Carrying an inverted pendulum on a bumpy road

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I. SOME BACKGROUND

THE INVERTED PENDULUM: SOME HISTORY

- Stephenson discovered in 1908 that an inverted pendulum (rod *above* the pivot) will not fall down if pivot is subjected to fast *vertical* vibrations.

- Idea useful in ‘stabilizing’ unstable equilibria in electromagnetic fields. Paul Physics Nobel prize 1989.

- Several mathematical approaches are possible (literature is huge); here (as in Paul’s Nobel Prize lecture) we use the theory of the Mathieu equation.

INVERTED PENDULUM EQUATION OF MOTION

• If q is the angle between the rod and the UPWARD vertical through the pivot, the (linearized) equation of motion (in absence of vibrations) is

$$\ell \frac{d^2 q}{dt^2} = +g\, q$$

or, as first order system, $dq/dt = p$, $\ell dp/dt = gq$.

• If $a(t)$ is the (upwards) acceleration of the pivot wrt. laboratory,

$$\frac{d^2}{dt^2} q = \ell^{-1}(g + a(t))q.$$

• Assume that $a(t)$ is sinusoidal

$$a(t) = v_{max}\omega \cos(\omega t), \qquad v_{max} > 0.$$

• The (vertical) pivot velocity $v(t)$ and pivot displacement $s(t)$ are given by

$$v(t) = v_{max} \sin(\omega t), \qquad s(t) = -\frac{1}{\omega} v_{max} \cos(\omega t).$$

• We'll be interested in the case where $\omega \gg 1$; with respect to this large parameter, a, v and s are therefore of sizes $O(\omega)$, $O(1)$ and $O(1/\omega)$ respectively.

• After change of variables $\omega t = 2x$, eqn of motion reads

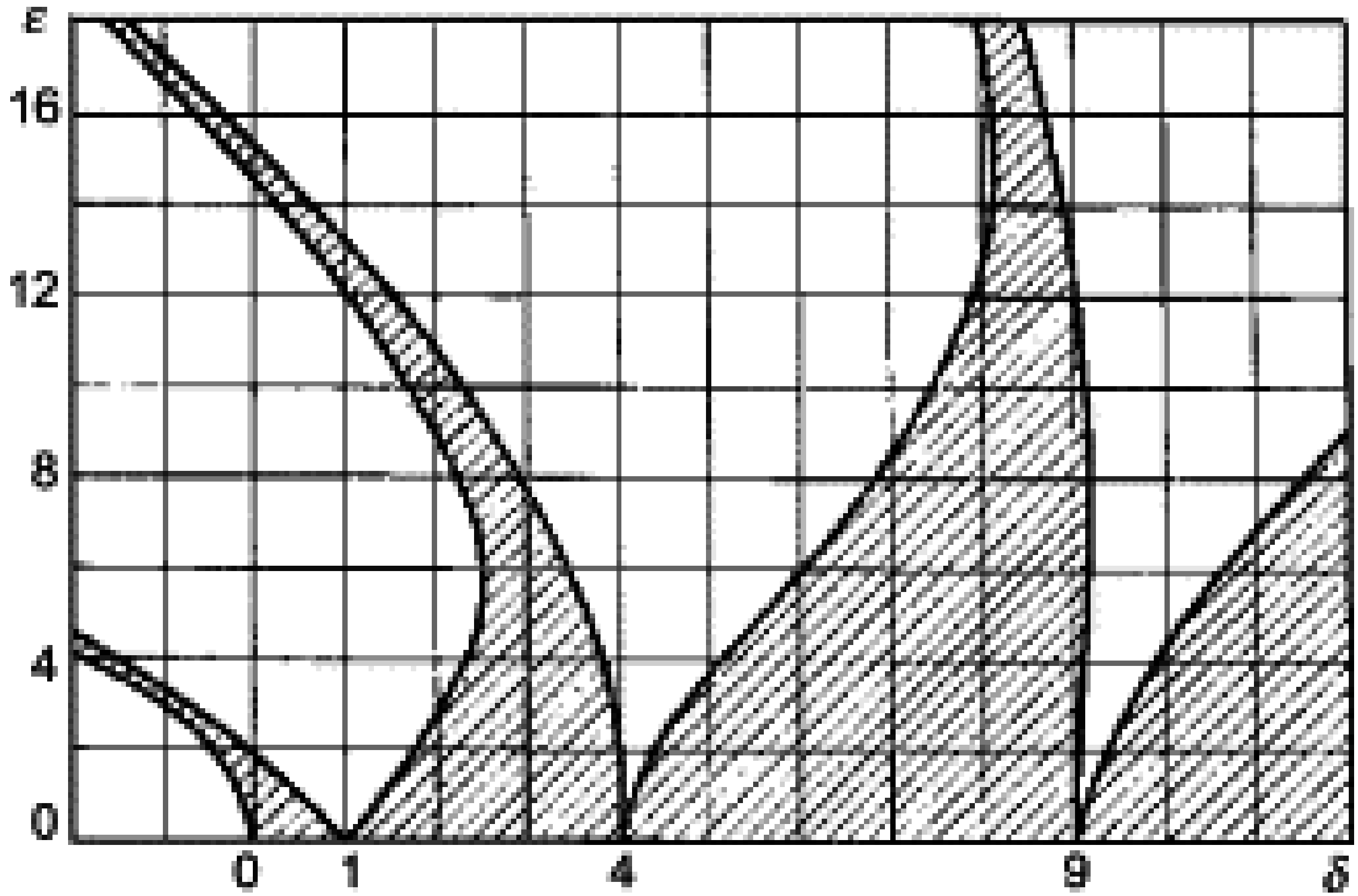
$$\frac{d^2}{dx^2}q + \left(-\frac{4g}{\ell\omega^2} - \frac{4v_{max}}{\ell\omega}\cos(2x)\right)q = 0,$$

