Homogenization of thin random structures and measures

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Motivation

- **Dimension reduction.** Shells, skeletons, rod structures $\rightarrow$ surfaces and segments structures;

- **Reduction of the number of parameters.** Asymptotic problems with two small parameters (the microscopic length scale of the medium and the structure thickness) $\rightarrow$ problems with only one parameter;

- **Porous and perforated media with rough geometry.** Random geometries: percolation cluster, Poisson cloud, Voronoi tessellation of a stationary point process, etc.
Homogenization of periodic singular measures and structures have been originally developed in the works of V. Zhikov, G. Bouchitte and I. Fragala, G. Bouchitte and G. Boutazzo, V. Zhikov and S. Pastukhova.

Homogenization of singular random measures and structures has been studied in the work of A. P. and V. Zhikov, M. Heida.
Let $\mu(x)$ be a positive finite Borel measure on a standard $n$-dimensional torus $\mathbb{T}^n \equiv \mathbb{R}^n/\mathbb{Z}^n$ or in $\mathbb{R}^n$. We identify $\mu$ with the corresponding periodic measure in $\mathbb{R}^n$. Without loss of generality, we may assume that

$$\int_{\mathbb{T}^n} d\mu(x) = 1.$$ 

Then the scaled measures $\mu^\varepsilon = \varepsilon^n \mu(\frac{x}{\varepsilon})$ converge weakly, as $\varepsilon \to 0$, to the Lebesgue measure in $\mathbb{R}^n$, that is for any continuous function $\psi = \psi(x)$ with a finite support we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \psi(x) d\mu^\varepsilon(x) = \int_{\mathbb{R}^n} \psi(x) dx.$$
To clarify the idea of introducing Sobolev spaces with respect to a measure, consider a simple example. Let $\mu$ be a positive finite Borel measure in a smooth bounded domain $G$. Consider the variational problem

$$\inf_{\varphi \in C_0^\infty(G)} \int_G \left( a(x) \nabla \varphi(x) \cdot \nabla \varphi(x) + \varphi^2(x) - 2f(x)\varphi(x) \right) d\mu(x),$$

where $a(x)$ is a continuous positive definite matrix in $\overline{G}$ and $f(x)$ is a continuous function in $\overline{G}$. Our goal is to introduce a Sobolev space with respect to the measure $\mu$ in such a way that the minimum is attained and a minimizer can be found as a solution to the corresponding Euler equation.
Definition

We say that a function \( u \in L_2(\mathbb{T}^n, \mu) \) belongs to the space \( H^1(\mathbb{T}^n, \mu) \) if there exists a vector-function \( z \in (L_2(\mathbb{T}^n, \mu))^n \) and a sequence \( \varphi_k \in C^\infty(\mathbb{T}^n) \) such that

\[
\varphi_k \longrightarrow u \quad \text{in } L_2(\mathbb{T}^n, \mu) \text{ as } k \to \infty,
\]

\[
\nabla \varphi_k \longrightarrow z \quad \text{in } (L_2(\mathbb{T}^n, \mu))^n \text{ as } k \to \infty.
\]

The function \( z(x) \) is called the gradient or \( \mu \)-gradient of \( u(x) \) and is denoted by \( \nabla^\mu u \).

Similarly, we can define the spaces \( H^1(\mathbb{R}^n, \mu) \), \( H^1_{\text{loc}}(\mathbb{R}^n, \mu) \) and also the space \( H^1(G, \mu) \) for an arbitrary domain \( G \subset \mathbb{R}^n \) and a (locally) finite Borel measure \( \mu \) on \( G \).
Examples. A segment

The gradient of a $H^1(\mathbb{T}^n, \mu)$ function need not be unique. In particular, the zero function can have a nontrivial gradient. We illustrate this with

**Example**

In the square $[-1/2, 1/2]^2$, we consider the segment $\{-1/4 \leq x_1 \leq 1/4, x_2 = 0\}$ and introduce

$$d\mu = 2\chi(x_1)\,dx_1 \times \delta(x_2),$$

(1)

where $\chi(t)$ is the characteristic function of the segment $[-\frac{1}{4}, \frac{1}{4}]$ and $\delta(t)$ is the Dirac mass at zero.

Let $\psi(x) \in C_0^\infty$ coincide with a function of the form $\theta(x_1)x_2$ in a small neighborhood of the segment. Then $\psi = 0$ in $L_2(\mathbb{T}^2, \mu)$. Choosing $\varphi_k(x) = \psi(x)$ for all $k$ in the definition of $\mu$-gradient, we find $z(x) = \nabla^\mu \psi(x) = (0, \theta(x_1))$. Thus, any vector-valued function of the form $(0, \theta(x_1))$ with smooth $\theta(s)$ serves as the $\mu$-gradient of zero. In fact, this assertion is valid for any $\theta(s)$ from $L_2(-1/4, 1/4)$. 
Gradients of zero

The gradients of zero form a closed subspace of \((L_2(\mathbb{T}^n, \mu))^n\), denote it \(\Gamma_\mu(0)\). The set of the gradients of any \(H^1(\mathbb{T}^n, \mu)\)-function is the sum of its arbitrary gradient and \(\Gamma_\mu(0)\).

**Example (Segment)**

Consider the space \(H^1(\mathbb{T}^n, \mu)\) (or \(H^1(\mathbb{R}^n, \mu)\)) for 1D Lebesgue measure \(\mu\) on the segment \(I = \{x \in \mathbb{R}^n: 0 \leq x_1 \leq a, x_2 = x_3 = \cdots = x_n = 0\}\).

**Proposition**

The space \(H^1(\mathbb{T}^n, \mu)\) consists of all Borel functions \(u(x)\) such that \(u(s, 0, 0, \ldots, 0) \in H^1(0, a)\). Moreover,

\[
\nabla^\mu u(x) = (u'_{x_1}(x_1, 0), \psi_2(x_1), \ldots, \psi_n(x_1)),
\]

where

\[
u'_{x_1} \equiv \frac{d}{ds} u(s, 0, 0, \ldots, 0) \bigg|_{s=x_1}, \text{ and } \psi_2, \psi_3, \ldots, \psi_n \text{ are arbitrary functions in } L_2(0, a).
\]
Examples. Hedgehog

Example ("Hedgehog")

Consider the segments $I_1, I_2, I_N$ starting at the origin and directed along vectors $v_1, v_2, \ldots, v_N$. Let $\mu_1, \mu_2, \ldots, \mu_N$ be the standard 1D Lebesgue measures on the segments $I_1, \ldots, I_N$ respectively, and let $\lambda_1, \ldots, \lambda_N$ be arbitrary positive numbers. We set

$$\mu = \sum_{j=1}^{N} \lambda_j \mu_j.$$ 

A function $u(x)$ belongs $H^1(\mathbb{T}^n, \mu)$ if and only if $u|_{I_j} \in H^1(I_j)$, and the values of the restricted functions at the origin coincide for all segments (recall that an $H^1$-function of a single variable is continuous).
Example (Reinforced shells)

Let \( \Pi_0 = \{ x \in \mathbb{T}^n : x_1 = 0 \} \). We set

\[
d\tilde{\mu}(x) = \delta(x_1) \times dx' + dx, \quad x' = (x_2, \ldots, x_n).
\]

A function \( u(x) \) belongs to \( H^1(\mathbb{T}^n, \tilde{\mu}) \) if and only if \( u \in H^1(\mathbb{T}^n) \) and the trace \( u(x)|_{\Pi_0} \in H^1(\mathbb{T}^{n-1}) \).

