Special Finite Element Shape Function Based on Component Mode Synthesis

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Outline

- Motivation
- Component Mode Synthesis
- New Special Finite Element Method
- Relation to Other Conforming Methods
- Conclusion
Problem of interest

\[
\begin{cases}
-\nabla \cdot (c(x) \nabla u(x)) = f(x) & \text{in } \Omega \\
u(x) = 0 & \text{on } \partial \Omega
\end{cases}
\]

Example:

- subsurface flows in porous media
- thermal conduction in composite materials
- ...

Difficulties arise when the coefficient \( c \) is rough.

Objective

Find a good approximation subspace for the boundary value problem: find $u \in H^1_0(\Omega)$

\[
\begin{cases}
-\nabla \cdot (c(x)\nabla u(x)) = f(x) & \text{in } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Numerous existing techniques (conforming)

- Generalized finite element GFEM (I. Babuska, J. Osborn, M. Melenk, ...)
- Multiscale finite element MFEM (T. Hou, Y. Efendiev, ...)
- ...

Idea: Adapt component mode synthesis techniques
Component mode synthesis

Goal: Compute the first eigenmodes of a structure that can be divided into substructures on which modes are known.

1. Compute a few local eigenmodes in $\Omega_1$ and in $\Omega_2$.
2. Compute a few coupling eigenmodes on $\Gamma$.
3. Approximate the global eigenmodes with the computed modes.

References: R. Craig and M. Bampton (1968), F. Bourquin (1991), J. Bennighof (1998), ...
Partition the domain $\Omega$ into $J$ non intersecting subdomains $\Omega_j, j = 1, \cdots, J$, that share the interface $\Gamma$. 

![Diagram of a grid partitioned into subdomains](image-url)
Component mode synthesis

- Partition the domain $\Omega$ into $J$ non intersecting subdomains $\Omega_j, j = 1, \ldots, J$, that share the interface $\Gamma$
- Denote $V_{\Omega_j} = \{ v \in H^1_0(\Omega) \mid v|_{\Omega \setminus \Omega_j} = 0 \}$.
- Introduce $V_\Gamma = \{ E_\Omega \tau \in H^1_0(\Omega) : \tau \in H^{1/2}_{00}(\Gamma) \}$.

The energy-minimizing extension $E_\Omega$ is defined by

\[
\begin{cases}
- \nabla \cdot (c(x) \nabla E_\Omega \tau(x)) = 0 & \text{in } \Omega_j, \text{ for all } j, \\
E_\Omega \tau = \tau & \text{on } \Gamma, \\
E_\Omega \tau = 0 & \text{on } \partial \Omega.
\end{cases}
\]

or

\[
a(E_\Omega \tau, E_\Omega \tau) = \inf_{\nu \in H^1_0(\Omega)} a(\nu, \nu) \text{ subject to } \nu|_\Gamma = \tau.
\]

where

\[
a(u, v) = \int_\Omega \nabla u(x)^T \cdot c(x) \cdot \nabla v(x) dx
\]
Component mode synthesis

- With \( V_{\Omega_j} = \{ v \in H^1_0(\Omega) : v|_{\Omega\setminus\Omega_j} = 0 \} \) and
  \( V_{\Gamma} = \{ E_\Omega \tau \in H^1_0(\Omega) : \tau \in H^{1/2}_{00}(\Gamma) \} \), we have

  \[
  H^1_0(\Omega) = \left( \bigoplus_{j=1}^J V_{\Omega_j} \right) \oplus V_{\Gamma}
  \]

- The decomposition is **orthogonal** with respect to \( a(\cdot, \cdot) \)

  \[
  a(v_i, v_j) = 0 \quad \forall v_i \in V_{\Omega_i} \text{ and } v_j \in V_{\Omega_j} \ (i \neq j)
  \]

  \[
  a(v_i, v_\Gamma) = 0 \quad \forall v_i \in V_{\Omega_i} \text{ and } v_\Gamma \in V_{\Gamma}
  \]

- The minimization satisfies

  \[
  \min_{v \in H^1_0(\Omega)} \left( \frac{1}{2} a(v, v) - (f, v) \right) = \sum_{j=1}^J \min_{v \in V_{\Omega_j}} \left( \frac{1}{2} a(v, v) - (f, v) \right) + \min_{v \in V_{\Gamma}} \left( \frac{1}{2} a(v, v) - (f, v) \right).
  \]
Component mode synthesis

\[ H_0^1(\Omega) = \left( \bigoplus_{j=1}^{J} V_{\Omega_j} \right) \oplus V_\Gamma \]

