Scaling limits for microscopic interface models *

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Plan of talk

1. Effective microscopic interface models
   1.1. 2D Ising model \Rightarrow \text{Effective interface models}
   1.2. $\nabla \varphi$ interface model

2. Scaling limits
   2.1. Static theory
       - Large deviation principle
       - Free boundary problem (Alt-Caffarelli)
       - Wulff shape, Winterbottom shape
   2.2. Dynamic theory — Hydrodynamic limit
       - Motion by mean curvature with anisotropy
       - Evolutionary variational ineq. (obstacle)
       - Stochastic PDE with reflection
   2.3. SOS model (motion of Young diagrams)
1.1. 2D Ising model $\Rightarrow$ Effective interface models

- $s = \{s_x\}_{x \in \Lambda_{\ell}} \subset \{0, 1\}^{\Lambda_{\ell}}$: config. on a large box $\Lambda_{\ell} \subset \mathbb{Z}^2$
- $\gamma = \gamma(s)$: contours separating two regions
- Energy (Hamiltonian) of $s$:

$$H(s) = |\gamma| (\text{+ constant})$$

1. Possible config.
2. Neglected config.
(1) Config.'s in Fig. 2 (with bubbles) appear with very small probability at low temperature.

(2) One can disregard such config.'s and assume that only config.'s in Fig. 1 can appear.

(3) Such config.'s \( s \) are represented by height variables
\[
\phi = \{\phi_i\} : [-\ell, \ell] \cap \mathbb{Z} \to \mathbb{Z}
\]
which measure distances of \( \gamma \) from one fixed reference axis (hyperplane).

(4) For such \( \phi \), the energy \( H(s) \) has another form:
\[
H(\phi) = \sum_{\langle i,j \rangle \subset [-\ell, \ell] \cap \mathbb{Z}} |\phi_i - \phi_j| (+ \text{ constant})
\]
• The model of random interfaces $\phi$ with the energy $H$ is called **SOS (Solid on Solid) model**.

• $\nabla \varphi$ interface model is obtained by replacing "$\phi_i \in \mathbb{Z}$" with "$\phi_i \in \mathbb{R}$", and $|\phi_i - \phi_j|$ with $V(\phi_i - \phi_j)$.  
1.2. $\nabla \varphi$ interface model

Surface (interface) in $\mathbb{R}^{d+1}$ separating $A/B$ phases (no hang-over nor bubble)

$D \subset \mathbb{R}^d$: macroscopic region (reference hyperplane)

$D_N = ND \cap \mathbb{Z}^d$: discrete microscopic region

$N$: size of microscopic system

$\phi_i$: height of micro interface (of atomic level) at $i \in D_N$
Surface Energy of $\phi = \{\phi_i\}_{i \in D_N} : D_N \to \mathbb{R}$

$$H(\phi) = \sum_{\langle i,j \rangle \subset D_N} V(\phi_i - \phi_j)$$
(with boundary condition)

Equilibrium State (Gibbs measure):

$$d\mu = \frac{1}{Z} e^{-H(\phi)} \prod_{i \in D_N} d\phi_i$$
$Z$: normalization

$V: \mathbb{R} \to \mathbb{R}$, potential
2. Scaling Limits.

**Micro:** Large-scale system

\[ \downarrow \quad \text{local equilibria (local ergodicity)} \]

**Macro:** Variational principle, Nonlinear PDE

Static theory: **Scaling limit in space**

Hydrodynamic limit: **Space-time scaling limit**
2.1. Static theory

Scaling (Micro $\rightarrow$ Macro):

$$h^N(x) = \frac{1}{N} \phi[Nx], \quad x \in D$$

Surface Tension: $\sigma = \sigma(u), \ u \in \mathbb{R}^d$

$\sigma$ = macro energy for a surface with slide $u$

(determined by averaging or ergodicity)
• Large deviation principle (Deuschel-Giacomin-Ioffe ’00)

\[ \mu \left( h^N \sim h \right) \underset{N \to \infty}{\sim} \exp\left\{ -N^d \Sigma_D(h) \right\}, \quad h : D \to \mathbb{R}, \text{ given} \]

\[ \Sigma_D(h) = \int_D \sigma(\nabla h(x)) \, dx \quad (+\text{const}) \]

: Total surface tension

• Add weak pinning (self potential) \( W \) to \( \mu \) (F-Sakagawa ’04):

\[ \mu^W \left( h^N \sim h \right) \sim \exp\left\{ -N^d \Sigma_D^W(h) \right\} \]

\[ \Sigma_D^W(h) = \Sigma_D(h) - A \left| \{ x \in D ; h(x) \leq 0 \} \right| \quad (+\text{const}) \]

\[ A = W(+\infty) - W(-\infty) \]

\( \mu^W \): Gibbs measure defined from

\[ H^W(\phi) = H(\phi) + \sum_{i \in D_N} W(\phi_i) \]
(1) LLN under $\mu^W$: $h_N \rightarrow \bar{h}$: minimizer of $\Sigma_D^W$

(2) Such variational problems were discussed by Alt-Caffarelli ('81), A-C-Friedman ('84), Weiss ('95), ... Free boundary problem (Young’s relation)

(3) Wulff Shape (under Wall+Volume constraint)
Winterbottom Shape (under Wall+Volume constraint +$\delta$-pinning, Bolthausen-Ioffe '97)
2.2. Dynamic theory.