Remark

If the co-dimension of a plane \( \Pi \subset \mathbb{R}^n \) is greater than one, then the trace of a \( H^1(\mathbb{R}^n) \)-function on \( \Pi \) is not well-defined. Therefore, if \( \mu \) is the surface Lebesgue measure on \( \Pi \) and \( d\tilde{\mu} = d\mu + dx \), then \( H^1(\mathbb{T}^n, \tilde{\mu}) \) is isomorphic to the direct sum of the spaces \( H^1(\mathbb{R}^n) \) and \( H^1(\mathbb{R}^n, \mu) \).

We denote

\[
H(\mathbb{R}^n, \mu) = \{(u, z) : u \in H^1(\mathbb{R}^n, \mu), z = \nabla^\mu u \}.
\]
Suppose that Radon measures $\mu_k$ weakly converges, as $k \to \infty$, to $\mu$ in $\mathbb{R}^n$.

**Definition**

We say that $g_k \in L^2(\mathbb{R}^n, \mu_k)$ weakly converge in $L^2(\mathbb{R}^n, \mu_k)$ to $g \in L^2(\mathbb{R}^n, \mu)$, as $k \to \infty$, if

1. $\|g_k\|_{L^2(\mathbb{R}^n, \mu)} \leq C$;

2. $\lim_{k \to \infty} \int_{\mathbb{R}^n} g_k(x) \varphi(x) d\mu_k(x) = \int_{\mathbb{R}^n} g(x) \varphi(x) d\mu(x)$

for all $\varphi \in C_0(\mathbb{R}^n)$. 
Convergence in variable spaces

**Definition**

A sequence \( \{g_k\} \) converges strongly to \( g(x) \in L_2(\mathbb{R}^n, \mu_k) \) if it weakly converges and

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} g_k(x) h_k(x) d\mu_k(x) = \int_{\mathbb{R}^n} g(x) h(x) d\mu(x)
\]

for any sequence \( \{h_k(x)\} \) weakly converging to \( h(x) \in L_2(\mathbb{R}^n, \mu) \) in \( L_2(\mathbb{R}^n, \mu_k) \).

**Lemma**

Let \( \{g_k\} \) weakly converge to \( g(x) \) in \( L_2(\mathbb{R}^n, \mu_k) \). Then \( \{g_k\} \) converges strongly if and only if

\[
\lim_{k \to \infty} \|g_k\|_{L_2(\mathbb{R}^n, \mu_k)} = \|g\|_{L_2(\mathbb{R}^n, \mu)}.
\]

**Lemma**

Let \( \{\mu_k\} \) converge weakly to \( \mu \). Then any bounded sequence \( \{g_k(x)\} \), \( \|g_k\|_{L_2(\mathbb{R}^n, \mu_k)} \leq C \), converges weakly along a subsequence in \( L^2(\mathbb{R}^n, \mu_k) \) towards some function \( g(x) \in L_2(\mathbb{R}^n, \mu) \).
Potential and solenoidal fields

Definition

The space $L^\text{pot}_2(\mathbb{R}^n, \mu)$ is the closure of the linear set $\{\nabla \varphi : \varphi \in C^\infty_0(\mathbb{R}^n)\}$ in the $(L_2(\mathbb{R}^n, \mu))^n$-norm.

Definition

The space $L^\text{pot}_2(\mathbb{R}^n, \mu)$ of solenoidal vector-valued functions is the orthogonal complement to the space $L^\text{pot}_2(\mathbb{R}^n, \mu)$ in $(L_2(\mathbb{R}^n, \mu))^n$. 
Let \( K(x) \geq 0 \) be a \( C_0^\infty \) function such that \( \int K(x) dx = 1 \) and \( K(-x) = K(x) \). For a Radon measure \( \mu(x) \) in \( \mathbb{R}^n \) or on \( \mathbb{T}^n \) we set
\[
d\mu^\delta(x) = \rho^\delta(x) dx, \quad \rho^\delta(x) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{x-y}{\delta}\right) d\mu(y).
\]
The measures \( \mu^\delta \) locally weakly converge in \( \mathbb{R}^n \) to \( \mu \). We also use the usual smoothening
\[
\varphi^\delta(x) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{y}{\delta}\right) \varphi(x-y) dy.
\]
Then
\[
\int_{\mathbb{R}^n} \varphi^\delta(x) d\mu(x) = \int_{\mathbb{R}^n} \varphi(x) d\mu^\delta(x)
\]

**Lemma**

*For every \( v \in L_2(\mathbb{R}^n, \mu) \) there is \( v^\delta \in L_2(\mathbb{R}^n, \mu) \) such that*
\[
\int_{\mathbb{R}^n} v^\delta(x) \varphi(x) d\mu^\delta(x) = \int_{\mathbb{R}^n} v(x) \varphi^\delta(x) d\mu(x)
\]
*for all \( \varphi \in C_0(\mathbb{R}^n) \). The family \( v^\delta(x) \) strongly converges to \( v(x) \) in \( L_2(\mathbb{R}^n, \mu^\delta) \) as \( \delta \to 0 \).*
Divergence operator

Definition

Let $g \in L_2(\mathbb{R}^n, \mu)$ and $v \in (L_2(\mathbb{R}^n, \mu))^n$. We say that $g(x) = \text{div}^\mu v(x)$ if

$$
\int_{\mathbb{R}^n} g(x) \varphi(x) d\mu(x) = - \int_{\mathbb{R}^n} v(x) \cdot \nabla \varphi(x) d\mu(x)
$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. 

