Resonant effects in random dielectric structure

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1. Metamaterials in optic
2. Homogenization of Maxwell equations
3. Dielectric random structure
4. Stochastic Framework
5. Dissipation Limit
6. Perspectives
1. Metamaterials in optic
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The behavior of materials is described by two parameters:

- **Permittivity**: \( \varepsilon_r = \varepsilon_r(\omega) \)
- **Permeability**: \( \mu_r = \mu_r(\omega) \)

**Optic regime**: frequency between 375 and 750 THz (wavelength in [400, 800] nm).

- \( \mu_r(\omega) \approx 1 \)
- \( \varepsilon_r(\omega) = \varepsilon' + i\varepsilon'' \)
  - **Metal**: \( \varepsilon' \in \mathbb{R} \), \( \varepsilon'' > 0 \) (\( \varepsilon'' \approx 1 \))
  - **Dielectric**: \( \varepsilon' > 0 \), \( \varepsilon'' = 0 \) linked to dissipation.

**Goal**: build artificial composite structures made of metal (\( \varepsilon'' \gg 1 \)) or dielectric (\( \varepsilon' \gg 1 \)) such that for some frequencies:

- \( \varepsilon_{\text{eff}}(\omega) < 0 \)
- \( \mu_{\text{eff}}(\omega) < 0 \) (magnetic activity)

Both (negative refractive index)
Electromagnetic parameters

The behavior of materials is described by two parameters:

\[ \varepsilon_r = \varepsilon_r(\omega) \quad , \quad \mu_r = \mu_r(\omega) . \]

**Optic regime:**

frequency between 375 and 750 THz (wavelength in [400, 800] nm).

\[ \mu_r(\omega) \simeq 1 \quad , \quad \varepsilon_r(\omega) = \varepsilon' + i\varepsilon'' \]

- **Metal:** \( \varepsilon' \in \mathbb{R} \quad , \quad \varepsilon'' > 0 \quad (\varepsilon'' \gg 1) \)
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- **Both** (negative refractive index)
Metamaterial with $\varepsilon < 0$ and $\mu < 0$ (in microwave regime)

Negative refraction

Application 1: plane lens Veselago

Theoric description

Numerical simulation

Invisibility device

Structure permitting invisibility (microwave)

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The small parameter $\eta$ (the period) characterize the structure (in practice $\eta \sim \frac{\text{wavelength}}{10}$).

Permittivity $\varepsilon_{\eta}$ is very high on inclusion in metallic case: filling ratio must be infinitesimal (to keep finite dissipation).
Maxwell system

Total Field \((E_\eta, H_\eta)\) solve \(\mathbb{R}^3\)

\[
\begin{align*}
\text{rot } E_\eta &= i\omega \mu_0 H_\eta \\
\text{rot } H_\eta &= -i\omega \varepsilon_0 \varepsilon_\eta(x) E_\eta 
\end{align*}
\]

+ outgoing wave conditions of Silver-Müller \(|x| \to +\infty\):

\[
(E^d_\eta, H^d_\eta) = O \left( \frac{1}{|x|} \right), \quad \omega \varepsilon_0 \left( \frac{x}{|x|} \wedge E^d_\eta \right) - k_0 H^d_\eta = o \left( \frac{1}{|x|} \right).
\]

\((E^d_\eta, H^d_\eta) = (E_\eta - E^i, H_\eta - H^i)\) is the diffracted field.

**Goal**: Pass to the limit \(\eta \to 0\)

We have to take account fast oscillations of electromagnetic field
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The structure is invariant in direction $e_3$ and we describe only its intersection by an horizontal plane.

$\Omega$ is the probability space and $\omega \in \Omega$ a fixed random event.

$$D_{\eta}(\omega) = \bigcup_{i \in J_\eta} D^i_{\eta}(\omega), \quad D^i_{\eta} := \eta[i - y(\omega) + B(\theta_i(\omega), \rho_i(\omega))]$$

$y(\omega)$ consists in a random translation of the lattice.
Physical parameters

\[ \mu = 1 \]

\[ \varepsilon_{\eta}(x, \omega) := 1_{\mathbb{R}^2 \setminus \mathcal{D}_{\eta}}(x) + \sum_{i \in J_{\eta}} \frac{\varepsilon_i(\omega)}{\eta^2} 1_{\mathcal{D}_i}(x), \quad a_{\eta}(x, \omega) := 1/\varepsilon_{\eta}. \]

- The diffracting obstacle is illuminated by a monochromatic incident wave travelling in the $H\parallel$ mode.
- The magnetic field takes the form $\mathbf{H}(x, t) = u(x_1, x_2) e^{-i\omega t} \mathbf{e}_3$.

