \quad \text{upto subsequences and (up to subsequences) and} \]

\[ d \leftarrow \epsilon d, \quad n \leftarrow \epsilon n \]  

uniqueness for $u$ and $p$ for all times.

For all $\epsilon > 0$, $(d^\epsilon, n^\epsilon)$ is uniquely defined for all times (i.e. solution of NS if)

A sequence $(d^\epsilon, n^\epsilon)_\epsilon \geq 0$ is said to be a LES.

**Example:** Galerkin approximations are regularizing techniques.

**Does a Galerkin solution converge to a suitable weak solution?**

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EXAMPLE 1: Leray’s regularization is a LER technique (1934)!

However the “model” is not frame invariant (is it important?)

EXAMPLE 1: Leray’s regularization is a LES technique (1934)

EXAMPLE 2: NS-a model (Holmes–Marsden–Ratius (1998), Holmes et al. (1999)).

The equation is frame-indifferent.

It is a Leray regularization where the nonlinear term is

\[ \frac{1}{\epsilon n} \Delta u - \nu \Delta + \frac{1}{\epsilon n} (n \times \Delta) \]

Alternative interpretation

\[ \frac{1}{\epsilon n} \Delta u - \nu \Delta + \frac{1}{\epsilon n} (n \times \Delta) + \frac{1}{\epsilon n} \Theta \]

\[ \frac{1}{\epsilon n} \Delta u - \nu \Delta + \frac{1}{\epsilon n} (n \times \Delta) - \nu \Delta \cdot \left( \frac{1}{\epsilon n} \right) + \frac{1}{\epsilon n} \Theta \]

\[ \frac{1}{\epsilon n} \Delta u - \nu \Delta + \frac{1}{\epsilon n} (n \times \Delta) + \frac{1}{\epsilon n} \Theta \]

\[ \frac{1}{\epsilon n} \Delta u - \nu \Delta + \frac{1}{\epsilon n} (n \times \Delta) - \nu \Delta \cdot \left( \frac{1}{\epsilon n} \right) + \frac{1}{\epsilon n} \Theta \]

\[ \frac{1}{\epsilon n} \Delta u - \nu \Delta + \frac{1}{\epsilon n} (n \times \Delta) + \frac{1}{\epsilon n} \Theta \]
EXAMPLES OF LES

EXAMPLE 3: Ladyzhenskaja (1967) proposed

\[ \frac{\partial}{\partial t} u^2 + u^2 \frac{\partial}{\partial r} u^2 + r \frac{\partial}{\partial r} \left( \rho u^2 + \frac{\partial}{\partial r} \right) = f \]

where operator \( T \) is non-linear, and \( D = \frac{1}{2} \left( r u^2 + \frac{\partial}{\partial r} \right) T \).

For instance \( T(\psi) = \bar{\psi} \left( \frac{1}{r} \psi \right) \bar{\psi} \) with \( \psi = 0 \) is possible. For instance \( \frac{\partial}{\partial t} u^2 = 0 \)

where operator \( T \) is non-linear, and

\[
\begin{aligned}
\cdot 0 n &= 0 = \frac{\partial}{\partial t} \psi n \\
\text{or} & \quad n \text{ is periodic,} \\
0 &= \frac{\partial}{\partial t} \psi n \\
0 &= \psi n \cdot \Delta \\
I &= \left( (D)_{\psi} + \psi n \Delta \psi \right) \cdot \Delta - \psi d \Delta + \psi n \Delta \cdot \psi n + \psi n^2 \psi
\end{aligned}
\]

THEOREM: (Ladyzhenskaja (1967)) The modified NS equations yield Smagorinsky’s model.

EXAMPLES OF LES

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EXAMPLE 4:

HYPERVISCOSITY
HYPERVISCOSITY

EXAMPLE 4: Lions (1959) proposed to use hyperviscosity.

HYPERVISCOSITY

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HYPERVISCOSITY: A PRIORI ESTIMATES

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Part III

\[ I = \frac{d}{T} + \frac{d}{T} \int \zeta \frac{d\tau}{T} \| n \Delta \| \zeta \frac{d\tau}{T} \| n \| \sim \rho \frac{\zeta}{T} \| n \Delta \| \zeta \| n \| \int \]

Using Holder's inequality, we write

\[ \int \frac{\zeta}{T} \| f \| + \rho \frac{\zeta}{T} \| n \Delta \| \zeta \| n \| \int \frac{\zeta}{T} \| 0 \| \| n \| \| \rho \| \| n \| \int \frac{\zeta}{T} \| T \| n \| \]