\( \phi(t) = \{\phi_i(t)\}_{i \in D_N} \): Microscopic dynamics (with equilibrium state \( \mu \)) is introduced via Langevin equation (\( \equiv \) SDEs):

\[
d\phi_i(t) = -\frac{\partial H}{\partial \phi_i}(\phi(t))\,dt + \sqrt{2}\,dw_i(t), \quad i \in D_N,
\]

where

\[
\frac{\partial H}{\partial \phi_i}(\phi) = \sum_{j:|i-j|=1} V'(\phi_i - \phi_j)
\]

\( \{w_i(t)\} \): independent Brownian motions

Space-time diffusive scaling:

\[
h^N(t, x) = \frac{1}{N} \phi_{[Nx]}(N^2t)
\]
• **Hydrodynamic limit**: \( h^N(t, x) \rightarrow h(t, x) \) The limit \( h(t, x) \) is a unique weak solution of the nonlinear PDE (Motion by mean curvature with anisotropy, F-Spohn '97):

\[
\frac{\partial h}{\partial t}(t) = \text{div} \{ \nabla \sigma(\nabla h(t)) \}
\]

\[
\equiv \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \frac{\partial \sigma}{\partial u_k}(\nabla h(t)) \right\}
\]

\[\iff\quad \text{Gradient flow for } \Sigma_D:\]

\[
\frac{\partial h}{\partial t} = -\frac{\delta \Sigma_D}{\delta h(x)}(h)
\]

\[\sigma(u) = \sqrt{1 + |u|^2} \Rightarrow \text{MMC}\]
Dynamics on a wall: \( \phi(t) = \{\phi_i(t) \geq 0\}_{i \in D_N} \) (with repulsion from a wall put at level 0):

\[
d\phi_i(t) = \left[ -\frac{\partial H}{\partial \phi_i}(\phi(t)) dt + \sqrt{2} dw_i(t) \\
+ \frac{1}{N} f\left(\frac{t}{N^2}, \frac{x}{N}, \frac{\phi_i(t)}{N}\right) dt + d\ell_i(t) \right]
\]

\( \phi_i(t) \geq 0, \quad \ell_i(t) \nearrow, \quad \int_0^\infty \phi_i(t) d\ell_i(t) = 0 \)

- \( \ell_i(t) \) increases only when \( \phi_i(t) = 0 \) \( (d\ell_i(t) \sim \delta_0(\phi_i(t)) dt) \)
- \( f = f(t, x, h) \): macroscopic external field
Hydrodynamic limit (on $D = \mathbb{T}^d$, F '03):

The limit $h(t, x)$ is a unique solution of the Evolutionary Variational Inequality (MMC with reflection (obstacle)):

$V = H^1(\mathbb{T}^d), V' = H^{-1}(\mathbb{T}^d), (\cdot, \cdot) =_{V'} (\cdot, \cdot)_V$ or $(\cdot, \cdot)_{L^2(\mathbb{T}^d)}$

$h \in L^2(0, T; V), \frac{\partial h}{\partial t} \in L^2(0, T; V') \quad \forall T > 0,$

$\left(\frac{\partial h}{\partial t}(t), h(t) - v\right) + (\nabla \sigma(\nabla h(t)), \nabla h(t) - \nabla v) \leq \left(f(t, h(t)), h(t) - v\right), \text{ a.e. } t$

$\forall v \in V : v \geq 0,$

$h(t, x) \geq 0 \quad \text{a.e.,}$

$h(0, x) = h_0(x)$
• **Equilibrium fluctuation** on $D = [0, 1]$ (F-Olla '01)

$$u^N(t, x) := \sqrt{N}h^N(t, x) (\geq 0) \Rightarrow u(t, x)$$

The limit $u(t, x)$ is a unique weak stationary solution of the **SPDE with reflection** (Nualart-Pardoux type):

$$\frac{\partial u}{\partial t}(t, x) = q\frac{\partial^2 u}{\partial x^2}(t, x) + \dot{B}(t, x) + \xi(t, x)$$

$$u(t, x) \geq 0, \quad \int_0^\infty \int_0^1 u(t, x) \xi(dt dx) = 0$$

$$u(t, 0) = u(t, 1) = 0, \quad \xi: \text{random measure}$$

where $\dot{B}(t, x)$ is a space-time white noise and $q > 0$. (We assume $f = 0$).