an instance of Mathieu's equation [(1868) Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique]

$$\frac{d^2}{dx^2}q + \left(\delta + \epsilon\cos(2x)\right)q = 0.$$

• For $\epsilon \neq 0$, this may be *unstable* even if $\delta > 0$ (parametric resonance/botafumeiro) and *stable* even if $\delta < 0$.

• Stable regions in the plane of the paramenters δ and ϵ appear shaded in the following Strutt diagram [Van der Pol and Strutt (1928)].


ε
16
12
8
4
0
0
1
4
9
δ

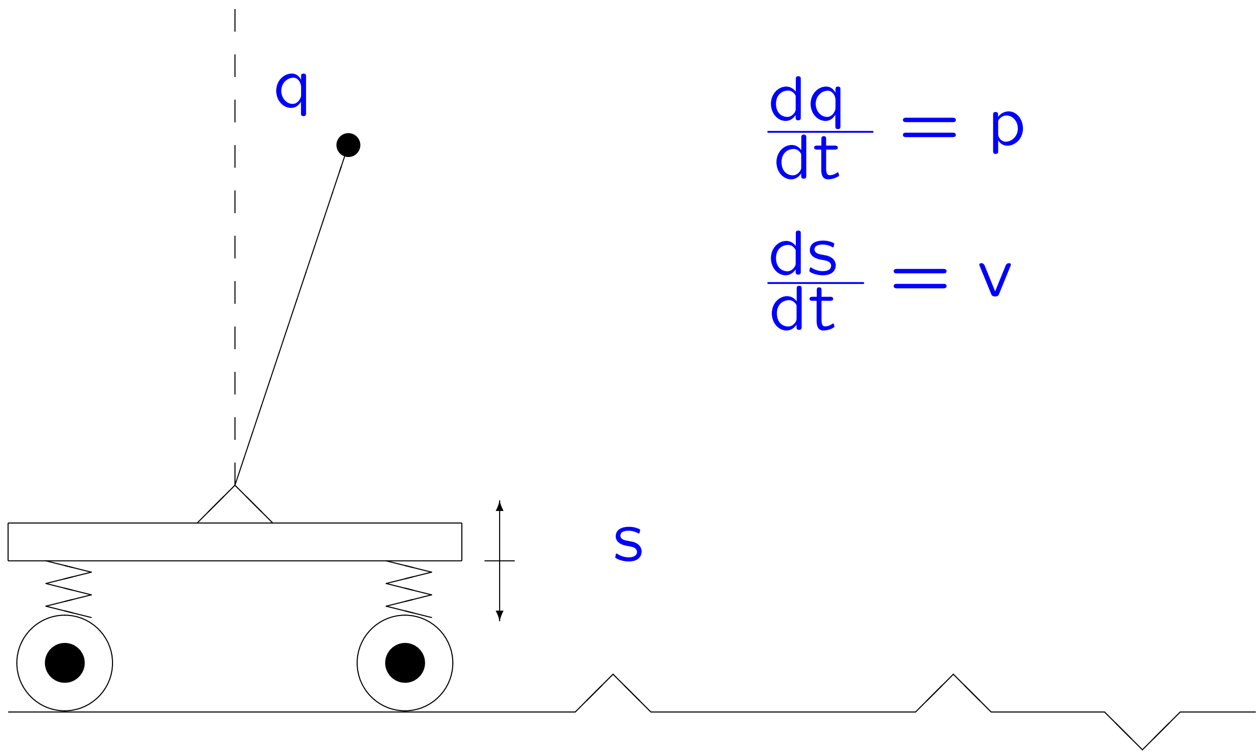
- Near origin there is a small stabilty region where $\delta < 0$. The left boundary of this region is found to be $\delta = -(1/8)\epsilon^2 + o(\epsilon^2)$.

- With $\epsilon = -4v_{max}/(\ell\omega)$ and $\delta = -4g/(\ell\omega^2)$, the conditions $\epsilon \ll 1$ and $\delta \ll 1$ correspond to $v_{max}/\omega \ll \ell$ (small amplitude of pivot vibration) and $\omega \gg \sqrt{(g/\ell)}$ (fast vibration).

- Then stability boundary corresponds to $g < v_{max}^2/(2\ell)$ or (angles mean time-average) $\langle v^2 \rangle/\ell > g$.

- Note value of frequency of vibration ω is irrelevant (provided is large enough). This suggests stabilizing by (random) noise. SS (2008): stabilization by impulses.

II. THE INVERTED PENDULUM CARRIED ON A BUMPY ROAD



$$\frac{dq}{dt} = p$$

$$\frac{ds}{dt} = v$$

CARRIAGE MOTION:

- Carriage receives vertical shocks at times $t_0 = 0 < t_1 < \cdots < t_n < \ldots$ that follow a Poisson process with intensity $1/\tau$.

- Carriage vertical displacement s and velocity v obey

$$\frac{dv}{dt} = -\gamma v - ks, \quad \frac{ds}{dt} = v, \quad t_n < t < t_{n+1}, \quad n = 0, 1, \ldots$$

- At shock times ($n = 0, 1, \ldots$)

$$s(t_n) = s(t_n+) = s(t_n-), \quad v(t_n+) = v(t_n-) + w\theta_n$$

- θ_n id. dist. mutually indpndnt and indpndnt of t_n-process; $E(\theta_n) = 0$, $E(\theta_n^2) = 1$, $E(\theta_n^3) = 0$, $E(\theta_n^4) < \infty$.
- $w > 0$ a parameter.

CARRIAGE STATIONARY PROCESS:

- Can show (by explicitly finding characteristic function) that the equations of motion possess a stationary solution $(v(t-), s(t))$.

- The stationary solution is ergodic.

