Dispersive methods and application to the Vlasov-Poisson equation.

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Introduction and goals

Vlasov-Poisson equation in 3 dimensions:

- Model the evolution in phase space (position, velocity), of a system of interacting particles in a force field E, the intensity of which depends on the particle density in the whole space.
- Nonlinear equation
- The dynamic of these equations is formally given by the trajectories associated to the equations.

Goals

- Build new dispersive tools for the equation TF.
- Apply those tools to prove news existence and uniqueness results for weak solutions of the Vlasov-Poisson equation.

dispersive methods

Simple case of the free transport equation ($F = 0$).

$$\partial_t f + \xi \cdot \nabla_x f = 0$$

General principle:
Exploits the way the solution spreads across space with time ⇒

- Regular solutions with rough initial data (gain of smoothness)
- At the cost of time averaging and space localization

Main use: These estimates are very useful to prove existence and uniqueness results for nonlinear dispersive equations.

Three categories of dispersive estimates

- 1. Dispersion estimates (works for very few dispersive estimates)
- 2. Strichartz estimates (average in time)
- 3. Moments effects (average in time and localize in space)

1 ⇒ 2 and 1 ⇒ 3; but the reciprocal is false.

One of the major difficulties is to prove 2. or 3. for dispersive equations for which 1. does not hold.

Dispersion estimate

- Conversation: $\|f(t)\|_{L^p_{x,\xi}} = \|f^{in}\|_{L^p_{x,\xi}}$
- Dispersion: $\|f(t)\|_{L^\infty_x(L^1_\xi)} \leq |t|^{-d}\|f^{in}\|_{L^1_x(L^\infty_\xi)}$

Estimate used to study the Vlasov-Poisson equation (propagation of moments (Lions-Perthame))

Strichartz estimates (F. Castella and B. Perthame).

Strichartz estimates	conservative properties
$\|f\|_{L_t^p(L_x^q(L_\xi^r))} \leq C\|f^{in}\|_{L^a}$	$\|f\|_{L_t^\infty(L_x^q(L_\xi^q))} \leq C\|(1+\|\xi\|)^q f^{in}\|_{L_{x,\xi}^q}$
$\frac{2}{p} = d\left(\frac{1}{r} - \frac{1}{q}\right),\ \frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{q}\right),\ p > 2 \geq a.$	$\|f\|_{L_t^\infty(L_x^q(L_\xi^r))} \leq C\|(1+\|\xi\|)^{\frac{2}{p}} f^{in}\|_{L_{x,\xi}^q}$

- Win $\frac{2}{p}$ moments with respect to the Hölder's inequalities.
- Estimates only used for a chemotaxis model by Bournaveas, Calvez, Gutierrez, Perthame (2005).

Moments effect.

$$\| |\xi|^{\frac{1}{p}} \gamma f \|_{L^p_{t,x,\xi}} \leq C \| f^{in} \|_{L^p_{x,\xi}}$$

- Estimates a lot less than stringent than ponctual dispersion
- Gain moments if we accept to be localized in space and if we average over time.
- Estimates used for the Vlasov-Poisson equation by Gasser, Jabin and Perthame for the propagation of moments (2003).

Known results: existence

Basic properties inspired by corresponding laws of physics

- Mass conservation and the Liouville principle i.e for all $t \in \mathbb{R}$, $p \in [1, +\infty]$

$$\|f(t)\|_{L^p_{x,\xi}} = \|f^{in}\|_{L^p_{x,\xi}}$$

- energy conservation

$$\int \frac{|\xi|^2}{2} f(t,x,\xi)dxd\xi \pm \int \frac{1}{2}|E(t,x)|^2 dx = C.$$

$\Rightarrow$ Global existence of weak solutions

$$f^{in} \in L^1_{x,\xi} \cap L^\infty_{x,\xi} \quad \text{et} \quad \int \frac{|\xi|^2}{2} f^{in}(x,\xi)dxd\xi < +\infty \tag{1}$$

(A. A. Arsen'ev, E. Hörst et R. Hunze)

• global existence of renormalized solutions (DiPerna, Lions) for $f^{in} \in L^1_{x,\xi}$, $f^{in} log(f^{in}) \in L^1_{x,\xi}$ and $|\xi|^2 f^{in} \in L^1_{x,\xi}$.

The question is hence, are those weak solutions unique?

Known results: uniqueness

We don't know if uniqueness of solutions can be proved using only estimates inspired by physics
$\Rightarrow$ We have to deal with another information of the solution.
Sufficient condition given by Loeper in 2005.

