\[
\begin{aligned}
\rho_t + \text{div}(\rho \vec{v}) &= 0 \\
\vec{v} &= -\nabla K * \rho
\end{aligned}
\]

\(\rho(x, t)\): density
\(\vec{v}(x, t)\): velocity field
\(x \in \mathbb{R}^d, t > 0\)
Aggregation Equation

\[ \begin{align*}
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\end{align*} \]

\( K : \mathbb{R}^d \to \mathbb{R} \)

“interaction potential”

\( \rho(x, t) : \text{density} \)
\( \vec{v}(x, t) : \text{velocity field} \)
\( x \in \mathbb{R}^d, \ t > 0 \)

\( -\nabla K : \mathbb{R}^d \to \mathbb{R}^d \)

“attracting field”
Aggregation Equation

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\begin{aligned}
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\(K : \mathbb{R}^d \rightarrow \mathbb{R}\)

“interaction potential”

\(\vec{v}(x, t)\): velocity field
\(x \in \mathbb{R}^d, \ t > 0\)

\(\rho(x, t)\): density

\(-\nabla K : \mathbb{R}^d \rightarrow \mathbb{R}^d\)

“attracting field”

For which interaction potentials do we get finite time blowup?
Aggregation for particles

One particle attracted by a fixed location $x = a$

$$\dot{X} = -\nabla K(X - a)$$

Multiple particles attracted by one another

$$\dot{X}_i = -\sum_{j \neq i} m_j \nabla K(X_i - X_j)$$
Continuum model

\[ \rho(x, t) = \text{density of particle at time } t \]

\[
\dot{X}_i = - \sum_{j \neq i} \nabla K(X_i - X_j) \ m_j
\]

\[ \vec{v}(x) = - \int_{\mathbb{R}^d} \nabla K(x - y) \ \rho(y) \, dy \]
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So \[ \vec{v} = -\nabla K \ast \rho \]

\[ \begin{cases} 
\rho_t + \text{div} (\rho \vec{v}) = 0 \\
\vec{v} = -\nabla K \ast \rho 
\end{cases} \]
Patlak-Keller-Segel model for chemotaxis

Model collective motion of cells which are attracted by self-emitted chemical substance (and move with Brownian motion).

\[ \rho(x, t) \]: density of cells
\[ c(x, t) \]: density of chemical substance
Cells move toward region with high concentration of chem.

\[ \rho_t + \text{div} (\rho \vec{v}) = \Delta \rho \]
\[ \vec{v} = \nabla c \]
\[ c_t - \Delta c = \rho \]
• Cells moves toward region with high concentration of chem.

\[ \rho_t + \text{div} (\rho \, \vec{v}) = \Delta \rho \]
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• Chemical substance “moves” faster than cells

\[ \rho_t + \text{div} (\rho \, \vec{v}) = \Delta \rho \]
\[ \vec{v} = \nabla c \]
\[ - \Delta c = \rho \quad (\Rightarrow c = N \ast \rho) \]
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So we get:

\[ \rho_t + \text{div} (\rho \vec{v}) = \Delta \rho \]
\[ \vec{v} = \nabla N \ast \rho \]
Interaction energy

\[ E_K(\rho) = \frac{1}{2} \iint K(x - y) \, \rho(x) \, dx \, \rho(y) \, dy \]

\[ = \frac{1}{2} \iint k(|x - y|) \, \rho(x) \, dx \, \rho(y) \, dy \]

\[ K(x) = k(|x|) \]

\[ 0 \leq E_K(\rho) \leq \frac{1}{2} k(\text{diam}) \]

\[ E_K(\rho) = 0 \iff \rho = \delta_0 \]
Interaction energy

\[ E_K(\rho) = \frac{1}{2} \iint K(x - y) \rho(x) \rho(y) \, dx \, dy \]