In a similar manner, testing by \( n \) yields

\[ \int \frac{\zeta}{T} \| f \| + \frac{\zeta}{T} \| 0 \| \| n \| \sim \rho \frac{\zeta}{T} \| n \| \int \frac{\zeta}{T} \| T \| n \|

Integration in time, gives the following estimate:

\[ \frac{(\zeta e, f)}{T} = \frac{\zeta}{T} \| n \| \frac{\zeta}{T} (\Delta - \| e + \frac{\zeta}{T} \| n \| \frac{\zeta}{T} \| \| n \| \frac{\zeta}{T} \| e \|

Testing the momentum equation by \( n \) yields
HYPERVISCOSITY: APRIORI ESTIMATES

Owing to Sobolev inequalities we also have

$\frac{d\tau}{\tau} \lesssim \frac{\nu}{\Lambda} \frac{\partial u}{\tau} \lesssim \frac{\nu}{\Lambda} \frac{\partial u}{\tau}$

These two conditions yield

$\nu H \| \epsilon n \| \lesssim \frac{d\tau}{\tau} \| \epsilon n \Delta \|

and

$\frac{\nu}{\Lambda} \frac{\partial u}{\tau} \lesssim \frac{\nu}{\Lambda} \frac{\partial u}{\tau}$

Owing to Sobolev inequalities we also have

Existence of solutions in $L^1(0,T; H^1)$ is proved by

$\forall H^1, \forall L^2(0, T) \| \delta n \| + (\int L^2(0, T) \| \delta n \|)$

and since $\int L^2(0, T) \| \delta n \|$

These two conditions yield

$\frac{d\tau}{\tau} \lesssim \frac{\nu}{\Lambda} \frac{\partial u}{\tau}$

The Galerkin technique using the a priori estimates.

$\forall H^1, \forall L^2(0, T) \| \delta n \| + (\int L^2(0, T) \| \delta n \|)$

and since $\int L^2(0, T) \| \delta n \|$

These two conditions yield

$\frac{d\tau}{\tau} \lesssim \frac{\nu}{\Lambda} \frac{\partial u}{\tau}$

The Galerkin technique using the a priori estimates.
SPECTRAL HYPERVISCOSITY
DEFINITION:

\[ H \phi = f \quad \text{with} \quad f = \sum k \in \mathbb{Z} u_k e^{ikx}; u_k = u_{-k}; \]

\[ N \phi = \sum k \in \mathbb{Z} (1 + jk^2) \phi_k e^{ikx}; \phi_k = \phi_{-k}; \]

Pressure space

Velocity space

\[ \{ \phi \in H^1(\mathbb{R}^3) \mid \sum k \in \mathbb{Z} (1 + jk^2) \phi_k e^{ikx} = (x) \} = N \phi \]

\[ \sum k \in \mathbb{Z} \phi_k \sum = an \int _{\mathbb{R}^3} (\nu \phi) = (a', n) \]

\[ \{ \phi \in L^2(\mathbb{R}^3) \mid \sum k \in \mathbb{Z} \phi_k e^{ikx} \sum = n \} = (\nu)_s H \]

DEFINITION:

FOURIER APPROXIMATION
\[ s^H \| \alpha \|_{s-H} N \approx n^H \| \alpha \|_{N^H} \| \alpha \| \|_{N^A} \| A \|_{N^A} \| A \| \| 5 \|
\]
\[ s^H \| \alpha \|_{s-H} N \approx n^H \| \alpha \|_{N^H, \alpha} - \alpha \| \| \alpha \| \|_{N^H, \alpha} - \alpha \| \| \alpha \| \|_{N^A} \| A \| \| 4 \|
\]

**Lemma:** \( P \) satisfies the following properties:

1. \( P \) is the restriction on \( T \) of the projection onto \( \mathcal{U} \).
2. \( P \) commutes with differentiation operators.
3. \( P \) is the restriction on \( T \) of the projection onto \( \mathcal{U} \).
4. \( P \) commutes with differentiation operators.
5. \( P \) commutes with differentiation operators.

\[ s^H \| \alpha \|_{s-H} N \approx n^H \| \alpha \|_{N^H} \| \alpha \| \|_{N^A} \| A \| \| 3 \|
\]

\[ s^H \| \alpha \|_{s-H} N \approx n^H \| \alpha \|_{N^H, \alpha} - \alpha \| \| \alpha \| \|_{N^H, \alpha} - \alpha \| \| \alpha \| \|_{N^A} \| A \| \| 2 \|
\]
ANAIVEHYPERVISCOSITYMODEL