Maxwell’s equations reduce to

\[
\begin{cases}
\text{div} \left( a_{\eta}(x, \omega) \nabla u_{\eta}(x, \omega) \right) + k_0^2 u_{\eta}(x, \omega) = 0 \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u_{\eta}^d}{\partial r} - i k_0 u_{\eta}^d \right) = 0
\end{cases}
\]

$u_{\eta}^d$ is the diffracted field.
Outline

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We consider for \( \delta > 0 \) and \( Y = [0, 1]^2 \) the set \( M \) describing each inclusion

\[
M := \left\{ \left( \theta, \rho, \varepsilon \right) \in Y \times \left[ 0, \frac{1}{2} \right] \times \mathbb{C}^+ : d(\theta, \partial Y) \geq \rho + \delta \right\}.
\]

The law of repartition of radius, center and permittivity is given by a probability \( p \) on \( M \). We introduce

\[
\Omega := \left( \prod_{\mathbb{Z}^2} M \right) \times Y, \quad \mathbb{P} := \left( \bigotimes_{\mathbb{Z}^2} p \right) \otimes L^2
\]

So

\[
\omega \in \Omega \iff \omega = \left( (m_j)_{j \in \mathbb{Z}^2}, y \right)
\]

with \( m_j = (\theta_j, \rho_j, \varepsilon_j) \in M \) \( \forall j \in \mathbb{Z}^2, \ y \in Y \).
Dynamical system

On the probability space \((\Omega, \mathbb{P})\) we define the group of transformations \(T_x : \Omega \rightarrow \Omega, \quad x \in \mathbb{R}^2\)

\[
T_x \left( (m_j)_{j \in \mathbb{Z}^2}, y \right) := \left( (m_j + [x + y])_{j \in \mathbb{Z}^2}, x + y - [x + y] \right).
\]

Now let

\[
\Sigma := \{\omega \in \Omega : |y - \theta_0| < \rho_0\}, \quad \Sigma^* := \Omega \setminus \Sigma.
\]

Properties

- The dynamical system preserves the measure and is ergodic.
- \(x \in D_\eta(\omega) \iff T_x^{\eta} \omega \in \Sigma.\)
- \(\partial_i^s f(\omega) = \lim_{h \to 0} \frac{f(T_{hei} \omega) - f(\omega)}{h} = \partial_y f(m, y), \quad \omega = (m, y).\)
Two scale convergence

**Definition (Stochastic two scale convergence)**

\[ f_\eta(x) \overset{\ast}{\rightharpoonup} f_0(x, \omega), \text{ if for some } \tilde{\omega} \in \Omega \text{ it holds} \]

\[
\int_{\mathbb{R}^2} f_\eta(x) \varphi(x, T^x_\eta \tilde{\omega}) \, dx \rightarrow \int_{\mathbb{R}^2 \times \Omega} f_0(x, \omega) \varphi(x, \omega) \, dx \, \mathbb{P}(d\omega)
\]

for any \( \varphi \in C_c^\infty(\mathbb{R}^2; C^1(\Omega)) \).

**Properties**

- \( \| f_\eta \|_{L^2(\Omega)} \leq C \implies \exists f_0 \text{ such that } u_\eta \rightharpoonup u_0 \text{ (up to a subsequence)}. \)
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for any \( \varphi \in C^\infty_c(\mathbb{R}^2; C^1(\Omega)) \).

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**A priori Hypothesis**

For almost every \( \tilde{\omega} \in \Omega, \)

\[
\int_B |u_\eta(x, \tilde{\omega})|^2 dx < +\infty.
\]
Identification of tow-scale limits

Thus \( \forall x \in \mathcal{B} \)

\[
a_\eta(x, \omega) = \left( \frac{\eta^2}{\varepsilon_0(T_{x, \omega})} 1\Sigma(T_{x, \omega}) + 1\Sigma^*(T_{x, \omega}) \right).
\]

Problem become

\[
\text{div} \left( a_\eta(x, \omega) \nabla u_\eta(x, \omega) \right) + k_0^2 u_\eta(x, \omega) = 0
\]

Two-scale limits

- \( u_\eta(x, \tilde{\omega}) \rightarrow u_0(x, \omega) \)
- \( \eta \nabla u_\eta(x, \tilde{\omega}) \rightarrow \nabla_s u_0(x, \omega) \)
- \( 1_{B_R \setminus \mathcal{D}_\eta(\tilde{\omega})}(x) \nabla u_\eta(x, \tilde{\omega}) \rightarrow \chi_0(x, \omega) \)
Identification of limit

Micro-resonator problem

\[
\begin{cases}
    u_0(x, \omega) = u(x) & \text{in } B \times \Sigma^* \\
    \Delta_s u_0(x, \omega) + \varepsilon_0(\omega) k_0^2 u_0(x, \omega) = 0 & \text{in } B \times \Sigma,
\end{cases}
\]

\(\Im(\varepsilon_0(\omega)) > 0\) is important here

We denote by \(0 < \lambda_1 < \cdots\), eigenvalues of Dirichlet problem on the ball \(B(0, 1)\)

\[-\Delta \varphi_n = \lambda_n \varphi_n\]

and \(\{\varphi_n, n \in \mathbb{N}\}\) an associated orthonormal basis.

