A numerical scheme for the growth of sandpiles on a table

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Outline

• Granular matter. Characteristics, approaches, math interest.

• A simple problem: sandpile growth on a plane support; comparison of recent differential models.

• Equilibria and asymptotic behavior of dynamical models.

• Numerical schemes in 1D and their properties.

• 2D extensions and examples.

• Developments.
Granular matter (neither solid, nor fluid or gas)
Examples: sand, sugar, rice, gravel, snow, ...

- **microscopic scale**: classical mechanics (single grains)
- **mesoscopic scale**: pattern formation phenomena, instability of homogeneous states
- **macroscopic scale**: formation of dunes or avalanches; identical piles have different physical properties if differently built up
Granular matter (neither solid, nor fluid or gas)
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**Self-Organized Critical Systems**
(Bak-Tang-Wiesenfeld '88, Dhar '90, Puhl '92)
Large interactive dynamical systems naturally evolving toward a critical state in which a minor event can lead to a catastrophe
Other applications: dynamic of earthquakes, economic markets, ecosystems, turbulence
Dynamics for granular matter

No universally accepted models; different approaches:

- algebraic-combinatorics (cellular automata)
Dynamics for granular matter

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- algebraic-combinatorics (*cellular automata*)

- stochastic (*particle methods*)
Dynamics for granular matter

No universally accepted models; different approaches:

- algebraic-combinatorics (*cellular automata*)
- stochastic (*particle methods*)
- differential (*ODE/PDE models*)
Sandpile growth on a plane support

Differential approach


Sandpile growth on a plane support: connections with other mathematical research fields

- *Optimal mass transport, Monge-Kantorovich problem*

- *Limit of $p$-Laplacians, infinity Laplacian, absolute minimizers and optimal Lipschitz extensions of given boundary data*

- *Hamilton-Jacobi equations, nonlocal geometric motion*

- *Weak KAM theory for Hamiltonian dynamics*
Sandpile growth on a plane support

NOTATIONS

- $\Omega \subseteq \mathbb{R}^2$: plane bounded support
- $f \geq 0$: vertical source (event. time-dependent), $D_f = \{x : f > 0\}$
- $u(x, t)$, pile height in $x \in \Omega$ at time $t$; $u_0(x)$: initial profile (here $u_0 \equiv 0$)
- $a = \tan(\alpha)$: critical slope (here $a = 1$)
  - $|\nabla u| < a$: sand piles up, $|\nabla u| \geq a$: sand instantly slides
- $d(x) = \text{dist}(x, \partial \Omega)$: distance from $\partial \Omega$
- $S$: cut locus of $\Omega$ (singular set of $d$)
I) Aronsson-Evans-Wu, '96

(Fast/slow diffusion and growing sandpiles)

\[
\begin{align*}
\begin{cases}
\partial_t u_p - \Delta_p u_p &= f & \text{in } R^2 \times (0, \infty), \\
u_p(0) &= 0 & \text{in } R^2
\end{cases}
\end{align*}
\]

\((\Delta_p w \equiv \nabla \cdot (|\nabla w|^{p-2} \nabla w), \ p\text{-Laplacian})\). When \(p \to \infty, u_p \to u\), solution of

\[
\begin{align*}
\begin{cases}
f - \partial_t u &\in \partial I_K[u] & t > 0 \\
u &= 0 & t = 0
\end{cases}
\end{align*}
\]

where \(\partial I_K\) denotes the subdifferential of the indicator function of the convex set \(K = \{v \in W^{1,\infty} : |\nabla v| \leq 1\} \). Equivalent formulation:

\[u \in K, \quad (\partial_t u - f, \phi - u) \geq 0, \quad \forall \phi \in K \quad (V.I.)\]
II) Prigozhin, '93-'96

(Variational model of sandpile growth)

Same conclusions of [AEW], starting from physical considerations:

\( u \) solves the system (equivalent to (V.I.))