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\[ E_K(\rho) = 0 \iff \rho = \delta_{x_0} \]
\[
E_K(\rho) = \frac{1}{2} \iint K(x - y) \rho(x) \rho(y) \, dx \, dy
\]

\[
\frac{d}{dt} E_K(\rho) = -\int \rho |\vec{v}|^2 \, dx
\]
\[ E_K(\rho) = \frac{1}{2} \int \int K(x - y) \rho(x) \rho(y) \, dx \, dy \]

\[ \frac{d}{dt} E_K(\rho) = - \int \rho |\vec{v}|^2 \, dx \]

center of mass = \[ \int \vec{x} \rho(x, t) \, dx \]

\[ \frac{d}{dt} \int \vec{x} \rho(x, t) \, dx = 0 \]
Summary

\[ \rho_t + \text{div} (\rho \vec{v}) = 0 \]

\[ \vec{v} = -\nabla K * \rho \]

**Question:** What is the (sharp) condition on the potential in order to have finite time blowup?
\[ \dot{X} = -\nabla K(X - a) \]

Question: how long does it take for a particle to reach the bottom of a fixed potential?

\[
\begin{cases}
\dot{r} = -k'(r) \\
r(0) = L
\end{cases}
\]

Answer: \[ T = \int_0^L \frac{dr}{k'(r)} \]

because to move by a distance \( dr \), it takes the particle a time \( \frac{dr}{k'(r)} \).
Main result

Sharp condition on the interaction potential in order to get blowup

If \( \int_0^L \frac{dr}{k'(r)} = +\infty \), then we have global existence in
\[
C([0, \infty), L^1 \cap L^p) \cap C^1([0, \infty), W^{-1,p}) \quad \text{for } p > \frac{d}{d-1}.
\]

\( L^1 \cap L^\infty \) (Bertozzi, C., Laurent; Nonlinearity (2009))
\( L^1 \cap L^p \) (Bertozzi, Laurent, Rosado; preprint)

If \( \int_0^L \frac{dr}{k'(r)} < +\infty \), then \( \rho(t) \to \delta x_0 \) in finite time.

(C., Di Francesco, Figalli, Slepcev, Laurent; preprint UAB)
The Osgood condition is sharp in the class of potential satisfying

- $\exists \delta > 0$ such that $k''(r)$ is monotone in $(0, \delta)$

- $\exists \delta > 0$ such that $rk''(r)$ is monotone in $(0, \delta)$
Why do we work with densities in $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1}$?

\[ \rho_t + \text{div}(\rho \vec{v}) = 0 \]
\[ \vec{v} = -\nabla K * \rho \]

$\nabla K \in W^{1,q}(\mathbb{R}^d)$ for $q < d$

$\rho \in L^p$ and $\nabla K \in W^{1,q}$

$\nabla K * \rho \in C^1$
Why do we work with densities in $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1}$?

\[ \begin{aligned}
\nabla K &\in W^{1,q}(\mathbb{R}^d) \text{ for } q < d \\
\rho &\in L^p \quad \text{and} \quad \nabla K \in W^{1,q} \\
\Rightarrow \quad \nabla K * \rho &\in C^1
\end{aligned} \]

\[ \begin{array}{l}
\left\{ \\
\rho_t + \text{div} (\rho \vec{v}) = 0 \\
\vec{v} = -\nabla K * \rho
\end{array} \]

$\Rightarrow$ Local existence
\[ \int_0^L \frac{dr}{k'(r)} = +\infty \implies \text{global existence} \]

- We want an apriori bound of \( \|\rho(t)\|_{L^p} \) for all time.
\[ \int_0^L \frac{dr}{k'(r)} = +\infty \quad \Rightarrow \quad \text{global existence} \]

- We want an apriori bound of \( \|\rho(t)\|_{L^p} \) for all time.

- The only thing we can use is that solutions of
  \[ \dot{r} = -k'(r) \]
  can not go to 0 in finite time.
\[ \int_0^L \frac{dr}{k'(r)} = +\infty \quad \Rightarrow \quad \text{global existence} \]

- We want an apriori bound of \( \|\rho(t)\|_{L^p} \) for all time.

- The only thing we can use is that solutions of

\[ \dot{r} = -k'(r) \]

can not go to 0 in finite time.