• **Dynamic large deviation** and Relation to static LD (F-Nishikawa '01)
2.3. SOS decreasing surfaces, 2D Young diagrams *

- Vershik ('96) discussed scaling limits for 2D Young diagrams under several types of statistics and derived the so-called Vershik curves in the limit.

- Our goal is to establish corresponding dynamical theory by means of hydrodynamic limit.

- Our model is viewed as describing a motion of (decreasing) interfaces in 2D space, called SOS dynamics.

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2.3.1. Ensembles for 2D Young diagrams

2.3.2. Scaling limits, Vershik curves

2.3.3. Dynamics (**WAZRP**,** WASEP** with stochastic reservoir at boundary)

2.3.4. Hydrodynamic limits

2.3.5. Problems
2.3.1. Ensembles for 2D Young diagrams

- Partitions of integers:
  \[ \mathcal{P}_n = \{ p = (p_1, p_2, \ldots) \mid p_1 \geq p_2 \geq \cdots \geq 0, n(p) = n \}, \]
  \[ n(p) := \sum_{i=1}^{\infty} p_i \]
  \[ \mathcal{Q}_n = \{ q = (q_1, q_2, \ldots) \in \mathcal{P}_n \mid q_i > q_{i+1} \text{ if } q_i > 0 \}, \]
  \[ \mathcal{P} = \bigcup_n \mathcal{P}_n, \quad \mathcal{Q} = \bigcup_n \mathcal{Q}_n \]

- For \( p \in \mathcal{P} \), we associate a Young diagram:
  \[ \psi_p(u) := \sum_{i=1}^{\infty} 1_{\{u < p_i\}}, \quad u > 0. \]
\[ p \in \mathcal{P}: \]

\[ n(p) = 22 \]

\[ p = (7, 6, 4, 4, 1, 0, 0, \ldots) \]

\[ q \in \mathcal{Q}: \]

\[ n(q) = 21 \]

\[ q = (7, 6, 4, 3, 1, 0, 0, \ldots) \]
• **canonical ensembles:**
  Uniform statistics (U-statistics)
  \[ \mu^n_U := \text{uniform prob. meas. on } \mathcal{P}_n \]
  Restricted uniform statistics (RU-statistics)
  \[ \mu^n_R := \text{uniform prob. meas. on } \mathcal{Q}_n \]

• **grandcanonical ensembles:** \( 0 < \varepsilon < 1 \)
  U-statistics \( \mu^n_\varepsilon(p) := \frac{1}{Z_\varepsilon^n} \varepsilon^n(p), \quad p \in \mathcal{P} \)
  RU-statistics \( \mu^n_\varepsilon(q) := \frac{1}{Z_\varepsilon^n} \varepsilon^n(q), \quad q \in \mathcal{Q} \)
2.3.2. Scaling limits

- For $N > 0$, define $\varepsilon = \varepsilon(N) = \varepsilon_U(N), \varepsilon_R(N)$ by
  \[ E^{\mu_U}[n(p)] = N^2, \quad E^{\mu_R}[n(q)] = N^2. \]
  (i.e., the averaged areas of $\mathcal{YD} = N^2$). Then,
  \[ \varepsilon_U(N) = 1 - \frac{\alpha}{N} + \cdots, \quad \alpha = \frac{\pi}{\sqrt{6}}, \]
  \[ \varepsilon_R(N) = 1 - \frac{\beta}{N} + \cdots, \quad \beta = \frac{\pi}{\sqrt{12}}. \]
  (cf. Hardy-Ramanujan’s formula: $\#\mathcal{P}_n \sim \frac{1}{4\sqrt{3n}}e^{2\alpha\sqrt{n}}$)

- Scaling for Young diagrams: For $p \in \mathcal{P}$,
  \[ \tilde{\psi}^N_p(u) := \frac{1}{N}\psi_p(Nu), \quad u > 0. \]
  (i.e., the averaged areas of scaled $\mathcal{YD} = 1$).
Proposition 1. (Vershik, ’96, LLN under $\mu_U^{\epsilon(N)}, \mu_R^{\epsilon(N)}$)

$$\tilde{\psi}_p^N(u) \quad \overset{N \to \infty}{\longrightarrow} \quad \psi_U(u) \quad \text{in prob. under } \mu_U^{\epsilon(N)},$$

$$\tilde{\psi}_q^N(u) \quad \overset{N \to \infty}{\longrightarrow} \quad \psi_R(u) \quad \text{in prob. under } \mu_R^{\epsilon(N)},$$

where

$$\psi_U(u) = \frac{1}{\alpha} \log (1 - e^{-\alpha u}),$$

$$\psi_R(u) = \frac{1}{\beta} \log (1 + e^{-\beta u}), \quad u \geq 0.$$  

The limit shapes are called the Vershik curves.