Definition:

\[ Q(x) = \frac{1}{(2 \pi)^{3/2}} \int P_1 \cdot j_k j_1 \cdot N j_2 \otimes e_{ik} \cdot x \, d^3 j \] 

\[ Q \odot u_N(x) = \int Q(x-y) \, dy = X_1 \cdot j_k j_1 \cdot N j_2 \otimes u_k e_{ik} \cdot x \] 

\[ Q \odot u_N(x) \equiv (b, Nn \cdot \Delta) \]

\[ [L, 0] \equiv b \alpha N X \equiv a \alpha \] 

\[ (a \cdot f) = (a, Nn \cdot \nabla) \alpha + (a \Delta, Nn \cdot \nabla) \alpha + (a \cdot \Delta, Nn \cdot \nabla) \alpha \]

Find such that \((N X : [L, 0])_0 \in N d\) and \((N W : [L, 0])_1 \in N n\) such that

Let \(\varepsilon > 0\)

If \(\alpha\) is an integer

\[ Nn(x) \odot (\Delta - ) = (x) Nn \odot \nabla \]

\[ \int_{(N^\infty |y|>1)} Nn(x-y) \otimes (x-y) \nabla (x) Nn \int = (x) Nn \odot \nabla \]

\[ \int_{(N^\infty |y|>1)} x \cdot \nabla \cdot (x-y) \otimes (x-y) \nabla (x-y) = (x) \nabla \]

Definition: NAIVE HYPERVISCOSITY MODEL

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The hyperviscosity spoils the consistency.

Consistency error cannot be arbitrarily small.

The interpolation error can be arbitrarily small.

\[
(s - N)O = sH \| n\|_{s - N} \geq \tau T \| n^N_\tau - n\|
\]

But if \( s \) may be arbitrarily large if \( n \) is smooth, \( s \) cannot be too small.

Consistency error cannot be arbitrarily small.

But \( \varepsilon \) cannot be too small to play the regularizing effect we expect.

Consistency error in \( \varepsilon \) is small.

A NAIVE HYPERVISCOSITY MODEL
Actually it is not necessary to stabilize the low wave numbers since they should be controlled by means of the a priori estimate. Therefore it is possible to work with £N = 1, even though this requires a spectral hyperviscosity model.

\[ \exists \alpha \in \mathbb{R} \quad \text{such that} \quad \alpha > \left\{ \begin{array}{ll} \frac{2}{3} \quad \text{if} \quad \alpha \geq \frac{2}{3} \\ \frac{2}{\alpha - 1} \quad \text{otherwise} \end{array} \right. \quad \text{with} \quad \theta > 0 \quad \text{with} \quad \theta \text{N} = \varepsilon \]
Part IV
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A SPECTRAL HYPERVISCOsITY MODEL

One can also define the hypervisosity kernel as follows:

\[ \mathcal{Q} = (x) \mathcal{O} \]

where the viscosity coefficients \( \mathcal{O} \) are such that

\[ N \geq \infty |y| A \quad \frac{\nu Z}{\nu Z N} \gtrsim |\frac{\nu Z}{N} - I| \]

This definition has the practical advantage of ensuring a smooth transition of the viscosity coefficients across the threshold

The results stated hereafter hold also with this definition.
Admissible values of the parameters $\alpha$, $\beta$, and $\theta$.

$$\begin{array}{|c|c|c|c|c|}
\hline
\frac{13}{8} & > & \frac{11}{9} & > & \frac{9}{4} > \frac{7}{3} > \frac{6}{2} > \theta \\
\frac{13}{80} & > & \frac{11}{48} & > & \frac{3}{8} > \frac{7}{8} > \frac{2}{1} > \theta'
\hline
\end{array}$$

$$\alpha = \begin{cases} \frac{2a}{\beta} + \frac{3}{(1-n)(a-1)} & \text{if } \alpha \geq \frac{2}{3} \\ \frac{2}{\beta} & \text{otherwise} \end{cases}$$

But the condition where $n$ is dissipative enforces

$$\frac{2a-n}{\beta} < \frac{2}{\beta}$$

We'd like $\beta$ large so that $e^{-N}$ is small.
Note that $\mathcal{N} = \mathcal{N} - \mathcal{N}$.