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Elliptic equations

Let $a(x) = \{a_{ij}(x)\}$ be a symmetric $n \times n$-matrix,

$$\Lambda |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda^{-1}|\xi|^2, \quad \Lambda > 0, \quad \xi \in \mathbb{R}^n$$

$\mu$-a.e. in $\mathbb{R}^n$. Suppose that $f \in L_2(\mathbb{R}^n, \mu)$, and $\lambda > 0$.

**Definition**

We say that a pair $(u, \nabla^\mu u)$ with $u \in H^1(\mathbb{R}^n, \mu)$, satisfies the equation

$$-\text{div}^\mu (a(x)\nabla^\mu u(x)) + \lambda u(x) = f(x) \quad (2)$$

in $L_2(\mathbb{R}^n, \mu)$, if for any $v \in H^1(\mathbb{R}^n, \mu)$ and any of its gradient $\nabla^\mu v$ it holds:

$$\int_{\mathbb{R}^n} a(x)\nabla^\mu u(x) \cdot \nabla^\mu v(x) d\mu(x) + \lambda \int_{\mathbb{R}^n} u(x)v(x) d\mu(x) = \int_{\mathbb{R}^n} f(x)v(x) d\mu(x).$$
A function $u \in H^1(\mathbb{R}^n, \mu)$ is called a \textit{solution} if the last identity holds for some of its gradients.

\begin{lemma}
The above equation has a unique solution $(u, \nabla^\mu u)$, $u \in H^1(\mathbb{R}^n, \mu)$. Moreover, the choice of the $\mu$-gradient of $u$ is uniquely determined by the condition $a(x)\nabla^\mu u(x) \in (\Gamma_\mu(0))^\perp$.

In the special case $a(x) = \text{Id}$ the integral identity reads

$$
\int_{\mathbb{R}^n} \nabla^\mu u(x) \cdot \nabla^\mu v(x) d\mu(x) + \lambda \int_{\mathbb{R}^n} u(x) v(x) d\mu(x) = \int_{\mathbb{R}^n} f(x) v(x) d\mu(x).
$$

The expression $\text{div}^\mu \nabla^\mu u$ is called the $\mu$-\textit{Laplacian} of $u$.

A gradient $\nabla^\mu u$ of a function $u \in H^1(\mathbb{R}^n, \mu)$ is \textit{tangential} if it is orthogonal to $\Gamma_\mu(0)$. Thus tangential gradient of $u$ is the orthogonal projection of an arbitrary $\mu$-gradient of $u$ on $(\Gamma_\mu(0))^\perp$. 

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Two-connectedness (ergodicity) of measures

**Definition**

A periodic measure $\mu$ is said to be *two-connected* or *ergodic* if any function $u \in H^1(\mathbb{T}^n, \mu)$ such that $\nabla^\mu u = 0$ is equal to a constant $\mu$-a.e.

**Lemma**

Let a measure $\mu$ be 2-connected. Then the set $
\{ g(x) \in L_2(\mathbb{T}^n, \mu) : g(x) = \text{div}^\mu v(x) \}$ is dense in
\[ \left\{ u \in L_2(\mathbb{T}^n) : \int_{\mathbb{T}^n} u(x) d\mu(x) = 0 \right\}. \]

**Exercise**

Let $Q$ be an open connected subset of $\mathbb{T}^n$, and let $d\mu(x) = \chi_Q dx$. Then $\mu$ is 2-connected.
Variational problem

The equation $Au + \lambda u = f$ is an Euler equation of the variational problem

$$\inf_{u \in H^1(\mathbb{R}^n, \mu)} \left\{ \int_{\mathbb{R}^n} \left( a(x) \nabla^\mu u(x) \cdot \nabla^\mu u(x) + \lambda u^2(x) \right) d\mu(x) - \int_{\mathbb{R}^n} 2f(x)u(x) d\mu(x) \right\}$$

Proposition

Let $f \in L_2(\mathbb{R}^n, \mu)$. Then for each $\lambda > 0$ the above variational problem has a unique minimum point $(u, \nabla^\mu u) \in H^1(\mathbb{R}^n, \mu)$. It solves the equation $Au + \lambda u = f$.

Similarly, we can treat the variational problem for the functional

$$\inf_{u \in H^1(\mathbb{R}^n, \mu)} \left\{ \int_{\mathbb{R}^n} \left( a(x) \nabla^\mu u(x) \cdot \nabla^\mu u(x) + c(x)u^2(x) \right) d\mu(x) - \int_{\mathbb{R}^n} 2f(x)u(x) d\mu(x) \right\},$$

where $c(x)$ satisfies the estimate $\Lambda \leq c(x) \leq \Lambda^{-1}$. The Euler equation reads

$$-\text{div}^\mu(a(x) \nabla^\mu u(x)) + c(x)u(x) = f(x).$$
Let $G$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $\mu(dx)$ be a positive finite Borel measure on $G$.

**Definition**

We say that $u \in H^1(G, \mu)$, and $z \in (L_2(G, \mu))^n$ is the gradient of $u$ if there is a sequence $\varphi_k \in C^\infty_0(G)$ such that

$$\varphi_k \rightharpoonup u \quad \text{in } L_2(G, \mu) \quad \text{as } k \to \infty,$$

$$\nabla \varphi_k \rightharpoonup z \quad \text{in } (L_2(G, \mu))^n \quad \text{as } k \to \infty.$$

**Dirichlet problem**

$$-\text{div}(a(x)\nabla u(x)) + c(x)u(x) = f(x) \quad \text{in } L_2(G, \mu)$$

$$u|_{\partial G} = 0.$$

**Definition**

We say that $u \in H^1(G, \mu)$ is a solution to the Dirichlet problem if for any $v \in H^1(G, \mu)$

$$\int_G a(x)(\nabla u(x) \cdot \nabla v(x) + c(x)u(x)v(x)) d\mu(x) = \int_G f(x)v(x) d\mu(x).$$

The existence and the uniqueness of a solution can be established in the standard way.
Dual definition of Sobolev spaces

**Definition**

We say that $u(x) \in H^1(\mathbb{R}^n, \mu)$, and $z(x) \in (L_2(\mathbb{R}^n, \mu))^n$ is a $\mu$-gradient of $u(x)$ if

$$\int_{\mathbb{R}^n} u(x) g(x) d\mu(x) = - \int_{\mathbb{R}^n} z(x) \cdot v(x) d\mu(x),$$

for each $g(x)$ and $v(x)$ such that $g(x) = \text{div}^\mu v(x)$.