- **How to do it?**

Show that \( \frac{1}{\|\rho(t)\|_{L^p}} \) can not go to 0 in finite time.
$$\frac{d}{dt} \left\{ \frac{1}{\| \rho(t) \|_{L^p}^{q/d}} \right\} \geq -c \ k' \left( \frac{1}{\| \rho(t) \|_{L^p}^{q/d}} \right)$$

$$\dot{y} = -c \ k'(y)$$
\[
\frac{d}{dt} \left\{ \frac{1}{\|\rho(t)\|_{L^p}^{q/d}} \right\} \geq -c \quad k' \left( \frac{1}{\|\rho(t)\|_{L^p}^{q/d}} \right)
\]

\[
\dot{y} = -c \quad k'(y)
\]

By Gronwall inequality:

\[
\frac{1}{\|\rho(t)\|_{L^p}^{q/d}} \geq y(t)
\]
\[
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By Gronwall inequality:

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\frac{1}{\|\rho(t)\|_{L^p}^{q/d}} \geq y(t)
\]

But \(y(t)\) can not go to 0 in finite time.
Main result

Sharp condition on the interaction potential in order to get blowup

- If \( \int_0^L \frac{dr}{k'(r)} = +\infty \), then we have global existence in
  \[ C([0, \infty), L^p) \cap C^1([0, \infty), W^{-1,p}) \quad \text{for} \quad p > \frac{d}{d-1}. \]
  (Bertozzi, C., Laurent; Nonlinearity (2009))

- If \( \int_0^L \frac{dr}{k'(r)} < +\infty \), then \( \rho(t) \to \delta x_0 \) in finite time.
  (C., Di Francesco, Figalli, Slepcev, Laurent; preprint UAB)
Gradient flow of

\[ E_K(\rho) = \frac{1}{2} \int \int K(x - y) \rho(x) \rho(y) \, dx \, dy \]

with respect to the Wasserstein distance.
What is the Wasserstein distance?

The Wasserstein distance is a distance on the space of probability measure.

Example: What is the Wasserstein distance between $\rho_1$ and $\rho_2$?
Two piles of sand!

Energy needed to transport $m$ kg of sand from $x = a$ to $x = b$:

$$
\text{energy} = m |a - b|^2
$$

$d_W^2(\rho_1, \rho_2) = \text{minimum of the total energy to transport } \rho_1 \text{ to } \rho_2$
Two piles of sand!

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\[d_W^2(\rho_1, \rho_2) = \int |x - T(x)|^2 d\rho_1(x)\]
Transporting measures:

Given $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ measurable, we say that $\nu = T \# \mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all measurable sets $K \subset \mathbb{R}^d$, equivalently...
Transporting measures:

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$$\int_{\mathbb{R}^d} \varphi \, d\nu = \int_{\mathbb{R}^d} (\varphi \circ T) \, d\mu$$

for all $\varphi \in C_0(\mathbb{R}^d)$. 

---

\textsuperscript{a} C. Villani, AMS Graduate Texts (2003).
Transporting measures:

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Monge-Kantorovich-Rubinstein-Wasserstein... Distance:

$$d^2_W(\mu, \nu)$$
Transporting measures:

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for all \( \varphi \in C_c(\mathbb{R}^d) \).

Monge-Kantorovich-Rubinstein-Wasserstein... Distance:

\[
d^2_W(\mu, \nu) = \inf_{\pi} \left\{ \int\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x, y) \right\}
\]
Definition of the distance

Transporting measures:

Given $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ measurable, we say that $\nu = T \# \mu$, if $\nu[K] := \mu[T^{-1}(K)]$ for all measurable sets $K \subset \mathbb{R}^d$, equivalently

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Monge-Kantorovich-Rubinstein-Wasserstein... Distance:

$$d^2_W(\mu, \nu) = \inf_\pi \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x, y) \right\}$$

where the transference plan $\pi$ runs over the set of joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$. 
Three examples

\[ d^2_W(\delta_a, \delta_b) = |a - b|^2 \]

\[ d^2_W(\rho, \delta x_0) = \int |x_0 - y|^2 \, d\rho(y) \]

\[ = \text{Var} (\rho) \]
A discrete time approximation of the PDE is obtained by solving a sequences of variational problem.