A SPECTRAL HYPERVISCOUSITY MODEL

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PROPOSITION: The hyperviscosity perturbation is spectrally small in the sense that

\[ s \frac{H}{\gamma^2} \| \mathcal{N} n \|_{s_{\theta^2} - \mathcal{N}} \geq s \frac{\gamma}{\gamma^2} \| \gamma \| \sum_{\mathcal{N}} s_{\theta^2} - \mathcal{N} \mathcal{N} c e \frac{\mathcal{N}}{z} \geq \frac{\gamma}{\gamma^2} \| \mathcal{N} n \|_{\mathcal{O}} \| \mathcal{N} e \]

so that

\[ s_{\theta^2} - \mathcal{N} \mathcal{N} c e = s_{\theta^2 - \mathcal{N}} \frac{\gamma}{\gamma^2} \mathcal{N} \]

Moreover, from the definition of \( \mathcal{N} \), \( \mathcal{N} c e \), and \( \theta \), we have

\[ s \frac{\gamma}{\gamma^2} \| \gamma \| \geq \frac{\gamma}{\gamma^2} \| \gamma \| \geq \frac{\gamma}{\gamma^2} \mathcal{N} \]

Using the fact that \( s \geq \gamma \) and that \( s \geq \gamma \), we have:

\[ \frac{\gamma}{\gamma^2} \| \gamma \| \sum_{\mathcal{N}} \frac{\gamma}{\gamma^2} \| \gamma \| \geq \frac{\gamma}{\gamma^2} \| \gamma \| \sum_{\mathcal{N}} \frac{\gamma}{\gamma^2} \| \gamma \| = \frac{\gamma}{\gamma^2} \| \mathcal{N} n \|_{\mathcal{O}} \]

PROOF: Note that

\[ s \frac{\gamma}{\gamma^2} \| \gamma \| \sum_{\mathcal{N}} \frac{\gamma}{\gamma^2} \| \gamma \| \geq \frac{\gamma}{\gamma^2} \| \mathcal{N} n \|_{\mathcal{O}} \| \mathcal{N} e \]

small in the sense that

PROPOSITION: The hyperviscosity perturbation is spectrally
The result is a consequence of Gronwall’s Lemma.

\[
\begin{align*}
&\frac{(\tau T)T}{\tau} \| N n \| \left( \frac{1}{2} + \alpha \mathcal{E} \right) + (\tau T)T \| f \| \left( \frac{1}{2} + \tau T \| \theta n \| \right) \gtrsim \\
&\left( \alpha H \right)T \| N n \| N \varepsilon + (\tau H)T \| N n \| \lambda \left( \frac{\tau T}{\tau} \right) \| \n \| \n \n
\end{align*}
\]

Since \( N n \) is solenoidal and using the above bound, we obtain

\[
\begin{align*}
\frac{1}{\tau} \| N n \| \mathcal{E} &\gtrsim \left( \frac{1}{2} - \frac{1}{2} \right) \| N n \| \mathcal{E} \quad \text{and that} \quad \frac{1}{\tau} \| \mathcal{E} \|^2 \mathcal{E} \quad \text{that} \quad \frac{1}{\tau} \| \n \| \n \n
\end{align*}
\]

To estimate the last term in the above inequality, use the fact

\[
\begin{align*}
\alpha \mathcal{E} &\gtrsim \left( \alpha H \right)T \| N n \| N \varepsilon + (\tau H)T \| N n \| \lambda \left( \frac{\tau T}{\tau} \right) \| \n \| \n \n
\end{align*}
\]

**Proof: Observe that**

\[
\begin{align*}
&\frac{1}{\tau} \| N n \| \mathcal{E} \quad \text{and that} \quad \frac{1}{\tau} \| \n \| \n \n
\end{align*}
\]

**Lemma:** We have the a priori estimates.
\[ \frac{3}{8} = d, \quad \forall = b \] for \( \mathcal{N} \mathcal{n} \) and are solenoidal since \( \mathcal{F} \) and \( \mathcal{N} \mathcal{n} \) are solenoidal.

By using several integrals by parts, we obtain

By using several integrals by parts, we obtain.

\[ \langle \mathcal{N} d_{1-} (\tau \Delta) \Delta, \mathcal{N} n \times \mathcal{N} n \rangle = \langle \mathcal{N} d_{1-} (\tau \Delta) \Delta, (\mathcal{N} n \times \mathcal{N} n) \cdot \Delta \rangle = \]

\[ \langle \mathcal{N} d_{1-} (\tau \Delta) \Delta, \mathcal{N} d \Delta \rangle = \varepsilon_{\tau} \mathcal{T}_{\tau} \mathcal{N} \mathcal{d} \mathcal{d} \]

Note that \( \mathcal{N} \mathcal{X} = \mathcal{N} d_{1-} (\tau \Delta) \Delta \) is an admissible test function.

Then, we multiply the momentum equation by \( \mathcal{N} \mathcal{W} \). By \( \mathcal{N} \mathcal{W} \), we obtain that \( \mathcal{N} \mathcal{W} \) is bijective.