Remark 1.

(1) Similar results hold under canonical ensembles.

(2) $y = \psi_U(u) \Leftrightarrow e^{-\alpha u} + e^{-\alpha y} = 1$, $y = \psi_R(u) \Leftrightarrow e^{\beta y} - e^{-\beta u} = 1$.  

23(r)
Vershik curves
2.3.3. Dynamics invariant under grandcanonical ensembles
\[\xi(x) := \#\{i; p_i = x\} \quad (\equiv \psi_p(x - 1) - \psi_p(x): \text{gradient of } \psi_p),\]
\[\eta(x) := \#\{i; q_i = x\}, \quad x \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}.\]
• U-statistics: $\xi(x) \in \mathbb{Z}_+, x \in \mathbb{N} = \{1, 2, \ldots\}, \xi(0) = \infty$

Weakly asymmetric zero-range process with weakly asymmetric stochastic reservoir at $x = 0$, i.e.,

“one-particle jump rate $x \mapsto x + 1” = \varepsilon(\equiv \varepsilon(N)), x \in \mathbb{Z}_+$

(i.e. a square is created on the surface of YD at interval $[x, x + 1]$)

“one-particle jump rate $x \mapsto x - 1” = 1, x \in \mathbb{N}$

(i.e. a square on the surface at $[x - 1, x]$ is eliminated)

• RU-statistics: $\eta(x) \in \{0, 1\}, x \in \mathbb{N}, \quad \eta(0) = \infty$

Weakly asymmetric simple exclusion process with weakly asymmetric stochastic reservoir at $x = 0$, i.e., same jump rates as above under exclusion rule.
2.3.4. Hydrodynamic limit

Diffusive scaling in space and time:

- **U-case**
  \[ \xi_t = \{\xi_t(x)\}_{x \in \mathbb{Z}_+} \longrightarrow p_t \in \mathcal{P} \longrightarrow \psi_{p_t}(u) \]
  \[ \longrightarrow \tilde{\psi}_p^N(t, u) \text{ defined by} \]
  \[ \tilde{\psi}_p^N(t, u) := \tilde{\psi}_{p_{N2t}}^N(u) \equiv \frac{1}{N} \psi_{p_{N2t}}(Nu), \quad u > 0. \]

- **RU-case**
  \[ \eta_t \longrightarrow q_t \in \mathcal{Q} \longrightarrow \tilde{\psi}_q^N(t, u) \text{ defined similarly.} \]
Theorem 2. (U-case) If $\tilde{\psi}_p^N(0, u) \xrightarrow[N \to \infty]{} \psi_0(u)$, then

$$\tilde{\psi}_p^N(t, u) \xrightarrow[N \to \infty]{} \psi_U(t, u) \text{ in prob.}$$

The limit $\psi_U(t, u)$ is a solution of nonlinear PDE:

$$\partial_t \psi = \{\psi'/(1 - \psi')\}' + \alpha \psi'/(1 - \psi'), \quad u > 0,$$

$$\psi(0, \cdot) = \psi_0(\cdot),$$

$$\psi(t, 0+) = \infty, \quad \psi(t, \infty) = 0,$$

where $\partial_t \psi = \partial \psi/\partial t$, $\psi' = \partial \psi/\partial u (< 0)$.

Remark 2. Vershik curve $\psi_U$ is a stationary sol of this PDE.
Theorem 3. (RU-case) If $\tilde{\psi}_q^N(0, u) \xrightarrow{N \to \infty} \psi_0(u)$, then

$$\tilde{\psi}_q^N(t, u) \xrightarrow{N \to \infty} \psi_R(t, u) \text{ in prob.}$$

The limit $\psi_R(t, u)$ is a solution of nonlinear PDE:

$$\partial_t \psi = \psi'' + \beta \psi'(1 + \psi'), \quad u > 0,$$

$$\psi(0, \cdot) = \psi_0(\cdot),$$

$$\psi'(t, 0+) = -\frac{1}{2}, \quad \psi(t, \infty) = 0.$$

Remark 3. Vershik curve $\psi_R$ is a stationary sol of this PDE.
2.3.5. Problems

- Surface diffusion: conservative dynamics (associated with the canonical ensembles)

- 3D case: Limit shapes of scaled surfaces under static ensembles are studied by Cerf, Kenyon, Okounkov. Dynamics associated with the grandcanonical ensemble can be constructed: **Weakly asymmetric simple dimer process on a honeycomb lattice.**
End of slides. Click [END] to finish the presentation.