Thus \(u_0(x, \omega) = u(x)\Lambda(\omega)\) with

\[
\Lambda(\omega) = \begin{cases}
    1 & \text{if } \omega \in \Omega \setminus \Sigma \\
    1 + \sum_{n \in \mathbb{N}} \frac{k_0^2 \varepsilon_0(\omega) \rho_0^2(\omega)}{\lambda_n - k_0^2 \varepsilon_0(\omega) \rho_0^2(\omega)} \int_{B_1} \varphi_n(y) dy \varphi_n \left(\frac{y - \theta_0(\omega)}{\rho_0(\omega)}\right) & \text{if } \omega \in \Omega \setminus \Sigma
\end{cases}
\]
Effective parameters

- **Permeability tensor**

\[
\mu^{\text{eff}}(k_0) = \mathbb{E}(\Lambda) = 1 + \sum_n \mathbb{E} \left[ \frac{\varepsilon \rho^4 k_0^2}{\lambda_n - \varepsilon \rho^2 k_0^2} \right] \left( \int_Y \varphi_n \right)^2
\]

- **Permittivity tensor**

\[
\varepsilon^{\text{eff}} = \mathbb{E} \left[ \frac{1}{A(\rho)} \right] \text{ where }
A(\rho) = \inf_v \left\{ \int_{Y \setminus B(\theta, \rho)} |e + \nabla v|^2 : v \text{ Y-periodic} \right\}
\]

**Limit problem (deterministic)**

\[
\begin{cases}
\text{div} \left( \frac{1}{\varepsilon^{\text{eff}}(x)} \nabla u(x) \right) + k_0^2 \mu^{\text{eff}}(x, k_0) u(x) = 0 \quad \text{in} \quad \mathbb{R}^2 \\
(u - u^{\text{inc}}) \quad \text{satisfies Somerfield condition}
\end{cases}
\]
Hypothesis

- $\Im(\varepsilon) > 0$ almost surely (for uniqueness in the limit problem)

- $\exists \ h > 0$ such that

$$
\mathbb{E} \left[ \left| \frac{\varepsilon \rho}{\text{dist}(\varepsilon \rho^2 k_0^2, \sigma_0)} \right|^{2+h} \right] < \infty
$$

(1)

with $\sigma_0 = \{\lambda_n, n \in \mathbb{N}\}$ the spectrum of the Laplace operator in the unit disk.

Homogenization results

- Condition (1) implies $\int_{\mathcal{B}} |u_\eta(x, \tilde{\omega})|^2 \, dx < +\infty$ almost surely.

- We have strong two scale convergence

$$
\int_{\mathcal{B}} |u_\eta(x, \tilde{\omega}) - u_0(x, T_{\frac{x}{\eta}} \tilde{\omega})|^2 \, dx \to 0.
$$
Representation of $\mu^{\text{eff}}$ in term of the characteristic wavelength: the radius of inclusions follows an uniform law between 0, 3 and 0, 4 and permittivity law is a Dirac mass in $100 + 5i$. 
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What happens when the law of $\varepsilon(\omega)$ is supported in real axis?

We consider for $\varepsilon = a + ib$

- non-random geometry: $\theta = \theta_0$ et $\rho = \rho_0$
- Distribution of $\varepsilon = a + i0$ on real axes with density $g(a)$.

$$p_0(\theta, \rho, a + ib) = \delta(\theta - \theta_0, \rho - \rho_0) \otimes g(a) \, da \otimes \delta(b)$$

- Approached by the following sequence of law with dissipation

$$p_h := \delta(\theta - \theta_0, \rho - \rho_0) \otimes g(a) \, da \otimes \frac{1}{h} \zeta\left(\frac{b}{h}\right)$$

where $\zeta$ is a probability on $]0,1[$ compatible with (2).
For $h > 0$, the homogenization result yield to

$$\mu_h^{\text{eff}} := 1 + \sum_n l_{h,n} \left( \int_{B_1} \varphi_n \right)^2, \quad l_{h,n} := \int \frac{k_0^2 \varepsilon \rho^4}{\lambda_n - \rho^2 \varepsilon k_0^2} dp_h$$

Non-vanishing dissipation

$$\mu_h^{\text{eff}} \to \mu_0^{\text{eff}} := 1 + \sum_n l_n(k_0) \left( \int_{B_1} \varphi_n \right)^2$$

$$\Re(l_n(k_0)) = \text{VP} \left( \int \frac{a k_0^2 \rho_0^4}{\lambda_n - a k_0^2 \rho_0^2} g(a) \, da \right)$$

$$\Im(l_n(k_0)) = \frac{\pi \lambda_n}{k_0^2} g \left( \frac{\lambda_n}{k_0^2 \rho_0^2} \right) > 0 \quad \text{if} \quad \text{supp}(g) \ni \left\{ \frac{\lambda_n}{k_0^2 \rho^2} \right\}$$
$\Im(\varepsilon_r) = 0$
$\Im(\varepsilon_r) = 1$
$\Im(\varepsilon_r) = 5$

$\mu_h^{\text{eff}}$ and $\mu_0^{\text{eff}}$ when $h \to 0$
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Perspectives

Random 3D structure: Collaboration with G. Bouchitté and L. Manca

Comparaison effective law/real structure: Collaboration with D. Felbacq (GES. Montpellier)

- Multiscattering code for real structure

Diffracted field for homogenized structure (left) and for real structure (right).
More general structures