- The invariant distribution has finite moments of order ≤ 4 and

$$E(v_n) = E(s_n) = 0,$$

and

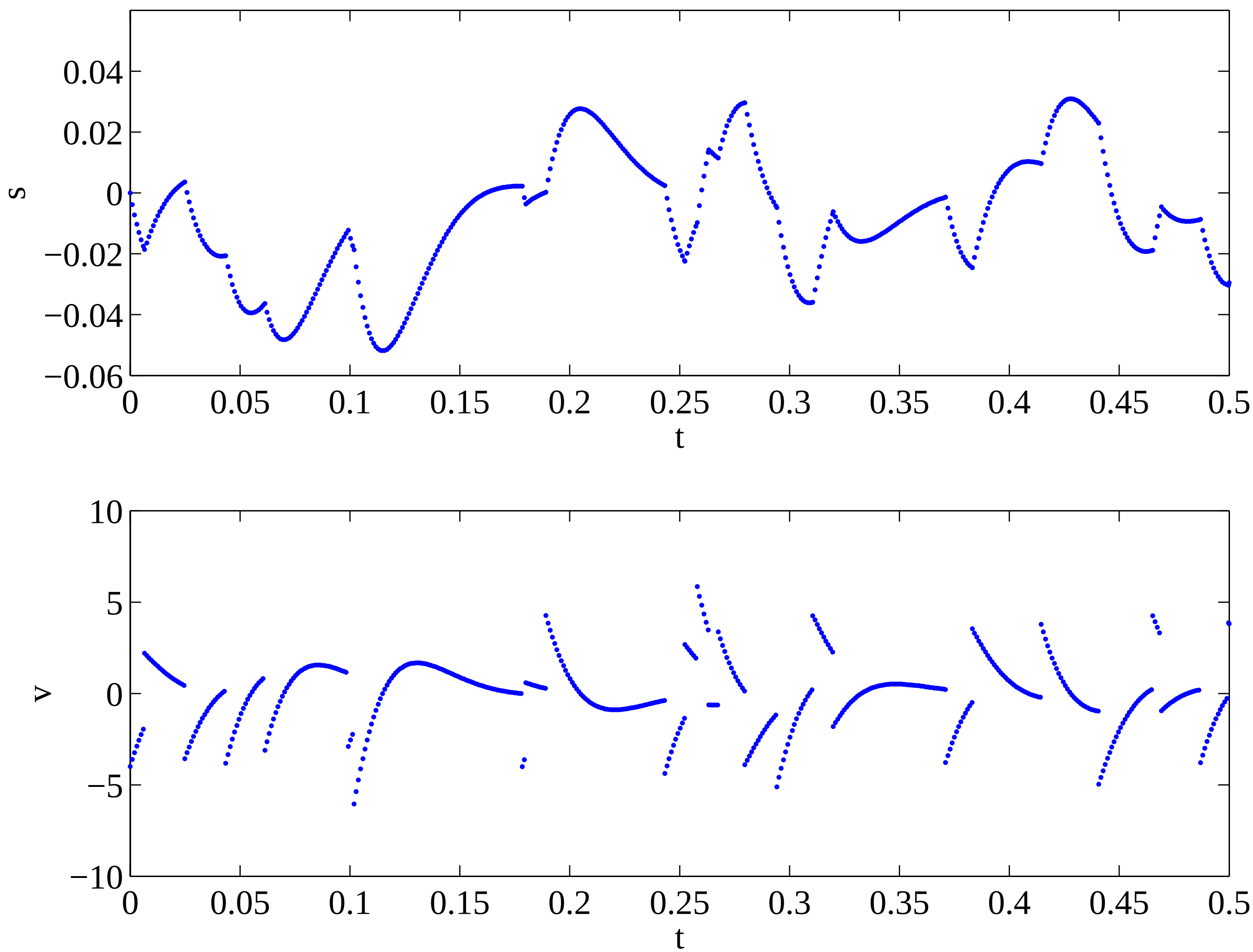
$$E(v_n^2) = \frac{w^2}{2\gamma\tau}, \quad E(v_n s_n) = 0, \quad E(s_n^2) = k^{-1} E(v_n^2).$$