\[
\begin{cases}
\partial_t u - \nabla \cdot (v \nabla u) = f & \text{in } \Omega \times (0, T) \\
|\nabla u| \leq 1, \quad |\nabla u| < 1 \Rightarrow v = 0 & \text{in } \Omega \times (0, T) \\
u = 0 & \text{on } \partial \Omega, \quad u(0, \cdot) = 0 & \text{in } \Omega,
\end{cases}
\]

- \( v(x, t) \geq 0 \) is an unknown auxiliary function which plays the role of a dynamic Lagrange multiplier for the constraint on \( \nabla u \).
- Existence and uniqueness of the solution (by penalty method)
- Numerical simulations (implicit semi-discretization in time + linear f.e.m. for stationary V.I. and constrained convex programming)
III) Hadeler-Kuttler, ’99

(Dynamical models for granular matter: a two-layer system)

\( u \) : standing layer, \( v \) : rolling layer, \( u + v \) : pile height

At any point the standing layer \( u \) grows until it reaches the critical slope: then the new sand pours down increasing the \( v \) layer and enlarging the pile base (de Gennes, ’96):

\[
\begin{align*}
\partial_t v &= \nabla \cdot (v \nabla u) - (1 - |\nabla u|)v + f & \text{in } \Omega \times (0, T) \\
\partial_t u &= (1 - |\nabla u|)v & \text{in } \Omega \times (0, T) \\
u(t, \cdot) &= 0 & \text{on } \partial \Omega , \quad u(0, \cdot) = 0 & \text{in } \Omega
\end{align*}
\]

Remark: \(|\nabla u| < 1\) does not imply \(v = 0\) !!
Equilibria

The (V.I.) and the (HK) models (even if with different dynamics) share the same equilibrium configurations, solutions of the system

\[
\begin{cases}
-\nabla \cdot (v\nabla u) = f & \text{in } \Omega \\
|\nabla u| = 1 & \text{in } \{v > 0\} \\
|\nabla u| \leq 1, \quad u, v \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

Remark. The system (E) is not able to determine \( u \) in regions where \( v = 0 \)!
Examples of equilibria 1

Point source \((f = f_0(t)\delta_{d_0}(x))\): a critical cone centered at \(d_0\) grows until it touches \(\partial\Omega\); all the additional sand then flows down from there. The final shape only depends from \(\text{dist}(d_0, \partial\Omega)\).
Examples of equilibria 2

Generic source \((f \geq 0, \ D_f \subset \Omega)\)

Minimal steady state

\[ u_*(x) = \max_{y \in D_f} \Gamma(x, y) \]

where

\[ \Gamma(x, y) = \{d(y) - |x - y|\}^+ \]
Examples of equilibria 2
Examples of equilibria 3

Maximal profile \((f > 0 \text{ on all } \Omega)\): it strongly depends on the shape of \(\Omega\); it is given by \(u(x) = d(x)\), viscosity solution of the eikonal equation

\[
|\nabla u| = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]
Examples of equilibria 3
Characterization of equilibria (Cannarsa-Cardaliaguet ’03)

[ Let $k(x)$ be the curvature of $\partial \Omega$ at the boundary projection of $x \in \Omega$, and $\tau(x) = \min\{t \geq 0 : x + t\nabla d(x) \in \overline{S}\} \]

Thm Let $\partial \Omega \in C^2$, $f \in C^0(\Omega)$; then:

- **Existence:** $(u, v)$ solves (E), where
  
  $$u = d \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \overline{S}$$

  $$v(x) = \int_0^{\tau(x)} f(x + t\nabla d(x)) \frac{1 - (d(x) + t)k(x)}{1 - d(x)k(x)} \, dt, \quad \forall x \in \Omega \setminus \overline{S}$$