- Choose a time step $\Delta t$. 
A discrete time approximation of the PDE is obtained by solving a sequence of variational problems.

- Choose a time step $\Delta t$.
- Solve

$$\rho_{k+1} = \arg \min_{\rho \in P_2(\mathbb{R}^d)} \left\{ \frac{1}{2 \Delta t} d_W^2(\rho, \rho_k) + E_K(\rho) \right\}$$

As $\Delta t \to 0$ it converges to the solution of a weak form of

$$\rho_t + \text{div}(\rho \vec{v}) = 0$$

$\vec{v} = -\nabla K^* \rho$ (see "Gradient Flow in Metric Spaces" by Ambrosio, Gigli, Savaré)
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(see ”Gradient Flow in Metric Spaces” by Ambrosio, Gigli, Savaré)
Let $K$ be a locally attractive potential such that is $\lambda$-convex:

$$K(x) - \frac{\lambda}{2}|x|^2$$

is convex.

Let $\partial K(x)$ be the (possibly multivalued) subdifferential of $K$ at the point $x$, namely the set

$$\partial K(x) = \{ \kappa \in \mathbb{R}^d : K(y) - K(x) \geq \kappa \cdot (y - x) + o(|x - y|) \ \forall \ y \in \mathbb{R}^d \}.$$

Let $\partial^0 K(x)$ be the element of $\partial K(x)$ with minimal norm. Our assumptions, $\partial^0 K(x) = \nabla K(x)$ for all $x \neq 0$ and $\partial^0 K(0) = 0$.

A vector field $w \in L^2(d\mu)$ is said to be an element of the subdifferential of $E_K$, i.e. $w \in \partial E_K$, if

$$E_K[\nu] - E_K[\mu] \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} w(x) \cdot (y - x) \, d\gamma(x, y) + o(dW(\nu, \mu))$$

for all $\gamma \in \Gamma_o(\mu, \nu)$.
Sub-differential Characterization

Let $K$ be a locally attractive potential such that is $\lambda$-convex: $K(x) - \frac{\lambda}{2}|x|^2$ is convex.

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J. A. Carrillo
Blowup in multidimensional aggregation equations
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for all $\gamma \in \Gamma_o(\mu, \nu)$. 
Sub-differential Characterization

Characterization of Sub-differential

Given a locally attractive potential, the vector field

\[ \kappa(x) := \int_{y \neq x} \nabla K(x - y) \, d\mu(y) \equiv (\partial^0 K * \mu)(x) \]

is the unique element of the minimal subdifferential of \( E_K \), i.e.

\[ \partial^0 K * \mu = \partial^0 E_K[\mu]. \]
An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a \textit{weak measure solution} with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 K \ast \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$
\int_0^t \int_{\mathbb{R}^d} \varphi_t(x, \tau) \, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x, 0) \, d\mu_0(x) = \\
\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \varphi(x, t) \cdot \partial^0 K(x - y) \, d\mu(t)(x) \, d\mu(t)(y),
$$

for all test functions $\varphi \in C_c^\infty([0, t) \times \mathbb{R}^d)$. 
Gradient Flow Solution

Concept of Solution

An absolutely continuous curve $\mu : [0, +\infty) \ni t \mapsto \mathcal{P}_2(\mathbb{R}^d)$ is said to be a weak measure solution with initial datum $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if and only if $\partial^0 K * \mu \in L^2(\mu(t))$ a.e. $\tau \in (0, t)$ and

$$\int_0^t \int_{\mathbb{R}^d} \varphi_t(x, \tau) \, d\mu(t)(x) + \int_{\mathbb{R}^d} \phi(x, 0) \, d\mu_0(x) =$$

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \varphi(x, t) \cdot \partial^0 K(x - y) \, d\mu(t)(x) \, d\mu(t)(y),$$

for all test functions $\varphi \in C^\infty_c([0, t) \times \mathbb{R}^d)$.