**Proposition**

*The two definitions of $H^1(\mathbb{R}^n, \mu)$ are equivalent.*
Consider the elliptic equation

$$-\text{div}^\mu a(x) \nabla^\mu u + \lambda u = f \quad \text{in } L_2(\mathbb{R}^n, \mu),$$

and the family of approximating equations of the form

$$-\text{div}^\mu a_\delta(x) \nabla^\mu u + \lambda u = f_\delta \quad \text{in } L_2(\mathbb{R}^n, \mu^\delta).$$

**Theorem**

Suppose that

$$\Lambda |\xi|^2 \leq a(x) \xi \cdot \xi \leq \Lambda^{-1} |\xi|^2, \quad \Lambda |\xi|^2 \leq a_\delta(x) \xi \cdot \xi \leq \Lambda^{-1} |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n,$$

$$a_\delta(x) \rightarrow a(x) \text{ strongly in } L_2(\mathbb{R}^n, \mu^\delta), \text{ and } f_\delta(x) \rightarrow f(x) \text{ strongly in } L_2(\mathbb{R}^n, \mu^\delta).$$

Then $$u_\delta(x) \text{ strongly converges to } u(x) \text{ in } L_2(\mathbb{R}^n, \mu^\delta) \text{ as } \delta \rightarrow 0.$$
Let $\mu$ be a periodic 2-connected measure in $\mathbb{R}^n$. For every $\xi \in \mathbb{R}^n$ consider the variational problem

$$\hat{A}\xi \cdot \xi = \min_{v \in L^2_{\text{pot}}(\mathbb{T}^n)} \int_{\mathbb{T}^n} (\xi + v(x)) \cdot (\xi + v(x)) d\mu(x).$$

Then $A\xi \cdot \xi$ is a nonnegative quadratic form in $\mathbb{R}^n$. The matrix of this quadratic form, denoted by $\hat{A}$, is called effective.

**Definition**

A periodic measure $\mu$ is *non-degenerate* if $\hat{A}$ is positive definite.

The kernel of $\hat{A}$ is denoted by $K^\mu$. 
Non-degenerate periodic measures

For a periodic matrix $a(x)$ such that

$$\Lambda |\xi|^2 \leq a(x)\xi \cdot \xi \leq \Lambda^{-1} |\xi|^2 \quad \mu - \text{a.e.}$$

define

$$\hat{A}_a \xi \cdot \xi = \min_{v \in L^2_{\text{pot}}(\mathbb{T}^n)} \int_{\mathbb{T}^n} a(x)(\xi + v(x)) \cdot (\xi + v(x)) d\mu(x).$$

**Proposition**

The kernel of $\hat{A}_a$ coincides with the kernel of $\hat{A}$.

$\hat{A}_a$ is called the **effective matrix** of the operator $-\text{div}^{\mu} (a(x) \nabla^{\mu} \cdot)$.
The Euler equation of the above variational problem reads:

\[
\text{find } v_\xi(x) \in L^\text{pot}_2(\mathbb{T}^n, \mu) \text{ such that } a(x)(\xi + v_\xi(x)) \in L^\text{sol}_2(\mathbb{T}^n, \mu).
\]

Denote by \( \Pi_\text{pot} \) the orthogonal projection in \((L_2(\mathbb{T}^n, \mu))^n\) on the subspace \( L^\text{pot}_2(\mathbb{T}^n, \mu) \). Then the Euler equation takes the form:

\[
\text{find } v_\xi(x) \in L^\text{pot}_2(\mathbb{T}^n, \mu) \text{ such that } \Pi_\text{pot}(a(x)v_\xi(x)) = -\Pi_\text{pot}(a(x)\xi).
\]

It is now clear that the operator mapping \( v \in L^\text{pot}_2(\mathbb{T}^n, \mu) \) to \( \Pi_\text{pot}(a(x)v_\xi(x)) \) is coercive in \( L^\text{pot}_2(\mathbb{T}^n, \mu) \).
The effective matrix $\hat{A}_a$ can be written in the form

$$\hat{A}_a \xi = \int_{\mathbb{T}^n} a(x)(v_\xi(x) + \xi))d\mu(x), \quad \xi \in \mathbb{R}^n.$$ 

Denote by $V(x)$ the matrix whose columns are formed by vector-functions $v_{e_1}(x), \ldots, v_{e_n}(x)$ ($\{e_j\}$ are the coordinate vectors in $\mathbb{R}^n$). Then

$$\hat{A}_a = \int_{\mathbb{T}^n} a(x)(Id + V(x)))d\mu(x).$$

**Proposition**

The kernel $K^\mu$ of $\hat{A}$ (or $\hat{A}_a$) coincides with the set of constant potential vectors.

A vector $\eta \in \mathbb{R}^n$ belongs to $(K^\mu)^\perp$ if and only if there is $v \in L^2_{\text{sol}}(\mathbb{T}^n, \mu)$ such that

$$\int_{\mathbb{T}^n} v(x)d\mu(x) = \eta.$$
Adapted cell problem

Consider the modified cell problem

Find \( \nu^+_{\xi}(x) \in L^2_{\text{pot}}(\mathbb{T}^n, \mu) \) such that \( a(x)(\Pi^{\text{eff}} \xi + \nu^+_{\xi}(x)) \in L^2_{\text{sol}}(\mathbb{T}^n, \mu) \),

\( \Pi^{\text{eff}} \) is the orthogonal projection on \((\mathcal{K}^\mu)^\perp\).

Corollary

The relation holds:

\[
a(x)(\xi + \nu_{\xi}(x)) = a(x)(\Pi^{\text{eff}} \xi + \nu^+).
\]

The effective matrix \( \hat{A}_a \) can be expressed by

\[
\hat{A}_a \xi = \int_{\mathbb{T}^n} a(x)(\nu^+_{\xi}(x) + \Pi^{\text{eff}} \xi))d\mu(x), \quad \xi \in \mathbb{R}^n.
\]
Two-scale convergence in variable spaces

Let \( \mu \) be a periodic measure in \( \mathbb{R}^n \). For \( \varepsilon > 0 \) we set \( \mu_\varepsilon(dx) = \varepsilon^n \mu\left(\frac{dx}{\varepsilon}\right) \), i.e., \( \mu_\varepsilon(B) = \varepsilon^n \mu(\varepsilon^{-1}B) \) for any Borel set \( B \subset \mathbb{R}^n \).

The measure \( \mu_\varepsilon \) weakly converge to the measure \( \mu(\square)dx, \quad \square = [0, 1)^n \), as \( \varepsilon \to 0 \). In particular, if \( \mu(\square) = 1 \) then \( \mu_\varepsilon \) converges weakly to the standard Lebesgue measure.

Let \( G \) be a Lipschitz domain in \( \mathbb{R}^n \).