- **Almost uniqueness:** if $(u', v')$ is another solution of (E), then
  
  $$v' = v \quad \text{in } \Omega, \quad u' = d \quad \text{in } \{x \in \Omega : v' > 0\}.$$
Asymptotics (when \( f = f(x) \))

\[
\begin{align*}
u_t & \geq 0 \quad , \quad u(\cdot, t) \leq d(\cdot) \quad \Rightarrow \quad u(x, t) \rightarrow \overline{u}(x)
\end{align*}
\]

Case 1: If \( \mathcal{S} \subset D_f \Rightarrow \overline{u} = d \quad \text{(unique solution for (E))} \)
Asymptotics \ (when \ f = f(x))

\[ u_t \geq 0 \ , \ u(., t) \leq d(.) \ \Rightarrow \ u(x, t) \to \overline{u}(x) \]

Case 1: If \ \overline{S} \subset D_f \Rightarrow \ \overline{u} = d \ \ (unique \ solution \ for \ (E))

Case 2: If \ \overline{S} \not\subset D_f \Rightarrow u_* \leq \overline{u} \leq d \ \ (non \ uniqueness)\n\]
\(u_*\) = minimal equilibrium and physical solution

(V.I.) : \ \overline{u} = u_*

(HK) : \ \overline{u} = ?? \ \ \Omega^+ = \{x \in \Omega : \overline{u} > 0\} ?? \ (active \ region)
A FD scheme for the two-layer system

\[ v_i^{n+1} = v_i^n + \Delta t \left[ v_i^n D^2 u_i^n + \tilde{D} v_i^n D u_i^n - (1 - |D u_i^n|) v_i^n + f_i \right] \]

\[ u_i^{n+1} = u_i^n + \Delta t (1 - |D u_i^n|) v_i^n \]

\[ u_i^0 = v_i^0 = 0 \quad (i = 1, \ldots, N) \quad , \quad u_1^n = u_N^n = 0 \quad \forall n. \]

where

\[ D u_i \equiv \maxmod\left(\frac{u_i+1 - u_i}{h}, \frac{u_i - u_i-1}{h}\right) \quad \text{(maxmod difference)} \]

\[ \tilde{D} v_i \equiv \begin{cases} \frac{v_i+1 - v_i}{h} & \text{if } D u_i \geq 0 \\ \frac{v_i - v_i-1}{h} & \text{elsewhere} \end{cases} \quad \text{(upwind difference)} \]
Properties of the FD scheme

Let $f \geq 0$ in $\Omega$, and $\frac{\Delta t}{h} \leq \min \left( \frac{1}{2}, \frac{c}{\|f\|_{\infty}} \right)$; then for any $n$:

- (Positivity and monotonicity in $u$) \hspace{1cm} 0 \leq u^n \leq u^{n+1}$

- (Positivity in $v$) \hspace{1cm} v^n \geq 0

- (Gradient constraint in $u$) \hspace{1cm} |Du^n| \leq 1

$\Rightarrow$ Under the previous stability conditions: $(u^n, v^n) \rightarrow (\bar{u}, \bar{v})$, equilibrium of the discrete system such that $(1 - |D\bar{u}|)\bar{v} = 0$. 
HK-1D : $f \equiv 0.5$, $D_f = [0, 1]$, discrete equilibria $u_h$ and $v_h$. 
HK-1D : \( f \equiv 0.5, \ D_f = [0, 0.4], \) discrete equilibria \( u_h \) and \( v_h \).
**HK-1D:** \( f \equiv 0.5 \), \( D_f = [0.2, 0.4] \cup [0.8, 1] \), discrete equilibria \( u_h \) and \( v_h \).
A monotone scheme for (V.I.)
(discrete transport + projection)

- \( u_i^{n+1/2} = u_i^n + \Delta t \bar{f}_i \) \((i = 2, ..., N - 1)\)

- \( u_i^{n+1} = P_{K_i^n}(u_i^{n+1/2}) \)

