A Continuous Time Approach for the Asymptotic Value in Two-Person Zero-Sum Repeated Games

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Joint work with Pierre Cardaliaguet and Sylvain Sorin

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The game is specified by a state space $\Omega$, move sets $I$ and $J$ and a transition probability $Q$ from $I \times J \times \Omega \rightarrow \Omega$. 
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For simplicity, all sets under consideration are supposed to be finite. $\Delta(X)$ is the set of probabilities over $X$. 

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For simplicity, all sets under consideration are supposed to be finite. $\Delta(X)$ is the set of probabilities over $X$.

The stochastic game is **absorbing** if only one state $\omega_0$ is non-absorbing, $Q(i,j,\omega)(\omega) = 1$ for all states except $\omega_0$. 

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How the game is played

- The game is of incomplete information: before the game starts (stage 0), nature chooses $k \in K$ according to $p \in \Delta(K)$ and chooses $l \in L$ according to $q \in \Delta(L)$, player I privately learns $k$ and player J learns $l$. 

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- Both know the initial state $\omega_1 = \omega$.

- Inductively, at stage $t = 1, \ldots,,$ knowing the past history $h_t = (\omega_1, i_1, j_1, \ldots, i_{t-1}, j_{t-1}, \omega_t)$, and each player his private information, player I chooses at random $i_t \in I$ according to $x_t \in \Delta(I)$ and player J chooses $j_t \in J$ according to $y_t \in \Delta(J)$.
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- The payoff at stage $t$ is $g(k, l, i_t, j_t, \omega_t) = g_t$. It is not observed.
- The new state $\omega_{t+1}$ is drawn according to a probability distribution $Q(i_t, j_t, \omega_t)(\cdot)$. 
How the game is played

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- The new state \( \omega_{t+1} \) is drawn according to a probability distribution \( Q(i_t, j_t, \omega_t)(\cdot) \).

- \((i_t, j_t, \omega_{t+1})\) are announced and the situation is repeated.
Evaluating Payoffs

Consider a probability distribution over the integers
\[ \mu = (\mu_1, \ldots, \mu_t, \ldots) : \mu_t \geq 0 \text{ and } \sum_t \mu_t = 1. \]
Evaluating Payoffs

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  \( \mu = (\mu_1, \ldots, \mu_t, \ldots) \): \( \mu_t \geq 0 \) and \( \sum_t \mu_t = 1 \).
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Main questions: (1) existence of \( \lim \nu_{\mu} \) as \( |\mu| := \sup_t \mu_t \to 0 \), (2) characterization of the limit.
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- **Main questions:** (1) existence of $\lim v_\mu$ as $|\mu| := \sup_t \mu_t \to 0$, (2) characterization of the limit.
- **Idea of the proof:** the game is extended to continuous time and viscosity solution tools are used.
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- **Idea of the proof:** the game is extended to continuous time and viscosity solution tools are used.
- The paper solves repeated games with incomplete information (\( \Omega \) is a singleton) and absorbing games (\( K \) and \( L \) are singleton) and splitting games (each player controls a martingale).
Asymptotic analysis: previous results

- Literature mainly concerns finitely repeated games ($\mu_k = 1/n$, for $k = 1, \ldots, n$) with values denoted $v_n$ and discounted games ($\mu_n = \lambda(1 - \lambda)^{n-1}$) with values denoted $v_\lambda$. 
Asymptotic analysis: previous results

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- For repeated games with incomplete information on one side, Aumann and Maschler proved the existence of $\lim v_n$ and $\lim v_\lambda$ using martingale tools, and identify it as $Cav_{\Delta(K)}u$, where $u$ is the value of non revealing game.
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- The result has been extended to incomplete information on both sides by Mertens and Zamir. They identified the limit as the unique solution of the system with unknown $\phi$:

\[
\phi(p, q) = Cav_{p \in \Delta(K)} \min \{\phi, u\}(p, q), \quad (1)
\]

\[
\phi(p, q) = Vex_{q \in \Delta(L)} \max \{\phi, u\}(p, q) \quad (2)
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  \]
- For stochastic game with complete information, existence is due to Bewley and Kohlberg using semi-algebraic tools.
**Aumann-Maschler: repeated game incomplete information**

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$p$

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$1 - p$
### Aumann-Maschler: repeated game incomplete information

The non-revealing game is:

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\[ p \]

\[ 1 - p \]