**Definition**

We say that \( u_\varepsilon \in L_2(G, \mu_\varepsilon) \) two-scale converge in \( L_2(G, \mu_\varepsilon) \) to \( u(x, y) \in L_2(G \times \square, dx \times \mu(y)) \), as \( \varepsilon \to 0 \), if

\[
\| u_\varepsilon \|_{L_2(G, \mu_\varepsilon)} \leq C, \quad \varepsilon > 0,
\]

and

\[
\int_G u_\varepsilon(x) \phi(x) \psi\left(\frac{x}{\varepsilon}\right) d\mu_\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} \int_G \int_G u(x, y) \varphi(x) \psi(y) dx d\mu(y)
\]

for any \( \varphi \in C_0^\infty(G) \) and \( \psi \in C^\infty_{\text{per}}(\square) \).
Properties of two-scale convergence

Proposition (weak compactness of a bounded sequence)

Suppose that
\[ \| u^\varepsilon \|_{L^2(G,\mu_\varepsilon)} \leq C. \]

Then, along a subsequence \( \varepsilon_k \to 0 \), the functions \( u^\varepsilon \) two-scale converge in \( L^2(G,\mu_\varepsilon) \) to some function \( u(x,y) \in L^2(G \times \Box, dx \times \mu(y)) \).

Proposition (lower semi-continuity of the norm)

Suppose that \( u^\varepsilon(x) \) two-scale converge in \( L^2(G,\mu_\varepsilon) \) to a function \( u(x,y) \). Then
\[
\liminf_{\varepsilon \to 0} \| u^\varepsilon \|_{L^2(G,\mu_\varepsilon)} \geq \| u(x,y) \|_{L^2(G \times \Box, dx \times \mu(y))}.
\]
Strong two-scale convergence

**Definition**

We say that $u^\varepsilon(x) \in L_2(G, \mu_\varepsilon)$ strongly two-scale converge to $u(x, y) \in L_2(G \times \square, dx \times \mu(y))$ in $L_2(G, \mu_\varepsilon)$ if $u^\varepsilon(x)$ two-scale converge to $u(x, y)$ and

$$\int_G u^\varepsilon(x)v^\varepsilon(x) \, d\mu_\varepsilon(x) \longrightarrow \int_{G \times \square} u(x, y)v(x, y) \, dx \, d\mu(y) \quad \text{as } \varepsilon \to 0.$$

for any $v^\varepsilon(x)$ which two-scale converges in $L_2(G, \mu_\varepsilon)$ to $v(x, y)$.

Equivalent definition reads

**Definition**

We say that $u^\varepsilon(x) \in L_2(G, \mu_\varepsilon)$ strongly two-scale converge to $u(x, y) \in L_2(G \times \square, dx \times \mu(y))$ in $L_2(G, \mu_\varepsilon)$ if $u^\varepsilon(x)$ two-scale converge to $u(x, y)$ in $L_2(G, \mu_\varepsilon)$ and

$$\lim_{\varepsilon \to 0} \int_G |u^\varepsilon(x)|^2 \, d\mu_\varepsilon(x) = \int_{G \times \square} |u(x, y)|^2 \, dx \, d\mu(y).$$
Properties of two-scale convergence

**Proposition**

Suppose that \( u^\varepsilon(x) \in H^1(G, \mu_\varepsilon) \) and
\[
\| u^\varepsilon \|_{L_2(G, \mu_\varepsilon)} \leq C, \quad \lim_{\varepsilon \to 0} \varepsilon \| \nabla^\mu u^\varepsilon(x) \|_{(L_2(G, \mu_\varepsilon))^n} = 0.
\]

Then, along a subsequence, \( u^\varepsilon \) two-scale converge in \( L_2(G, \mu_\varepsilon) \) to some function \( u^0(x) \) which does not depend of \( y \).

\( \mathcal{K}^\mu \) denotes the kernel of \( \hat{A} \), and \( \Pi^\text{eff} \) the operator of orthogonal projection in \( \mathbb{R}^n \) on \( (\mathcal{K}^\mu)^\perp \). We set \( \nabla^\text{eff} = \Pi^\text{eff} \nabla \) and
\[
H^\text{eff}(G) = \{ u \in L_2(G) : \Pi^\text{eff} \nabla u \in (L_2(G))^n \}.
\]

**Theorem**

Suppose that \( \| u^\varepsilon \|_{L_2(G, \mu_\varepsilon)} \leq C \) and \( \| \nabla^\mu u^\varepsilon \|_{(L_2(G, \mu_\varepsilon))^n} \leq C \). Then, along a subsequence,
\[
u^\varepsilon(x) \overset{2}{\to} u^0(x) \quad \text{two-scale in } L_2(G, \mu_\varepsilon) \text{ as } \varepsilon \to 0,
\]
\[
\nabla^\mu u^\varepsilon(x) \overset{2}{\to} \nabla^\text{eff} u^0(x) + u_1(x, y) \quad \text{two-scale in } (L_2(G, \mu_\varepsilon)) \text{ as } \varepsilon \to 0;
\]
with \( u^0 \in H^\text{eff}(G) \) and \( u_1 \in L_2(G; L^\text{pot}_2(\Box, \mu)) \).
Properties of two-scale convergence

Theorem

If

$$\| u^\varepsilon \|_{L_2(G, \mu_\varepsilon)} \leq C, \quad \varepsilon \| \nabla^\mu u^\varepsilon \|_{(L_2(G, \mu_\varepsilon))^n} \leq C,$$

then there is a subsequences $\varepsilon_k \to 0$ and a function $u^0(x, y) \in L_2(G; H^1_{\text{per}}(\Box, \mu))$ such that

$$u^\varepsilon(x) \stackrel{2}{\longrightarrow} u^0(x, y) \quad \text{two-scale in } L_2(G, \mu_\varepsilon),$$

$$\varepsilon \nabla^\mu u^\varepsilon(x) \stackrel{2}{\longrightarrow} \nabla^\mu y u^0(x, y) \quad \text{two-scale in } (L_2(G, \mu_\varepsilon)).$$
Let $\mu$ be a periodic measure in $\mathbb{R}^n$, and let $\mu_\varepsilon = \varepsilon^n \mu\left(\frac{dx}{\varepsilon}\right)$. Consider an elliptic equation

$$-\text{div}^{\mu_\varepsilon}\left(a\left(\frac{x}{\varepsilon}\right)\nabla^{\mu_\varepsilon} u\right) + c\left(\frac{x}{\varepsilon}\right) u = f^\varepsilon(x), \quad \text{in } L_2(\mathbb{R}^n, \mu_\varepsilon),$$

We assume that

$$\Lambda|\xi|^2 \leq a(y)\xi \cdot \xi \leq \Lambda^{-1}|\xi|^2, \quad \xi \in \mathbb{R}^n$$

$\mu$-a.e. We also assume that $0 < c_0 \leq c(y) \leq c_1 \mu$-a.e. We set

$$\hat{c} = \int c(y) d\mu(y).$$
The equation

\[- \text{div}(\hat{A}_a \nabla u) + \hat{c} u = f(x), \quad x \in \mathbb{R}^n,\]

is called homogenized. The solution to this equation is denoted by \(u^0(x)\). Under our assumptions, this equation has a unique solution in \(L_2(\mathbb{R}^n)\).