- \( u_i^0 = 0 \) \((i = 1, ..., N)\); \( u_1^n = u_N^n = 0 \) \( \forall n \)

where

\[
K_{i}^{n} = \{ v \in \mathbb{R} : |u_{i+1}^{n} - v| \leq h , |u_{i-1}^{n} - v| \leq h \}
\]

\[
P_{K_{i}^{n}}(v) = [(u_{i-1}^{n} \lor u_{i+1}^{n}) - h] \lor [v \land ((u_{i-1}^{n} \land u_{i+1}^{n}) + h)]
\]
\( \tilde{f}_i \) is the **corrected** source term:

\[
\tilde{f}_i = f(x_i) + v(x_i)
\]

where \( v(x_i) \) is the sand which slides from adjacent nodes, *instantly*, if the slope is critical in \( x_i \), to be determined through a control variable computed at each time-step.
(DV) \[ N=51, \Delta x=0.02, \Delta t=0.01, \text{supp}(f)=(0,0.4), \text{it}=841, \text{Tmax}=8.41 \]
VI-1D: \( f \equiv 0.5 \), left: \( D_f = [0, 0.4] \), right: \( D_f = [0.2, 0.4] \cup [0.8, 1] \).
2D Numerical Experiments

- Explicit formulas for the equilibria are not easy to implement

- FD scheme for the two-layer system (HK) easily extends, and again the equilibria are not the minimal ones (larger support)

- The 4-direction trivial extension of the monotone scheme for (V.I.) is highly inefficient: minimal support, but cones become pyramids

- Better results by extending the projection step to 16 directions (with a double stencil)
HK-2D : $\Omega = (0, 1) \times (0, 1)$, $D_f = \Omega$, $N = 101$. 
HK-2D : $\Omega = (0, 1) \times (0, 2)$, $D_f = \Omega$, $N = 101$, $M = 201$. 
HK-2D: $\Omega = (0, 1)^2$, $D_f = [0.45, 0.55]^2$, $N = 51$, $u_h$ versus $u_\ast$. 
HK-2D: left: $\Omega = (0, 1) \times (0, 2)$, right: $\Omega = (0, 1) \times (0, 4)$, $D_f$ small, $N = 51$. 

*soluzione numerica $u_h$*
HK-2D : \( \Omega = (0, 1) \times (0, 1) \), \( D_f \subset \Omega \), \( N = 51 \), \( u_h \) versus \( u_\ast \).
HK-2D : $\Omega = (0, 1) \times (0, 1)$, $D_f \subset \Omega$, $N = 51$, $v_h$ versus $v_\ast$. 
Future developments

• Precise stability and convergence properties for the schemes in 1D and 2D

• More accurate 2D schemes, and comparison with the direct implementation of the closed formula (efficient algorithms for the computation of $d$ are available from the control theory, [Falcone et al.])

• Schemes for different model problems (silos, collapsing sand-piles, obstacles, mixtures, river networks) and related theories (optimal mass transport, geometric evolution of fronts)
VI-2D : 16-directions stencil.
VI-2D : $\Omega = (0,1)^2$, $D_f = [.4,.6]^2$, $N = 81$, $u_h$ (16 dir) versus $u_\ast$. 
VI-2D : $\Omega = (0, 1)^2$, $D_f = [.4, .6]^2$, $N = 81$, level lines for $u_h$ (16 dir) and $u_\ast$. 
$\Omega = (0, 1) \times (0, 1)$: 3 walls, $N = 21$.

$\Omega = (0, 1) \times (0, 1)$: 2 opposite walls, $N = 21$. 
$\Omega = (0, 1) \times (0, 1)$: corner wall, $N = 21$.

$\Omega = (0, 1) \times (0, 2)$: corner wall, $N = 21$. 
$\Omega = (0,1) \times (0,1)$: one wall, $N = 21$.

$\Omega = (0,1) \times (0,2)$: one wall, $N = 21$. 