- \( V_{CMS} = \left( \bigoplus_{j=1}^{J} \text{span}\{z_{1,j}\} \right) \oplus \text{span}\{z_{n,\Gamma}; 1 \leq n \leq N_\Gamma \} \)
  - \( z_{1,j} \) is the first eigenmode for \( a(\cdot, \cdot) \) in \( V_{\Omega_j} \) (subdomain \( \Omega_j \)).
  - \( z_{n,\Gamma} \) are the first eigenmodes for \( a(\cdot, \cdot) \) in \( V_\Gamma \) (interface \( \Gamma \)).
    - Note that \( z_{\ast,\Gamma} = E_\Omega \tau_\ast \)
- Solve \( a(u, v) = (f, v) \) on the subspace \( V_{CMS} \).
Component mode synthesis

- $-\Delta u = f$ in $(0, 1)^2$ such that $u(x, y) = x(1-x)y(1-y)$

$V_{CMS}$ uses 7 times less degrees of freedom than $V_{Q1}$.

Fact

The global coupling eigenmodes are unknown or difficult to obtain numerically.
Use the same domain partition

Rectangles $\Omega_j$ with vertices $P$ and interior \textit{skeleton} $\Gamma$

$$V_{ACMS} = \left( \bigoplus_{j=1}^{J} \text{span}\{z_{1,j}\} \right) \oplus \left[ \left( \bigoplus_{e \subset T} \text{span}\{\psi_e\} \right) \oplus \left( \bigoplus_{P \in \mathcal{T}} \text{span}\{\varphi_P\} \right) \right]$$

- $z_{1,j}$ is the first fixed-interface mode in $\Omega_j$.
- The functions $\psi_e$ and $\varphi_P$ belong to $V_\Gamma$ and have \textit{local} support.

The space is \textit{consistent} with the decomposition

$$H^1_0(\Omega) = \left( \bigoplus_{j=1}^{J} V_{\Omega_j} \right) \oplus V_\Gamma.$$
The function $\psi_e$ belongs to $V_\Gamma$ and is the energy-minimizing extension of $\tau_e \in H_{00}^{1/2}(\Gamma)$, whose support is an interior edge, $e$.

$$a(E_\Omega \tau_e, E_\Omega \eta) = \lambda (E_\Omega \tau_e, E_\Omega \eta), \forall \eta \in H_{00}^{1/2}(\Gamma)$$

The function $\tau_e$ is an eigenfunction for the Steklov-Poincaré operator (or the Schur complement).
For any vertex $P$, $\varphi_P$ belongs to $V_\Gamma$

\[
\begin{cases}
-\nabla \cdot (c(x)\nabla \varphi_P(x)) &= 0 \quad \text{in } \Omega_j, \text{ for all } j, \\
\varphi_P(x) &\neq 0 \quad \text{on } \Gamma, \\
\varphi_P(x) &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Trace from Hou & Wu (1997):

\[
\int_{x_1}^{x} \frac{ds}{c(s, y_*)} / \int_{x_1}^{x_2} \frac{ds}{c(s, y_*)}
\]

and

\[
\int_{y_1}^{y} \frac{ds}{c(x_*, s)} / \int_{y_1}^{y_2} \frac{ds}{c(x_*, s)}
\]
Numerical experiment

- \(- \Delta u = f\) in \((0, 1)^2\) such that \(u(x, y) = x(1 - x)y(1 - y)\)

- \(V_{ACMS}\) uses 7 times less degrees of freedom than \(V_{Q1}\).

Fact

The basis functions for \(V_{ACMS}\) are defined locally.
Remarks

Recall

\[ H^1_0(\Omega) = \left( \bigoplus_{j=1}^{J} V_{\Omega_j} \right) \oplus V_\Gamma. \]

- \( V_{ACMS} \) is built from the decomposition.
  - Interior modes \( z_{1,j} \in V_{\Omega_j} \)
  - Edge functions \( \psi_e \in V_\Gamma \)
  - Vertex functions \( \phi_P \in V_\Gamma \)

- The basis functions are obtained numerically by solving local problems in every element \( \Omega_j \).
Numerical experiments

Goal: Compare different discretizations Q1, MFEM-O, CMS, ACMS

- $V_{Q1} = \bigoplus_{P \in \mathcal{T}} \text{span}\{N_P\}$
- $V_{MFEM-O} = \bigoplus_{P \in \mathcal{T}} \text{span}\{\phi_P\}$
- $V_{CMS} = \left( \bigoplus_{j=1}^{J} \text{span}\{z_{1,j}\} \right) \oplus \text{span}\{z_{n,\Gamma}; 1 \leq n \leq N_{\Gamma}\}$
- $V_{ACMS} = \left( \bigoplus_{j=1}^{J} \text{span}\{z_{1,j}\} \right) \oplus \left( \bigoplus_{e \subset \mathcal{T}} \text{span}\{\psi_e\} \right) \oplus \left( \bigoplus_{P \in \mathcal{T}} \text{span}\{\phi_P\} \right)$