**Theorem**

If \(f^\varepsilon(x)\) converge strongly (weakly) in \(L_2(\mathbb{R}^n, \mu_\varepsilon)\) to a function \(f(x) \in L_2(\mathbb{R}^n)\), then

\[u^\varepsilon(x) \rightarrow u^0(x) \quad \text{strongly (weakly) in } L_2(\mathbb{R}^n, \mu_\varepsilon) \text{ as } \varepsilon \rightarrow 0,\]

Moreover (flux convergence),

\[a\left(\frac{x}{\varepsilon}\right) \nabla^{\mu_\varepsilon} u^\varepsilon \rightharpoonup \hat{A}_a \nabla^{\text{eff}} u^0(x) \quad \text{weakly in } (L_2(\mathbb{R}^n, \mu_\varepsilon))^n \text{ as } \varepsilon \rightarrow 0.\]

**Proposition**

If \(f^\varepsilon\) converges to \(f\) strongly in \(L^2(\mathbb{R}^n, \mu_\varepsilon)\), then the energy converges:

\[
\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} a\left(\frac{x}{\varepsilon}\right) \nabla^{\mu_\varepsilon} u^\varepsilon(x) \cdot \nabla^{\mu_\varepsilon} u^\varepsilon(x) d\mu_\varepsilon(x) = \int_{\mathbb{R}^n} \hat{A}_a \nabla^{\text{eff}} u^0 \cdot \nabla^{\text{eff}} u^0 dx.
\]
Let $G$ be a Lipschitz bounded domain. Consider the Dirichlet problem

$$-\text{div}^{\mu_\varepsilon} \left(a \left(\frac{x}{\varepsilon}\right) \nabla^{\mu_\varepsilon} u^\varepsilon\right) + c \left(\frac{x}{\varepsilon}\right) u^\varepsilon = f^\varepsilon(x) \quad \text{in} \; L_2(G, \mu_\varepsilon),$$

$$u^\varepsilon \in H^1(G, \mu_\varepsilon).$$

and homogenized Dirichlet problem

$$-\text{div} \left(\hat{A}_1 \nabla^{\text{eff}} u^0\right) + \hat{c} u^0 = f \quad \text{in} \; G, \quad u^0 \in H^0_{\text{eff}}(G).$$

Both problems are well-posed, their solutions are denoted $u^\varepsilon$ and $u^0$.

**Theorem**

If $f^\varepsilon(x)$ strongly (weakly) converges in $L_2(G, \mu_\varepsilon)$ to $f(x) \in L_2(G)$, then,

$$u^\varepsilon(x) \longrightarrow u^0(x) \quad \text{strongly (weakly) in} \; L_2(G, \mu_\varepsilon) \; \text{as} \; \varepsilon \rightarrow 0,$$

Moreover, the flux convergence holds:

$$a \left(\frac{x}{\varepsilon}\right) \nabla^{\mu_\varepsilon} u^\varepsilon \rightharpoonup \hat{A}_1 \nabla^{\text{eff}} u^0(x) \quad \text{weakly in} \; (L_2(G, \mu_\varepsilon))^n \; \text{as} \; \varepsilon \rightarrow 0$$

and, in the case of the strong convergence of $f^\varepsilon$, the energy convergence holds:

$$\lim_{\varepsilon \rightarrow 0} \int_G a \left(\frac{x}{\varepsilon}\right) \nabla^{\mu_\varepsilon} u^\varepsilon(x) \cdot \nabla^{\mu_\varepsilon} u^\varepsilon(x) d\mu_\varepsilon(x) = \int_G \hat{A}_1 \nabla^{\text{eff}} u^0 \cdot \nabla^{\text{eff}} u^0 dx.$$
Let \((\Omega, \mathcal{F}, P)\) be the standard probability space with dynamical system \(T_x\) that is, a group of measurable maps \(T_x : \Omega \rightarrow \Omega\) such that

- \(T_{x+y} = T_x \cdot T_y, \quad x, y \in \mathbb{R}^n, \quad T_0 = Id,\)
- \(P(T_x^{-1}(A)) = P(A)\) for all \(x \in \mathbb{R}^n, \ a \in \mathcal{F},\)
- \(T_x(\omega)\) is a measurable map from \((\mathbb{R}^n \times \Omega, \mathcal{B} \times \mathcal{F})\) to \((\Omega, \mathcal{F})\).

We assume that \(\Omega\) is a **compact metric space**, \(\mathcal{F}\) is the Borel \(\sigma\)-filed on \(\Omega\), and \(T_x\) is **continuous**. We also assume that \(T_x\) is ergodic. Non-ergodic case can also be treated. However, this leads to additional technical difficulties.
A random field $f(x, \omega)$ is stationary (statistically homogeneous) if there is a measurable $\tilde{f} = \tilde{f}(\omega)$ such that $f(x, \omega) = \tilde{f}(T_x \omega)$.

For a Radon measure $\mu$ on $\mathbb{R}^n$ denote by $T_x \mu$ the shift of $\mu$:

$$T_x \mu(B) = \mu(B + x), \quad x \in \mathbb{R}^n, \ B \in \mathcal{B}(\mathbb{R}^n).$$

**Definition**

A family $\mu_\omega$, $\omega \in \Omega$, of Radon measures on $\mathbb{R}^n$ is called a stationary random measure if for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ the random function

$$F_\varphi(x, \omega) = \int_{\mathbb{R}^n} \varphi(x - y) d\mu(y)$$

is measurable and stationary, that is, $F_\varphi(x, \omega) = \tilde{F}_\varphi(T_x \omega)$ with

$$\tilde{F}_\varphi(\omega) = \int_{\mathbb{R}^n} \varphi(y) d\mu(y).$$
Random measures

Let $\Box = [0, 1)^n$. The quantity

$$m = \mathbb{E}(\mu_\omega(\Box)) = \int_\Omega \int_\Box d\mu_\omega(x) dP(\omega)$$

is called the *intensity* of $\mu_\omega$.