- Hereafter assume that motion of carriage comes from stationary process and furthermore

$$\gamma = \gamma^* \tau^{-1}, \qquad k = k^* \tau^{-2},$$

with γ^* and k^* *constant*. This leaves τ and w as only carriage *parameters*, governing resp. frequency and strength of bumps.

- Numerical simulations have $\theta = \pm 1$ with probability $1/2$ each. Further $\gamma^* = 2$ and $k^* = 2$.



0.04
0.02
0
−0.02
−0.04
−0.06
s
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t
10
5
0
−5
−10
v
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t

PENDULUM MOTION:

- Pendulum differential equations:

$$\frac{d^2q}{dt^2} = -\nu\frac{dq}{dt} + \ell^{-1}(g + a(t))q, \quad t_n < t < t_{n+1}, \quad n = 0, 1, \ldots,$$

or

$$\frac{dp}{dt} = -\nu p + \ell^{-1}(g + \frac{dv}{dt})q, \qquad \frac{dq}{dt} = p, \qquad t_n < t < t_{n+1}$$

(ν friction, g gravity, ℓ rod length. In simulations 5, 9.8, 0.20.)

- At jumping times q remains continuous but

$$p(t_n+) = p(t_n-) + \ell^{-1}\Delta v(t_n)q(t_n), \qquad n = 0, 1, \ldots$$

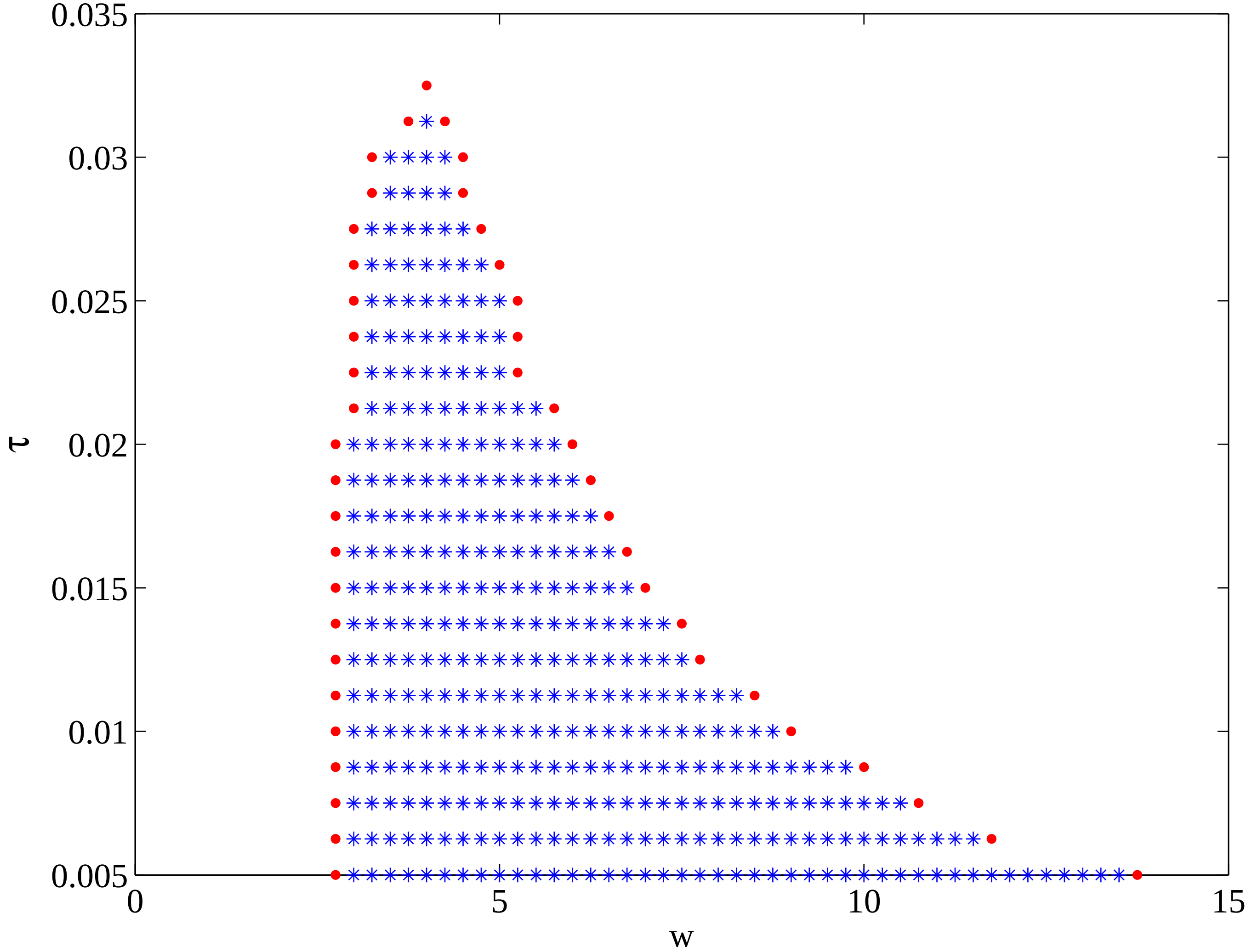
III. SIMULATIONS

- For grid in (w, τ) plane, we computed numerically 50 samples of the process for $0 \leq t \leq 40$.

- A sample was taken to be 'stable' if $\mid q(40) \mid \leq 0.0001 \mid q(0) \mid$.