Recall

$$H^1_0(\Omega) = \left( \bigoplus_{j=1}^{J} V_{\Omega_j} \right) \oplus V_{\Gamma}.$$
Problem

\[-\nabla \left( \frac{1}{1.2 + \cos(32\pi x(1 - x)y(1 - y))} \nabla u \right) = f \quad \text{in} \quad \Omega = (0, 1)^2\]

\[u = 0 \quad \text{at} \quad \partial \Omega\]

Exact solution

\[u(x, y) = 38.4\pi x(1 - x)y(1 - y) + \sin(32\pi x(1 - x)y(1 - y))\]

Quality measure: Energy

\[E - E^* = \frac{a(u - u^*, u - u^*)}{2}\]

- $E^*$ is the exact minimal energy.

Plot convergence curves in terms of degrees of freedom.
Numerical experiment

- $V_{MFEM-O}$ uses 15 times less degrees of freedom than $V_{Q1}$ and $V_{ACMS}$ 50 times less.
Recall

\[ H_0^1(\Omega) = \left( \bigoplus_{j=1}^{J} V_{\Omega j} \right) \oplus V_{\Gamma} \]

- **Vertex** functions \( \varphi_P \in V_{\Gamma} \).
- **Edge** functions \( \psi_e \in V_{\Gamma} \).
- **Interior** modes \( z_{1,j} \in V_{\Omega j} \).

Define the spaces

- \( V_{MFEM-O} = \bigoplus_{P \in \mathcal{T}} \text{span}\{ \varphi_P \} \)
- \( V_{MFEM-O-INT} = \bigoplus_{P \in \mathcal{T}} \text{span}\{ \varphi_P \} \oplus \left( \bigoplus_{j=1}^{J} \text{span}\{ z_{1,j} \} \right) \)
- \( V_{MFEM-O-E} = \bigoplus_{P \in \mathcal{T}} \text{span}\{ \varphi_P \} \oplus \left( \bigoplus_{e \subset \mathcal{T}} \text{span}\{ \psi_e \} \right) \)
- \( V_{ACMS} = \left( \bigoplus_{P \in \mathcal{T}} \text{span}\{ \varphi_P \} \right) \oplus \left( \bigoplus_{j=1}^{J} \text{span}\{ z_{1,j} \} \right) \oplus \left( \bigoplus_{e \subset \mathcal{T}} \text{span}\{ \psi_e \} \right) \)
Impact of basis functions $\psi_e$ and $z_{1,j}$
Impact of trace for \( \phi_P \)

\[
(\phi_P)|_{\Gamma} = \frac{\int_{x_1}^{x} \frac{ds}{c(s,y_*)}}{\int_{x_1}^{x_2} \frac{ds}{c(s,y_*)}}
\]

Energy-minimizing extension
Gives \( V_{ACMS} \)

\[
(\phi_P^L)|_{\Gamma} = \frac{x-x_1}{x_2-x_1}
\]

Energy-minimizing extension
Gives \( V_{ACMS-L} \)
Impact of trace for $\varphi_P$
Effect of subcell mesh size
Numerical experiment

Problem

\[-\nabla \cdot \left[ \left( \frac{2 + 1.8 \sin(25\pi x)}{2 + 1.8 \cos(25\pi y)} + \frac{2 + \sin(25\pi y)}{2 + 1.8 \sin(25\pi x)} \right) \nabla u \right] = -1\]

\[u = 0\]

Quality measure: Energy

\[\mathcal{E} - \mathcal{E}^* = \frac{a(u - u^*, u - u^*)}{2}\]

- \(\mathcal{E}^*\) is obtained via Richardson extrapolation.
Numerical experiment
Numerical experiment
On a given partition $\mathcal{T}$, the finite dimensional subspaces satisfy
\[ \dim V_{Q1} = \dim V_{MFEM-O} < \dim V_{ACMS}. \]

Per degree of freedom, the approximating subspace $V_{ACMS}$ gives more accuracy than $V_{MFEM-O}$ and $V_{Q1}$.

All components of $u$ should be approximated.

- Else lacking components may limit the accuracy.
- A finer partition would remove this limitation.