The value of the non-revealing game is \[ u(p) = p(1 - p) \].
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\[
\begin{array}{c|cc}
  & L & R \\
\hline
T & 1 & 0 \\
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\end{array}
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\[\lim v_\lambda(p) = \lim v_n(p) = Cavu(p) = p(1 - p)\]
Sorin, Big-Match with incomplete information

\[
\begin{array}{c|cc}
 & L & R \\
\hline
T & 1^* & 0^* \\
B & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
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\hline
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$$\lim_{\lambda} v_\lambda(p) = \lim_{n} v_n(p) = (1 - p)(1 - \exp(-\frac{p}{1-p})).$$ This is not an algebraic function!
The dynamic programming principle

- $\mu = (\mu_1, \ldots, \mu_t, \ldots)$ induces a partition $\Pi = \{t_n\}$ of $[0, 1]$ with $t_0 = 0$ and $t_n = \sum_{m=1}^{n} \mu_m$. 

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- Let \( v_{\Pi} = v_{\mu} \) denotes its value.
- The Shapley dynamic programming principle shows that:
  \[
  v_{\Pi}(p, q, \omega) = \text{Val}_{(x,y) \in \Delta(I)^{K} \times \Delta(J)^{L}} \{ t_1 g_1 + (1-t_1) E_{x,y} v_{\Pi_{t_1}}(\tilde{p}, \tilde{q}, \tilde{\omega}) \}
  \]
The dynamic programming principle

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$$

and $\Pi_{t_1}$ is the normalization on $[0, 1]$ of $\Pi$ on $[t_1, 1]$. 
Define $W_{\Pi}(t_n)$ as the value of the game starting at time $t_n$ with duration is $1 - t_n$.

$$W_{\Pi}(t_n)(p, q, \omega) = \max_{x \in \Delta(I)^K} \min_{y \in \Delta(J)^L} \{ \mu_{n+1} g_1 + E_{x, y} W_{\Pi}(t_{n+1})(\tilde{p}, \tilde{q}, \tilde{\omega}) \}.$$
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By taking the linear extension of $\{W_\Pi(t_n, p, q, \omega)\}_n$, we define a function $W_\Pi(t, p, q, \omega)$ on $[0, 1] \times \Delta(K) \times \Delta(L) \times \Omega$. 
Continuous time extension

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- Standard arguments shows that the family of functions $W_\Pi$ is concave in $p$, convex in $q$ and uniformly Lipschitz in $(p, q)$. 

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Standard arguments shows that the family of functions $W_\Pi$ is concave in $p$, convex in $q$ and uniformly Lipschitz in $(p, q)$.

It may be proved that when $\mu_m$ is decreasing in $m$, $W$ is uniformly Lipschitz in $t$. 

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Main result

Here $\Omega$ is reduced to a singleton. Let

$$u(p, q) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \left( \sum_{i, j, k, l} p^k q^l x(i)y(j)g(k, l, i, j) \right).$$
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**Theorem**

The equi-continuous family $W_\Pi$ has a unique cluster point.
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The equi-continuous family $W_\Pi$ has a unique cluster point. The uniform limit is the unique continuous function that satisfies:
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Theorem

*The equi-continuous family $W_\Pi$ has a unique cluster point. The uniform limit is the unique continuous function that satisfies: for all $(t, p, q)$ and all $C^1$ test function $\phi : [0, 1] \to \mathbb{R}$:*
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The equi-continuous family $W_\Pi$ has a unique cluster point. The uniform limit is the unique continuous function that satisfies: for all $(t, p, q)$ and all $C^1$ test function $\phi : [0, 1] \to \mathbb{R}$:

- **P1**: If, $p$ is extreme of the hypograph of $W(t, ; q)$ and $W(\cdot, p, q) - \phi(\cdot)$ has a global maximum at $t$, then $u(p, q) + \phi'(t) \geq 0$. 
Main result

Here $\Omega$ is reduced to a singleton. Let

$$u(p, q) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \left( \sum_{i, j, k, l} p^k q^l x(i) y(j) g(k, l, i, j) \right).$$

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As a corollary, one obtains the Mertens-Zamir’s result.
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Model

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- At stage $t = 1, 2, ..., \text{player I}$ chooses $i_t \in I$ using some lottery $x_t \in \Delta (I)$. Simultaneously, \text{player J} chooses $j_t \in J$ using some lottery $y_t \in \Delta (J)$.
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- with probability $1 - \pi(i_t, j_t)$ the game is absorbed and the payoff in all future stages is $g(i_t, j_t)$. 