We assume that

$$0 < m < \infty$$

and that $\tilde{F}_\varphi \in L^\infty(\Omega, P)$ for any $\varphi \in C_0^\infty$. 
Definition

The *Palm measure* of a random measure $\mu_\omega$ is the measure $\tilde{\mu}$ on $(\Omega, F)$ defined by

$$
\tilde{\mu}(A) = \int_\Omega \int_{\mathbb{R}^n} 1_A(T_x \omega) d\mu_\omega(x) dP(\omega).
$$

Campbell formula.

Theorem

$\tilde{\mu}$ is a finite Borel measure on $\Omega$. For any $f(x, \omega)$ non-negative or $dx \times \tilde{\mu}$-integrable it holds

$$
\int_\Omega \int_{\mathbb{R}^n} f(x, T_x \omega) d\mu_\omega(x) dP(\omega) = \int_\Omega \int_{\mathbb{R}^n} f(x, \omega) d\tilde{\mu}(\omega) dx.
$$
Let $\mu_\omega(x)$ be absolutely continuous w.r.t. the Lebesgue measure $dx$, that is, $d\mu_\omega(x) = \rho(T_x \omega)dx$ with $\rho \in L^1(\Omega, P)$. Then $\tilde{\mu} = \rho(\omega)P$. Indeed,

$$\int_\Omega \int_{\mathbb{R}^n} f(x, T_x \omega) \rho(T_x \omega) dx dP(\omega) = \int_{\mathbb{R}^n} \left( \int_\Omega f(x, T_x \omega) \rho(T_x \omega) dP(\omega) \right) dx$$

$$\int_{\mathbb{R}^n} \int_\Omega f(x, \omega) \rho(\omega) dP(\omega) dx = \int_{\mathbb{R}^n} \int_\Omega f(x, \omega) d\tilde{\mu}(\omega) dx.$$ 

If $\mu_\omega$ is not absolutely continuous then the structure of $\tilde{\mu}$ might be more tricky.
Let $K(y)$ be a $C_0^\infty(\mathbb{R}^n)$ function such that $K \geq 0$, $K(-y) = K(y)$, and \( \int_{\mathbb{R}^n} K(y) dy = 1 \). For any $\delta > 0$ denote

$$
\tilde{\rho}^\delta(\omega) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{y}{\delta}\right) d\mu_\omega(y), \quad \rho^\delta(x, \omega) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{x - y}{\delta}\right) d\mu_\omega(y).
$$

By the stationarity of the measure $\mu_\omega(x)$ we have

$$
\rho(x, \omega) = \tilde{\rho}(T_x \omega).
$$

Consider the measures

$$
d\mu^\delta_\omega(x) = \tilde{\rho}^\delta(T_x \omega) dx, \quad d\tilde{\mu}^\delta(\omega) = \tilde{\rho}^\delta(\omega) dP(\omega).
$$

**Lemma**

As $\delta \to 0$, the measure $\tilde{\mu}^\delta$ converges weakly to $\tilde{\mu}$ in $\Omega$, and the measures $\mu^\delta_\omega$ converges almost surely to the measures $\mu_\omega$ in $\mathbb{R}^n$.
Let $T_x$ be ergodic, and assume that $\mu_\omega$ has finite intensity $m > 0$. Then
\[
\lim_{t \to \infty} \frac{1}{t^n |A|} \int_{tA} g(T_x \omega) d\mu_\omega(x) = \int_{\Omega} g(\omega) d\tilde{\mu}(\omega) \quad \text{a.s. with respect to } P
\]
for all bounded Borel sets $A$, $|A| > 0$, and all $g \in L^1(\Omega, \tilde{\mu})$. 

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Lemma

Let $v \in L^2(\Omega, \tilde{\mu})$. The $P$-a.s. $v(T_x\omega)$ belongs to $L^2_{\text{loc}}(\mathbb{R}^n, \mu_\omega)$. Moreover,

$$
\mathbb{E} \int_A v^2(T_x\omega) d\mu_\omega(x) = |A| \|v\|_{L^2(\Omega, \tilde{\mu})}^2
$$

for any Borel set $A \subset \mathbb{R}^n$ of finite measure.

Proof.

This statement is an immediate consequence of the Campbell formula and the Fubini theorem.

Similar statements hold for $L^p(\Omega, \tilde{\mu})$ functions.
Let us construct \( H^1(\Omega, \tilde{\mu}) \). To this end we consider the set of continuous on \( \Omega \) functions such that the limit

\[
(\partial_i u)(\omega) := \lim_{\delta \to 0} \frac{u(T_{\delta e_i} \omega) - u(\omega)}{\delta}
\]

exists for all \( \omega \in \Omega \) and \( i = 1, 2, \ldots, n \), and the limit function is continuous on \( \Omega \). We denote this set of functions by \( C^1(\Omega) \).

**Lemma**

The space \( C^1(\Omega) \) is dense in \( L^2(\Omega, \tilde{\mu}) \).

**Proof.**

Since \( C(\Omega) \) is dense in \( L^2(\Omega, \tilde{\mu}) \), it is sufficient to approximate a \( C(\Omega) \) function by \( C^1(\Omega) \) functions. Using the continuity of \( T_x \), it is straightforward to check that for any \( \psi \in C(\Omega) \) the sequence

\[
\psi^\delta(\omega) = \delta^{-n} \int_{\mathbb{R}^n} K\left(\frac{y}{\delta}\right) \psi(T_y \omega) dy
\]

approximates \( \psi \), as \( \delta \to 0 \).
Sobolev spaces

Definition

We say that \( u = u(\omega) \) belongs to \( H^1(\Omega, \tilde{\mu}) \), and \( z \in (L^2(\Omega, \tilde{\mu}))^n \) is a gradient of \( u \), if \( u \in L^2(\Omega, \tilde{\mu}) \) and there is a sequence \( u_k \in C^1(\Omega) \) such that

\[
    u_k \longrightarrow u \quad \text{in} \quad L^2(\Omega, \mu), \quad \partial_i u_k \longrightarrow z_i \quad \text{in} \quad (L^2(\Omega, \mu))^n,
\]

as \( k \to \infty \), \( i = 1, 2, \ldots, n \).

The space of pairs \((u, z)\) is denoted by \( H(\Omega, \tilde{\mu}) \). For \( u \in H^1(\Omega, \tilde{\mu}) \) the symbol \( z = \partial u \) stands for a gradient of \( u \).

A gradient need not be unique. We denote be \( \Gamma_{\tilde{\mu}}(0) \) the set of gradients of zero function. This is a closed subspace in \((L^2(\Omega, \tilde{\mu}))^n\).
Lemma

Let $u$ is an element of $(H^1(\Omega, \bar{\mu}))^n$, and $z$ is its gradient. Then $\mathbb{P}$-a.s. $u(T_x\omega)$ belongs to $H^1_{\text{loc}}(\mathbb{R}^n, \mu_\omega)$, and $z(T_x\omega)$ is a gradient of $u(T_x\omega)$ regarded as a function of $x$.