- A point (w, τ) was taken to be 'stable' if all 50 samples were stable.

- Stable points are shown in the next diagram.


0.035
0.03
0.025
0.02
0.015
0.01
0.005
τ
0
5
10
15
w

IIII. ANALYSIS

Theorem 1. *Fix* w *such that* $w^2/(2\ell\gamma^*) = E(v^2)/\ell > g$, *then, for* τ *sufficiently small,* $q(t)$ *converges exponentially to* 0*:*

$$\limsup_{t\to\infty} \frac{1}{t} \log |q(t)| < 0, \quad \text{a.s.}$$

• *Proof:* Behaviour of paths $(p(t), q(t))$ investigated by sequence of invertible, symplectic, stationary changes of variables. (Cf. Ovseyevich 2006.)

- *First change.* Remove discontinuities from $v(t)$:

$$p = p_1 + \ell^{-1} v q_1, \qquad q = q_1.$$

This yields:

$$\begin{aligned} \frac{dp_1}{dt} &= -\nu p_1 + \ell^{-1}(g - \ell^{-1} v^2) q_1 - \ell^{-1} v p_1 - \nu \ell^{-1} v q_1, \\ \frac{dq_1}{dt} &= p_1 + \ell^{-1} v q_1. \end{aligned}$$

(Note $\ell^{-1} v^2$ opposing gravity.)

• *Second and third changes.* Remove $O(1)$ oscillatory terms $\nu\ell^{-1}vq_1$, $\ell^{-1}vp_1$, $\ell^{-1}vq_1$:

$$p_1 = p_2 - \nu\ell^{-1}sq_2, \qquad q_1 = q_2,$$

and

$$p_2 = \frac{1}{\chi(\ell^{-1}s)}p_3, \qquad q_2 = \chi(\ell^{-1}s)q_3, \qquad \chi(\sigma) = 1 + \sigma\mathrm{sech}(\sigma),$$

(Only need $\chi(\sigma) = 1 + \sigma + O(\sigma^2)$ for small σ.)