More refined, it is a gradient flow-type solution:

$$v(t) = -\partial^0 E_K[\mu(t)] = -\partial^0 K * \mu(t), \quad \|v(t)\|_{L^2(\mu(t))} = |\mu'(t)| \text{ a.e. } t > 0$$

with $\mu(0) = \mu_0$ and $v(t)$ is the tangent vector to the curve $\mu(t)$ with minimal norm.
Energy equality is satisfied:

\[
\int_a^b \int_{\mathbb{R}^d} |v(t)(x)|^2 \, d\mu(t)(x) \, dt + E_K[\mu(a)] = E_K[\mu(b)]
\]

holds for all \(0 \leq a \leq b < \infty\).
Well-posedness of Gradient Flow Solutions

Energy equality is satisfied:

\[ \int_{a}^{b} \int_{\mathbb{R}^d} |\nu(t)(x)|^2 \, d\mu(t)(x) \, dt + E_K[\mu(a)] = E_K[\mu(b)] \]

holds for all \(0 \leq a \leq b < \infty\).

**d_W-Expansion**

Given two gradient flow solutions \(\mu^1(t)\) and \(\mu^2(t)\) in the sense of the theorem above, then

\[ d_W(\mu^1(t), \mu^2(t)) \leq e^{-\lambda t} \, d_W(\mu^1_0, \mu^2_0) \]

for all \(t \geq 0\). In particular, we have a unique gradient flow solution for any given \(\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)\).
JKO scheme gives measure solutions.
Put particle model and continuum model in the same framework

- If the initial data is

\[ \rho_0 = \sum_{i=1}^{N} m_i \delta_{X_i} \]

- Then the solution given by the JKO scheme (as \( \Delta t \to 0 \)) is

\[ \rho(t) = \sum_{i=1}^{N} m_i \delta_{X_i(t)} \]

where

\[ \dot{X}_i = -\sum_{j \neq i} m_j \nabla K(X_i - X_j) \]
Proof of blowup using the particle model

We want to prove that

\[ \int_0^L \frac{dr}{k'(r)} < +\infty, \quad \implies \quad \rho(t) \to \delta_{x_c} \text{ in finite time} \]
Proof of blowup using the particle model

We want to prove that

\[ \int_0^L \frac{dr}{k'(r)} < +\infty, \Rightarrow \rho(t) \to \delta_{x_c} \text{ in finite time} \]

Find a bound (independent of the nb. of particles) for the time it takes for all the particles to arrive at \( X_0 \).
\[ \dot{X}_i = -\sum_{j \neq i} m_j \nabla K(X_i - X_j) = -\sum_{j \neq i} m_j \frac{X_i - X_j}{|X_i - X_j|} k'(|X_i - X_j|) \]
\[ \dot{X}_i = - \sum_{j \neq i} m_j \nabla K(X_i - X_j) = - \sum_{j \neq i} m_j \frac{X_i - X_j}{|X_i - X_j|} k'(|X_i - X_j|) \]

\[ \frac{d}{dt} R(t)^2 = \frac{d}{dt} |X_i|^2 = 2 \dot{X}_i \cdot X_i = -2 \sum_{j \neq i} m_j \frac{(X_i - X_j) \cdot X_i}{|X_i - X_j|} k'(|X_i - X_j|) \]
\[
\dot{X}_i = - \sum_{j \neq i} m_j \nabla K(X_i - X_j) = - \sum_{j \neq i} m_j \frac{X_i - X_j}{|X_i - X_j|} k'(|X_i - X_j|)
\]

\[
\frac{d}{dt} R(t)^2 = \frac{d}{dt} |X_i|^2 = 2 \dot{X}_i \cdot X_i = -2 \sum_{j \neq i} m_j \frac{(X_i - X_j) \cdot X_i}{|X_i - X_j|} k'(|X_i - X_j|)
\]

\[(X_i - X_j) \cdot X_i \geq 0 \text{ for all } j \quad \text{Assume } \frac{k'(r)}{r} \text{ decreasing}
\]