Theorem (G. Loeper)

Let f^{in}be a bounded positive measure and let $T > 0$. Then, there exists at much one solution of the Vlasov-Poisson equaiton such that

$$\rho \in L^\infty([0, T] \times \mathbb{R}^3).$$

$\Rightarrow$ We are reduced to proving that the density ρ is bounded.

There exists two methods

- 1. Propagation of moments (Lions-Perthame 1992)
- 2. Combine dispersive estimates for transport equation with rough force fields and the propagation of moments (main theorem)

Uniqueness via the propagation of moments

Theorem (Lions-Perthame)

Assume that $f^{in} \in L^\infty_{x,\xi}$ and that for $m > 3$

$$\|(1+|\xi|)^{m_0} f^{in}\|_{L^1_{x,\xi}} < +\infty \quad \text{pour tout} \quad m_0 < m.$$

Then, there exists a weak solution of the V-P equation such that for all $T > 0$

$$\sup_{t\in[0,T]} \|(1+|\xi|)^{m_0} f(t)\|_{L^1_{x,\xi}} < C(T) \quad \text{and}$$

$$E \in \mathcal{C}(\mathbb{R}^+)(L^q(\mathbb{R}^3)) \quad \text{if} \quad \frac{3}{2} < q < \frac{3(3+m)}{6-m} \quad \text{et} \quad m < 6$$

$$E \in \mathcal{C}(\mathbb{R}^+)(\mathcal{C}^\alpha(\mathbb{R}^3)) \quad \text{if} \quad \alpha < \frac{m-6}{3+m} \quad \text{and} \quad m > 6\cdot$$

The case $m = 6$ is critical:

- if $m < 6$ then the nonlinear term E is rough and not bounded
- if $m > 6$,then the nonlinear term E is regular.

The propagation of moments allows one to deal with the case $m > 6$.

Uniqueness via the propagation of moments

For $m > 6$ Lions and Perthame proved that the trajectories of the V-P are **a little perturbation of those given by the free transport equation**

$$X(t,x,\xi) = x + t\xi + R_1(t,x,\xi) \quad \text{et} \quad V(t,x,\xi) = \xi + R_2(t,x,\xi)$$

where

$$R_1(t,x,\xi) = \int_0^t (t-s)E(s,X(s))ds, \quad R_2(t,x,\xi) = \int_0^t E(s,X(s))ds.$$

If $m > 6$ $E \in L^\infty$ then, for all $T > 0$ and $t \in [0,T]$

$$\|R_1(t)\|_{L^\infty_{x,\xi}} \leq |t|^2 C(T) \text{ et } \|R_2(t)\|_{L^\infty_{x,\xi}} \leq |t| C(T).$$

Uniqueness via the propagation of moments

Theorem (Lions-Perthame)

Assume that $f^{in} \in L^\infty_{x,\xi}$ and that

$$\|(1+|\xi|)^{6+0} f^{in}\|_{L^1_{x,\xi}} < +\infty.$$

Assume moreover that for all $R > 0$, $T > 0$

$$supess\Big\{ f^{in}(y+t\xi, w), |x-y| \leq Rt^2, |\xi - w| \leq Rt \Big\} \in L^\infty([0,T] \times \mathbb{R}^3_x(L^1_\xi)). \tag{2}$$

Then, there exists a solution of the V-P equation such that for all $T > 0$, $\rho \in L^\infty([0,T] \times \mathbb{R}^3_x)$. Hence, by the Theorem of G. Loeper, this solution is unique

Main theorem

This theorem gives an existence and uniqueness result for the critical case $m = 6$ i.e when E is rough (m not too small).

Theorem

Let $\infty + > p \geq 3$ and $f^{in} \in L^{\infty}_{x,\xi}$. Then, there exists $m(p) < 6$ such that if

$$\|(1 + |\xi|)^{m(p)} f^{in}\|_{L^1_{x,\xi}} < +\infty$$

then the trajectories are a little perturbation of thise of the free tranport equation

$$|X(t,x,\xi) - x - t\xi| \leq C|t|^{\frac{1}{p'}+1} \quad \text{and} \quad |V(t,x,\xi) - \xi| \leq C|t|^{\frac{1}{p'}} \tag{3}$$

If, moreover $T > 0$, $R > 0$,

$$sup\Big\{f^{in}(y + t\xi, w), |x - y| \leq R|t|^{\frac{1}{p'}+1}, |\xi - w| \leq R|t|^{\frac{1}{p'}}\Big\} \in L^{\infty}_{T}(\mathbb{R}^3_x(L^1_\xi)) \tag{4}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ then, there exists a unique solution of the V-P such that $\rho \in L^{\infty}([0,T] \times \mathbb{R}^3_x)$ for all $T > 0$.
We can choose p such that $m(p) < 6 - \frac{1}{2}$.