The choice of trace for $\varphi_P$ has a big impact on the discretization.
Relation to other conforming methods
Classical finite element Q1

\[ H^1_0(\Omega) = \left( \bigoplus_{j=1}^{J} V_{\Omega_j} \right) \oplus V_\Gamma \quad \text{and} \quad V_{Q1} = \bigoplus_{P \in \mathcal{T}} \text{span}\{N_P\} \]

- Recall that the partition is made of rectangles.
- When \( c \) is constant, \( V_{Q1} \) is a subspace of \( V_\Gamma \).
- When \( c \) is not equal to a constant, the nodal shape function \( N_P \) has nonzero components in \( V_\Gamma \) and some \( V_{\Omega_j} \).
Relation to other conforming methods
Multiscale Finite Element Method (T. Hou, Y. Efendiev, ...)

$$H^1_0(\Omega) = \left( \bigoplus_{j=1}^J V_{\Omega_j} \right) \oplus V_{\Gamma} \quad \text{and} \quad V_{\text{MFEM}-O} = \bigoplus_{P \in \mathcal{T}} \text{span}\{\phi_P\}$$

- When $c$ is constant, $V_{\text{MFEM}-O} = V_{Q1}$ is a subspace of $V_{\Gamma}$.
- For any general $c$, $V_{\text{MFEM}-O}$ is a subspace of $V_{\Gamma}$.
- Components in $V_{\Omega_j}$ of the solution $u$ are not computed by $V_{\text{MFEM}-O}$.
  - This error may limit the accuracy of the computed solution.
  - A finer partition would remove this limitation.
Relation to other conforming methods
Generalized Finite Element Method (I. Babuska, J. Osborn, M. Melenk, ...)

\[ V_{GFEM} = \left\{ \sum_{j=1}^{N} \phi_j \xi_j : \xi_j \in S_j \right\} \]

- The functions \((\phi_j)_{j=1,\ldots,N}\) form a partition of unity with local support \(\omega_j\).
- The supports \((\omega_j)_{j=1,\ldots,N}\) cover the domain \(\Omega\).
- The finite dimensional subspace \(S_j\)

\[ S_j = \text{span} \{ \xi_{i,j} \in H^1(\omega_j) ; \xi_{i,j} = 0 \text{ on } \partial \omega_j \cap \partial \Omega \} \]

approximates the solution \(u\) on \(\omega_j\).

- Harmonic polynomials for Laplace (M. Melenk).
- Planar waves for Helmholtz (M. Melenk, ...).
- ...

Harmonic polynomials for Laplace (M. Melenk).
Planar waves for Helmholtz (M. Melenk, ...).
Relation to other conforming methods
Is the multiscale finite element method a generalized finite element method?

\[ V_{GFEM} = \left\{ \sum_{j=1}^{N} \phi_j \xi_j : \xi_j \in S_j \right\} \]

\[ V_{MFEM-O} = \bigoplus_{P \in \mathcal{T}} \text{span}\{\phi_P\} \]

- The functions \((\phi_P)_{P \in \mathcal{T}}\) form a partition of unity.
- The sets \((\omega_j)_{j=1,\ldots,N}\) are the supports of the functions \((\phi_P)_{P \in \mathcal{T}}\).
- The finite dimensional subspaces \(S_j\) are

\[ S_j = \begin{cases} 
\{0\} & \text{when } \overline{\omega}_j \cap \partial \Omega \neq \emptyset, \\
\text{span}\{1\} & \text{otherwise}.
\end{cases} \]

**Fact**

**MFEM** is a generalized finite element method where the local approximation functions are **constant** and where the partition of unity functions \(\phi_j\) are energy-minimizing extensions.
Relation to other conforming methods

Is our new special finite element method a generalized finite element method?

\[ V_{ACMS} = \left( \bigoplus_{j=1}^{J} \text{span}\{z_{1,j}\} \right) \oplus \left( \bigoplus_{e \subset \mathcal{T}} \text{span}\{\psi_e\} \right) \oplus \left( \bigoplus_{P \in \mathcal{T}} \text{span}\{\varphi_P\} \right) \]

- The functions \((\varphi_P)_{P \in \mathcal{T}}\) form a partition of unity.
- Consider the finite dimensional subspaces \(S_j\)

\[ S_j = \begin{cases} 
\{0\} \oplus \text{span}\{\psi_e; \ e \subset \bar{\omega}_j \cap \Omega\} \oplus \text{span}\{z_{1,k}; \ \Omega_k \subset \omega_j\} \\
\text{span}\{1\} \oplus \text{span}\{\psi_e; \ e \subset \bar{\omega}_j\} \oplus \text{span}\{z_{1,k}; \ \Omega_k \subset \omega_j\}
\end{cases} \]

Fact

\(V_{ACMS}\) is a proper subspace of a generalized finite element method.

- By construction, \(V_{ACMS}\) is conforming and does not need the pasting from the partition of unity.
Define a new discretization

- Combine ideas from eigenanalysis, domain decomposition, finite element.

Basis functions have knowledge about the PDE.

- They are defined by a local minimization problem.

Future works

- Analysis for choice for $\varphi_P$
- Analysis regarding subcell mesh size
- 3D, elasticity, time-domain