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A Continuous Time Approach for the Asymptotic Value in
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Absorbing games

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with probability $1 - \pi(i_t, j_t)$ the game is absorbed and the payoff in all future stages is $g(i_t, j_t)$.

with probability $\pi(i_t, j_t)$ the interaction is repeated.
Main Result

Theorem

\( v_\mu \) converges to \( v \) given by:

\[
VAL((x,\alpha),(y,\beta)) \in (\Delta(I) \times R_+^I) \times (\Delta(J) \times R_+^J) \quad \frac{f(x, y) + f^*(\alpha, y) + f^*(x, \beta)}{1 + \pi^*(\alpha, y) + \pi^*(x, \beta)}
\]
Theorem

\( \nu_\mu \) converges to \( \nu \) given by:

\[
VAL_{((x,\alpha),(y,\beta))}\in(\Delta(I)\times R^I_+)(\Delta(J)\times R^J_+) \quad \frac{f(x, y) + f^*(\alpha, y) + f^*(x, \beta)}{1 + \pi^*(\alpha, y) + \pi^*(x, \beta)}
\]

where

\[\pi^*(i, j) = 1 - \pi(i, j), \quad f^*(i, j) = \pi^*(i, j) \times g(i, j)\]
Main Result

Theorem

\( v_\mu \) converges to \( v \) given by:

\[
\text{VAL}((x, \alpha), (y, \beta)) \in (\Delta(I) \times R^I_+) \times (\Delta(J) \times R^J_+)
\]

\[
\frac{f(x, y) + f^*(\alpha, y) + f^*(x, \beta)}{1 + \pi^* (\alpha, y) + \pi^* (x, \beta)}
\]

where

\[
\pi^*(i, j) = 1 - \pi(i, j), \quad f^*(i, j) = \pi^*(i, j) \times g(i, j)
\]

and \( \varphi(\alpha, \beta) = \sum_{i \in I, j \in J} \alpha^i \beta^j \varphi(i, j) \).
Idea of the Proof

Denote by $W_\mu(t_m)$ the value of the game starting at time $t_m$. Then

$$W_\mu(t_m) = Val(x, y) \left[\mu_{m+1} f(x, y) + \pi(x, y) W_\mu(t_{m+1}) + (1 - t_{m+1}) f^*(x, y)\right].$$
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Define for any

$$(t, a, b, x, \alpha, y, \beta) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times \Delta(I) \times \mathbb{R}_+^I \times \Delta(J) \times \mathbb{R}_+^J,$$

$$h(t, a, b, x, \alpha, y, \beta) = f(x,y) + (1 - t)[f^*(\alpha,y) + f^*(x,\beta)] - [\pi^*(\alpha,y) + \pi^*(x,\beta)] a + b \frac{1 + \pi^*(\alpha,y) + \pi^*(x,\beta)}{1 + \pi^*(\alpha,y) + \pi^*(x,\beta)}.$$
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Define for any

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$$h(t, a, b, x, \alpha, y, \beta) =$$

$$f(x, y) + (1 - t)[f^*(\alpha, y) + f^*(x, \beta)] - [\pi^*(\alpha, y) + \pi^*(x, \beta)] a + b$$

$$1 + \pi^*(\alpha, y) + \pi^*(x, \beta)$$

The Hamiltonian of the continuous time game is $H(t, a, b) =$

$$VALUE_{((x, \alpha), (y, \beta)) \in (\Delta(I) \times \mathbb{R}_+^I) \times (\Delta(J) \times \mathbb{R}_+^J)} h(t, a, b, x, \alpha, y, \beta)$$
Theorem

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- **R1:** If $U(\cdot) - \phi(\cdot)$ admits a global maximum at $t \in [0, 1)$
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Theorem

The family $W_\mu$ uniformly converges to the unique continuous functions satisfying the two properties: for all $t$ and any $C^1$ function $\phi : [0, 1] \to \mathbb{R}$:

- **R1:** If $U(\cdot) - \phi(\cdot)$ admits a global maximum at $t \in [0, 1)$ then $H(t, U(t), \phi'(t)) \geq 0$.
- **R2:** If $U(\cdot) - \phi(\cdot)$ admits a global minimum at $t \in [0, 1)$ then $H(t, U(t), \phi'(t)) \leq 0$. 