Idea of the proof

We have reduced the study of the trajectories and (X, V) to proving that

$$\sup_{x_0,\xi_0} \int_0^t |E(s, X(s, x_0, \xi_0)|^p ds \leq C(t).$$

- 1.Moments effects for the transport equation with rough force field $\Rightarrow$ (with α small enough)

  $$\sup_{|B|\leq 1} \|E\|_{L^p_{T_0}(C^\alpha(B))} < +\infty$$

  and obtaining a first approximation of the trajectories of V-P by the free transport if T is small enough.
- 2. Second moment effects along the trajectories of the V-P equation $\Rightarrow$ Theorem for T small enough.
- 3. Interpolation between the results obtained via the dispersive method and the propagation of moments obtained by Lions and Perthame $\Rightarrow$ global estimate in time

Motivation and natural regularity on F.

$$E(t,x) = E^1(t,x) + E^2(t,x)$$

with

$$E^1(t,x) = \int \frac{y}{|y|^3}(1-\gamma)(y)\rho(t,x-y)dy.$$

$$\|E^1\|_{L^\infty(\mathbb{R}\times\mathbb{R}^3)} \le \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^3} \left| \int \frac{y}{|y|^3}(1-\gamma))\rho(x-y)dy \right| \le C\|f^{in}\|_{L^1_{x,\xi}}.$$

We have

$$|E^2(t,x)| \le C\|\tilde{\gamma}\frac{y}{|y|^3}\|_{L^{3/2-0}}\|\gamma(x-\cdot)\rho\|_{L^{3+0}}.$$

$$|E^2(t,x)| \le C\|\gamma(x-\cdot)(1+|\xi|)^{2+0}f\|_{L^{3+0}_{x,\xi}}.$$

We can only hope to control the term

$$S_{T,p}(E) = \sup_{|B|\le 1} \|E\|_{L^p_T(L^\infty(B))}$$

In the following we assume F belongs to $S_{T,p}(F) < +\infty$.

Dispersive results for the equation (TF).

Proposition

Let $T > 0$ and $p \geq 1$ such that $S_{T,p}(F) < +\infty$.Then, there exists C such that, for all $(t_1, t_2) \in [0, T]^2$, and $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$|X(t_1, t_2, x, \xi) - x + (t_1 - t_2)\xi| \leq C(1 + S_{T,p})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}}$$

and

$$|V(t_1, t_2, x, \xi) - \xi| \leq C(1 + S_{T,p})^{1+\frac{1}{p}}(1 + |\xi|)^{\frac{1}{p}}$$

Let $\gamma \in \mathcal{D}(\mathbb{R}^d)$. Then, for all $(q, p) \geq 1$, $\alpha \geq 0$, there exists a contant C such that for all balls $B \subset \mathbb{R}^d$ we have

$$\| \sup_{x \in B} \gamma(x - \cdot)(1 + |\xi|)^{\alpha + \frac{1}{qp'}} f(t)\|_{L^q_{T,x,\xi}} \leq C(1 + S_{T,p})^{1+\beta} \|(1 + |\xi|)^{\alpha} f^{in}\|_{L^q_{x,\xi}}.$$

Idea of the proof:

- We split in time with respect to the velocity ξ on time intervals of size $\frac{1}{|\xi|}$ and we prove that on each of those intervals, the trajectory X leaves in a little ball
- We put everything together

Conclusion of the first step

- If T is small enough, then there exists $m(p) < 6$ such that

$$\sup_{|B|\leq 1} \|E\|_{L^p_T}(C^\alpha(B)) \leq C$$

  for α small enough
- The trajectories of the V-P equation satisty

$$|X(t,x,\xi) - x - t\xi| \leq C(1 + S_{T,p}(E))^{1+\frac{1}{p}}(1+|\xi|)^{\frac{1}{p}}$$

  and

$$|V(t_1,t_2,x,\xi) - \xi| \leq C(1 + S_{T,p}(E))^{1+\frac{1}{p}}(1+|\xi|)^{\frac{1}{p}}.$$

In particular, as ξ grows the approximation with free transport trajectories gets worse ⇒ one must refine this first approximation.

Moments effects along trajectories.