This leads to:

$$\begin{aligned} \frac{dp_3}{dt} &= -\nu p_3 + \ell^{-1}(g - \ell^{-1}v^2)q_3 + O(\tau)p_3 + O(\tau)q_3, \\ \frac{dq_3}{dt} &= p_3 + O(\tau)p_3 + O(\tau)q_3. \end{aligned}$$

• *Fourth change.* To change v^2 above into $E(v^2)$ (constant coeff leading terms), we use

$$p_3 = p_4 + S(t)q_4, \qquad q_3 = q_4,$$

with

$$\frac{dS}{dt} = -\ell^{-2}(v^2 - E(v^2)).$$

- Taking

$$S(t) = \int_0^t [-\ell^{-2}(v^2(t') - E(v^2))]dt',$$

does not work.

- Rather define

$$S(t) = \frac{1}{2\ell^2\gamma}(v(t-)^2 + ks(t)^2)) - \frac{E(v^2)}{\ell^2}(J(t) - t),$$

where $J(t)$ is the smallest t_n with $t_n > t$.

- S is *stationary* and $dS/dt = -\ell^{-2}(v^2 - E(v^2))$.

- At jumping times $\Delta S(t_n) = -\Delta M(t_n)$ where M is a *cadlag* process whose paths are constant in the intervals $[t_n, t_{n+1})$. The jumps in S, M are $O(\tau)$.

- Moreover M is a square integrable martingale with respect to the filtration $\mathcal{F}_t$ generated by the variables $(v(t'-), s(t'))$ and $J(t')$, $0 \le t' \le t$.

- After this change of variables:

- At jumps:

$$\Delta p_4(t_n) = -\Delta S(t_n) q_4(t_n) = \Delta M(t_n) q_4(t_n).$$

- Away from jumps:

$$\begin{aligned} \frac{dp_4}{dt} &= -\nu p_4 - \Lambda q_4 + O(\tau) p_4 + O(\tau) q_4, \\ \frac{dq_4}{dt} &= p_4 + O(\tau) p_4 + O(\tau) q_4, \end{aligned}$$

where $\Lambda = -\ell^{-1}(g - \ell^{-1} E(v^2)) > 0$.

• The system for (p_4, q_4) (relabelled (p, q)) is analyzed through the positive-definite Lyapunov function

$$V = \frac{1}{2}p^2 + \frac{\nu}{2}pq + \left(\frac{\nu^2}{4} + \frac{\Lambda}{2}\right) q^2.$$

for which (ξ denotes a stationary, ergodic process)

$$t^{-1}\left(\log V(t+) - \log V(0+)\right) \leq -\kappa + C_1 \tau \frac{1}{t}\int_0^t \xi(\sigma)\, d\sigma$$
$$+ \frac{1}{t}\sum_{0<t_n\leq t} \frac{q\frac{\partial V}{\partial p}}{V}\bigg|_{t_n-} \Delta M(t_n) + C_2 \frac{1}{t}\sum_{0<t_n\leq t} [\Delta M(t_n)]^2.$$

• Integral and last sum obey ergodic theorem. Middle sum is an Ito integral and obeys LLN.