\[
\frac{d}{dt} R(t)^2 \leq -2 \frac{k'(2R(t))}{2R(t)} \sum_{j \neq i} m_j (X_i - X_j) \cdot X_i
\]
\[ \dot{X}_i = - \sum_{j \neq i} m_j \nabla K(X_i - X_j) = - \sum_{j \neq i} m_j \frac{X_i - X_j}{|X_i - X_j|} k'(|X_i - X_j|) \]

\[
\frac{d}{dt} R(t)^2 = \frac{d}{dt} |X_i|^2 = 2 \dot{X}_i \cdot X_i = -2 \sum_{j \neq i} m_j \frac{(X_i - X_j) \cdot X_i}{|X_i - X_j|} k'(|X_i - X_j|) 
\]

\[(X_i - X_j) \cdot X_i \geq 0 \text{ for all } j \quad \text{Assume } \frac{k'(r)}{r} \text{ decreasing} \]

\[
\frac{d}{dt} R(t)^2 \leq -2 \frac{k'(2R(t))}{2R(t)} \sum_{j \neq i} m_j (X_i - X_j) \cdot X_i 
\]

\[
\sum_{j \neq i} m_j (X_i - X_j) \cdot X_i = \sum_{j} m_j (X_i - X_j) \cdot X_i = |X_i|^2 = R(t)^2 
\]
\[ \dot{X}_i = -\sum_{j \neq i} m_j \nabla K(X_i - X_j) = -\sum_{j \neq i} m_j \frac{X_i - X_j}{|X_i - X_j|} k'(|X_i - X_j|) \]

\[ \frac{d}{dt} R(t)^2 = \frac{d}{dt} |X_i|^2 = 2 \dot{X}_i \cdot X_i = -2 \sum_{j \neq i} m_j \frac{(X_i - X_j) \cdot X_i}{|X_i - X_j|} k'(|X_i - X_j|) \]

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\[ \sum_{j \neq i} m_j (X_i - X_j) \cdot X_i = \sum_j m_j (X_i - X_j) \cdot X_i = |X_i|^2 = R(t)^2 \]

\[ \frac{d}{dt} R(t)^2 \leq -k'(2R(t)) R(t) \]

\[ \frac{d}{dt} R(t) \leq -\frac{1}{2} k'(2R(t)) \]
Confinement

Weak confinement:

\[
\lim_{r \to +\infty} w'(r) \sqrt{r} = +\infty.
\]
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Weak confinement:

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For every \( R > 0 \), there exists \( R_b \geq R \), depending only on \( R \) and \( K \), such that the following holds: Let \( x_i(t) \) be the solution of the ODE system

\[ \dot{x}_i = - \sum_{j \neq i} m_j \nabla W(x_j - x_i), \quad i = 1, \ldots, N, \]

with \( m_j > 0 \), \( \sum_j m_j = 1 \), and \( \sum_j m_j x_j(0) = 0 \).
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If \( |x_i(0)| \leq R \) for \( i = 1, \ldots, N \) then

\[
|x_i(t)| \leq R_b \quad \text{for all } t > 0 \text{ and } i = 1, \ldots, N.
\]
Finite time blow-up: estimate of the size of the support.

\[
\frac{d}{dt} R(t) \leq -\frac{1}{2} k'(2R(t))
\]

The behavior of \( \dot{y} = -k'(y) \) determine whether or not we have finite time blow up.

Control of the ODE systems uniformly on the number of particles by \( d_2 \)-stability leads to the main results.
Just for fun!

\[ \rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2 \Delta t} d_W^2(\rho, \rho_k) + E_K(\rho) \right\} \]

What does it happen when you put particles in the JKO scheme?
\[ \rho_{k+1} = \arg \min_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ \frac{1}{2 \Delta t} d_W^2(\rho, \rho_k) + E_K(\rho) \right\} \]

What does it happen when you put particles in the JKO scheme?

**Theorem:** they remain particles if

- \( K(x) - \frac{\lambda}{2} |x|^2 \) is convex
- \( \Delta t < \frac{1}{\lambda} \)