Proposition

let $f^{in} \in L^1_{x,\xi} \cap L^\infty_{x,\xi}$, and $\infty + > p \geq 3$. Assume $S_{p,T}(E) < +\infty$. Then there exists $m(p) < 6$ such that if

$$\|f^{in}\|_{L^{1,m(p)}_{x,\xi}} < +\infty,$$

then there exists some constant C such that for all $(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in [0,T]$,

$$\sup_{(x_0,\xi_0)\in\mathbb{R}^6} \|E^2(s, X(s,x_0,\xi_0))\|_{L^p([0,T])} \leq C.$$

$$|X(t,x,\xi) - x - t\xi| \leq C|t|^{\frac{1}{p'}+1} \quad \text{and} \quad |V(t,x,\xi) - \xi| \leq C|t|^{\frac{1}{p'}}$$

Moments effects along trajectories.

$$|E^2(s,X(s,x_0,\xi_0))| \leq C\|\gamma(X(s,x_0,\xi_0)-\cdot)\rho\|_{L^{3+0}}.$$

We will gain moments on the initial data uing the fact that

$$s \rightarrow X(s,0,x,\xi) - X(0,s,x_0,\xi_0)$$

never stays too long in any compact set.
Examining the (simpler) case of the free transport equation

$$\widetilde{X}(s,x,\xi) : s \rightarrow x - x_0 + s(\xi+\xi_0).$$

- if $\xi \sim -\xi_0 \Rightarrow$ no dispersion
- if ξ far from $-\xi_0 \Rightarrow$ dispersion.

We split

$$\rho(t,x) = \rho^1(t,x) + \rho^2(t,x) \quad \text{with} \quad \rho^1(t,x) = \int_{B(-\xi_0,|\xi_0|^\alpha)} f(t,x,\xi)d\xi.$$

Moments effects along trajectories.

$$\rho(t,x) = \rho^1(t,x) + \rho^2(t,x) \quad \text{with} \quad \rho^1(t,x) = \int_{B(-\xi_0,\beta|\xi_0|^\alpha)} f(t,x,\xi)d\xi.$$

- In the first zone, dispersion is weak, but its area is small ⇒ we gain moments using direct Hölder inequalities
- In the second zone ξ is far enough from $-\xi_0$, that we may prove moments effects.

Small pertubation of V-P trajectories over a suitable time interval.

Global in time estimates.

Problem

$$\sup_{|B|\leq 1} \|E\|_{L^p_{T_0}(L^\infty(B))} \leq C(T, \|f^{in}\|_{L^{1,m(p)}})(1+S_{T,p})^{1+\beta}.$$

We aim for

$$\sup_{|B|\leq 1} \|E\|_{L^p_{T_0}(L^\infty(B))} \leq C(T, \|f^{in}\|_{L^{1,m(p)}})(1+S_{T,p})^{1-0}.$$

We will interpolate the estimates obtained by the dispersive method with the propagation of moments

Problem: The nonlinear term E must be controlled for interpolation ⇒ We localize E in frequency using Littlewood-Paley decomposition

$$\|\Delta_k E\|_{L^\infty(B)} \leq \|\Delta_k E\|^{\theta}_{L^\infty(B)} \|\Delta_k E\|^{1-\theta}_{L^\infty(B)}.$$

there are two ways to estimate $\|\Delta_k E\|_{L^\infty(B)}$.

Estimations globales en temps.

The first method uses propagation of moments (Lions-Perthame) and Bernstein inequalities. For all $q \in]\frac{3}{2}, +\infty[$, there exists $m(q) < 6$ such that if

$$\|f^{in}\|_{L^{1,m(q)}_{x,\xi}} < +\infty,$$

then, for all $T > 0$, for all $k \in \mathbb{N}$

$$\|\Delta_k E\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq C2^{\frac{3k}{q}} \|\Delta_k E\|_{L^\infty_{[0,T]}(L^q)} \leq C2^{\frac{3k}{q}}. \tag{5}$$

Estimate (5) is valuable for two reasons.

- It holds for initial data with strictly less than 6 moments in $L^1_{x,\xi}$.
- The unwanted term $S_{T,p}$ does not come up.

The problem with this estimate is that it bounds E for a norm that is too irregular.

Global in time estimates.

The second method to estimate $\|\Delta_k E\|_{L^\infty(B)}$ uses a dispersive approach. For all $p \in]3, \frac{10+\sqrt{88}}{6}[$, $\exists\ m(p) < 6$ such that if

$$\|f^{in}\|_{L^{1,m(p)}_{x,\xi}} < +\infty,$$

then, for all $T_0 > 0$ and $t \in [0, T_0]$

$$\|\Delta_k E(t)\|_{L^\infty(B)} \leq C 2^{k(-1+\frac{3}{p+0})} \left[\sup_{x \in B} \|\widetilde{\gamma}(x - \cdot\,)\rho(t)\|_{L^{p+0}} + 1 \right] \tag{6}$$