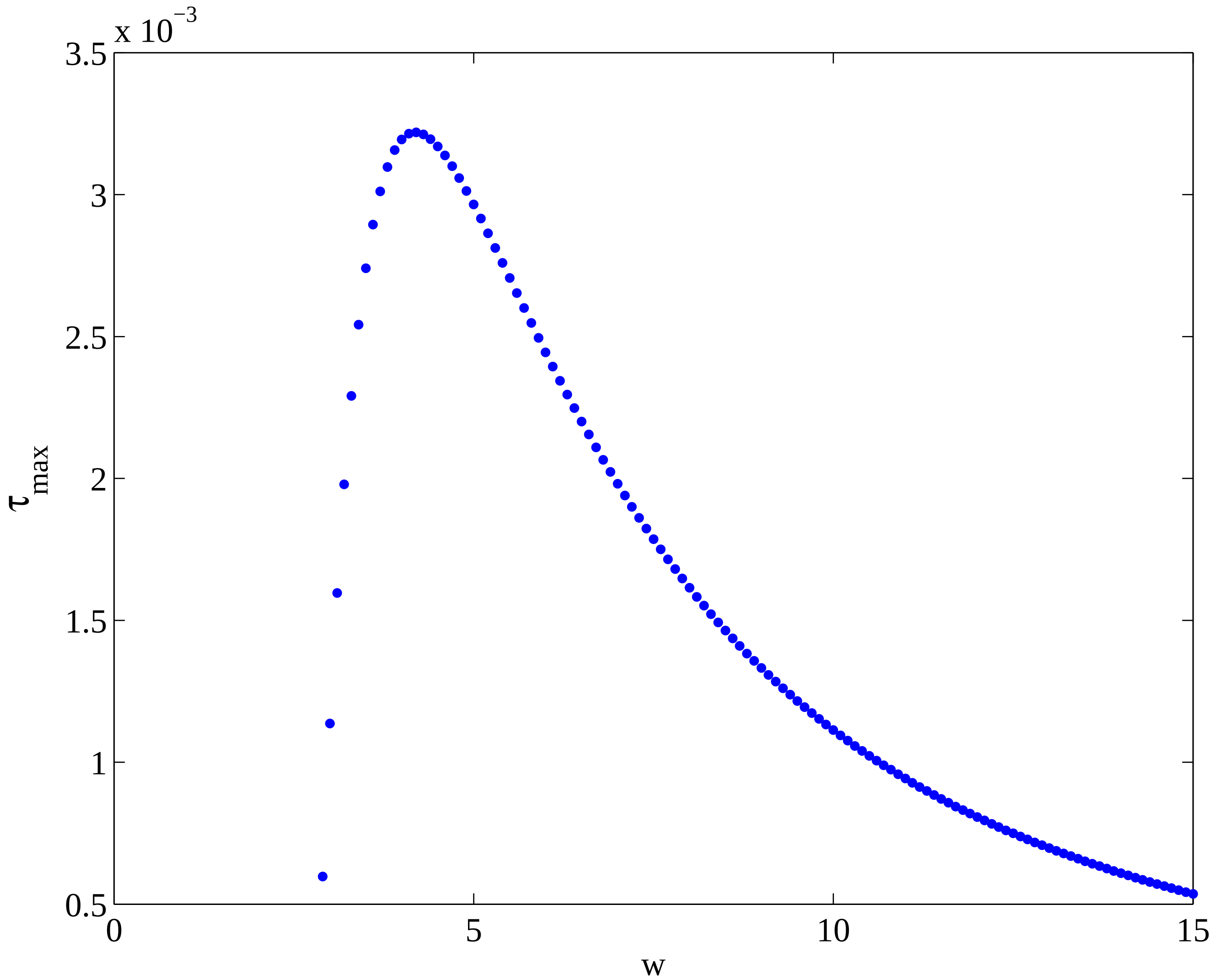
V. ASSESSING THE BOUNDS

• Last sum (with positive terms) most dangerous. Accounts by destabilization by large bumps.

• One finds

$$\begin{aligned} \kappa &= \nu - \frac{\nu^2}{\sqrt{4\Lambda + \nu^2}}, \\ C_2 &= \frac{4}{4\Lambda + \nu^2}, \\ E([M(t_n)]^2) &= \frac{2+\gamma^*}{4\ell^4\gamma^{*3}}\tau^2 w^4. \end{aligned}$$

• Next graph shows combinations of w and τ for which $-\kappa$ overcomes expectation of last sum in estimate.



x 10^−3
3.5
3
2.5
2
1.5
1
0.5
τ_max
0
5
10
15
w