**Proof:** First, we observe that

\[ c \geq \frac{(\tau^2 - b)\varepsilon}{b \varepsilon} \geq d \geq 1 \]

with \( c \geq \frac{(\tau b)\varepsilon}{b \varepsilon} \geq d \geq 1 \) and \( d \geq 1 \).

**Lemma:** We have...
\[ \mathcal{H} + (\phi \Delta ' Nn_N|Nn|_I^2) = \mathcal{H} + (\phi Nn ' Nn \Delta \cdot Nn) = ((\phi Nn) Nd ' Nn \Delta \cdot Nn) \]

**Corollary:** Let \( s > \frac{\alpha}{\varepsilon} + 1 \), \( \varepsilon > 1 \) be real numbers such that

\[ s \leq \frac{\varepsilon}{\varepsilon} < 1 \]

And because \( (\phi Nn - (\phi Nn) Nd ' Nn \Delta \cdot Nn) = \mathcal{H} \)

Define

\[ ((\phi Nn) Nd ' f) = ((\phi Nn) Nd ' Nn \cdot \nabla) + ((\phi Nn) Nd ' Nn \Delta) \cdot Nn \]

\[ ((\phi Nn) Nd \cdot \Delta ' Nn d) - ((\phi Nn) Nd ' Nn \Delta ' Nn) + ((\phi Nn) Nd ' Nn^3 \cdot \theta) \]

**Theorem:** Let \( s \) be a real number such that

\[ (s - H)\varepsilon \geq T \||Nn^3 \cdot \theta|| \]

\[ \sum \leq (s - H)\varepsilon \leq T \||Nn^3 \cdot \theta|| \]

**Corollary:**
At this point, there are two possibilities: either
\[ \frac{\alpha}{\beta} > 1 \text{ or } \frac{\alpha}{\beta} \leq 1. \]

\[ \mathcal{V} = \int \mathcal{V} \, \text{d}x, \quad \mathcal{V} = \mathcal{V} \int \mathcal{V} \, \text{d}x \]

For the remainder \( R \) we have

\[ |(\phi \mathcal{N}_n - (\phi \mathcal{N}_n \mathcal{N}_D)^\Delta \mathcal{N}_n \Delta \cdot \mathcal{N}_n)| = |R| \]
If \( \frac{\alpha}{\gamma} < \frac{1}{2} \), then

\[
\kappa u \gtrsim \kappa H \lesssim N \kappa \lesssim |H|_L^0. 
\]

That is to say, owing to the hypotheses

\[
s_H \| \phi \|_{\frac{\alpha H}{\gamma}} N n |_{\kappa + \frac{\gamma - \frac{\alpha}{\gamma}}{2} N} = s_H \| \phi \|_{\frac{\alpha H}{\gamma}} N n |_{\kappa - \frac{\gamma - \frac{\alpha}{\gamma}}{2} N} \gtrsim |H| 
\]

we have

\[
\frac{\gamma}{\gamma - \alpha H} \gtrsim \delta.
\]

Inverse inequality

\[
\begin{align*}
\left \| \kappa n \right \|_{\frac{\alpha H}{\gamma}} \left \| \kappa H \right \|_{\frac{\gamma - \frac{\alpha}{\gamma}}{2} N} \gtrsim \\
\left \| \kappa n \right \|_{\frac{\alpha H}{\gamma}} \left \| \kappa H \right \|_{\frac{\gamma - \frac{\alpha}{\gamma}}{2} N} \gtrsim \frac{\gamma}{\gamma - \alpha H} \left \| \kappa n \right \|.
\end{align*}
\]

Then, owing to the hypothesis

\[
\bar{\gamma} = \frac{\gamma}{\gamma - \alpha H} \gtrsim 5 \frac{2}{\gamma},
\]

we have

\[
\| \kappa n \| \lesssim \| \kappa H \| N n |_{\kappa - \frac{\gamma - \frac{\alpha}{\gamma}}{2} N} \lesssim |H|_L^0.
\]

If \( \alpha \gtrsim \frac{\gamma}{\gamma - \alpha H} \), then

\[
\bar{\gamma} \lesssim \delta.
\]
CONCLUSIONS
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- The new LES paradigm is constructive: i.e. enforcing the regularized solution to converge to a suitable weak solution implies strong constraints on the numerical methods.
- Extension to non periodic domains and finite elements is likely to be highly technical.
- There is no easy a priori estimate on the pressure.
  - There is no easy a priori estimate on $\partial_t \mathbf{u}_h$ (Fourier analysis gives $\mathbf{u}_h \in H^\frac{3}{8} \cap L^2$, this is not enough!)
  - The nonlinear term does not pose difficulties!

The trouble maker is the product $p_h \mathbf{u}_h$!