Proof.

Consider an approximating sequence $u_k \in C^1(\Omega)$. Then $u_k(T_x\omega) \in C^1_b(\mathbb{R}^n)$, and

$$\frac{\partial}{\partial x_i}u_k(t_x\omega) = (\partial_i u_k)(T_x\omega).$$

By the Campbell formula

$$E \int_A (u_k(T_x\omega) - u(T_x\omega))^2 d\mu_\omega(x) = |A| \int_\Omega (u_k(\omega) - u(\omega))^2 d\bar{\mu}(\omega) \to 0,$$

as $k \to \infty$. Then, along a subsequence, $\mathbb{P}$-a.s.

$$u_k(T_x\omega) \to u(T_x\omega) \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^n, \mu_\omega).$$

Similarly, $\nabla u_k(T_x\omega) \to z(T_x\omega)$.
Solenoidal and potential fields

Definition

The space $L^2_{\text{pot}}(\Omega, \tilde{\mu})$ of potential vectors is defined to be the closure of the set $\{\partial u : u \in C^1(\Omega)\}$ in $(L^2(\Omega, \tilde{\mu}))^n$.
The space $L^2_{\text{sol}}(\Omega, \tilde{\mu})$ of solenoidal vectors is defined to be the orthogonal complement of $L^2_{\text{pot}}(\Omega, \tilde{\mu})$ in $(L^2(\Omega, \tilde{\mu}))^n$.

Theorem

Let $v \in L^2_{\text{pot}}(\Omega, \tilde{\mu})$. Then $P$-a.s. the realizations $v(T_x \omega)$ belong to $L^2_{\text{pot}, \text{loc}}(\mathbb{R}^n, \mu_\omega)$.

$P$-a.s. the realization of a solenoidal vector-function $w \in L^2_{\text{sol}}(\Omega, \tilde{\mu})$ are solenoidal in the sense of the measure $\mu_\omega$. 

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**Definition**

We say that

\[ g = \text{div}_\omega \nu, \quad g \in L^2(\Omega, \tilde{\mu}), \quad \nu \in (L^2(\Omega, \tilde{\mu}))^n, \]

if

\[ \int_\Omega g(\omega) \psi(\omega) d\tilde{\mu}(\omega) = - \int_\Omega \nu(\omega) \cdot \partial \psi(\omega) d\tilde{\mu}(\omega) \quad \text{for all } \psi \in C^1(\Omega). \]

**Lemma**

If \( g = \text{div}_\omega \nu \), the P-a.s. \( \text{div} \nu(T_x \omega) = g(T_x \omega) \), that is

\[ \int_{\mathbb{R}^n} \nu(T_x \omega) \cdot \nabla \varphi(x) d\mu_\omega(x) = \int_{\mathbb{R}^n} g(T_x \omega) \varphi(x) d\mu_\omega(x), \quad \varphi \in C_0^\infty(\mathbb{R}^n). \]
2-connectedness

**Definition**

The measure $\tilde{\mu}$ is said to be 2-connected if $u = \text{const} \; \tilde{\mu}$-a.s. for every $u \in H^1(\Omega, \tilde{\mu})$ such that $\partial u = 0$.

**Crucial technical result.**

**Lemma**

If $\tilde{\mu}$ is 2-connected, then the set $\{g(\omega)\}$ of functions of the form $g(\omega) = \text{div}_\omega v(\omega)$ is dense in $\{u \in L^2(\Omega, \tilde{\mu}) : \int_\Omega u \, d\tilde{\mu} = 0\}$.
Consider the scaled measures

$$\mu_\omega^\varepsilon(x) = \varepsilon^n \mu_\omega\left(\frac{x}{\varepsilon}\right).$$

and a family of functions $v^\varepsilon \in L^2(G, \mu_\omega^\varepsilon)$.

**Definition**

We say that $v^\varepsilon$ weak two-scale converges, as $\varepsilon \to 0$, to a function $v = v(x, \omega)$, $v \in L^2(G \times \Omega, dx \times \tilde{\mu})$, if

- $\limsup \limits_{\varepsilon \to 0} \int_G |v^\varepsilon(x)| d\mu_\omega^\varepsilon < \infty$.

- For all $\varphi \in C_0^\infty(G)$ and $b \in C^1(\Omega)$ it holds

$$\lim \limits_{\varepsilon \to 0} \int_G v^\varepsilon(x) \varphi(x) b(T_{x/\varepsilon}) d\mu_\omega^\varepsilon(x) = \int_G \int_\Omega v(x, \omega) \varphi(x) b(\omega) d\tilde{\mu}(\omega) dx.$$
Lemma

P-a.s. every family of functions $v^\varepsilon$ such that $\limsup_{\varepsilon \to 0} \|v^\varepsilon\|_{L^2(G;\mu^\varepsilon_\omega)} < \infty$

converges along a subsequence to some $v = v(x, \omega)$ in the sense of weak two-scale convergence.

Lemma

Let

$$\|v^\varepsilon\|_{L^2(G;\mu^\varepsilon_\omega)} < C(\hat{\omega}), \quad \lim_{\varepsilon \to 0} \varepsilon \|\nabla v^\varepsilon\|_{(L^2(G;\mu^\varepsilon_\omega))^n} = 0.$$  

Then, along a subsequence,

$$v^\varepsilon \rightharpoonup v^0(x),$$

where $v^0$ does not depend on $\omega$. 

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Two-scale convergence

For $\xi \in \mathbb{R}^n$ consider

$$A^{\text{eff}}(\xi) = \min_{v \in L^2_{\text{pot}}(\Omega, \tilde{\mu})} \int_{\Omega} |\xi + v|d\tilde{\mu}(\omega).$$

The kernel of $A^{\text{eff}}$ consists of constant potential vectors $\xi \in L^2_{\text{pot}}(\Omega, \tilde{\mu})$. We denote it by $K_{\tilde{\mu}}$.

The orthogonal projection in $\mathbb{R}^n$ to the orthogonal complement of $K_{\tilde{\mu}}$ is denoted by $\Pi^{\text{eff}}$. We set $\nabla^{\text{eff}} = \Pi^{\text{eff}} \nabla$.

Lemma

Let

$$\|v^\varepsilon\|_{L^2(G; \mu^\varepsilon)} < C(\hat{\omega}), \quad \|\nabla_{\mu}^\varepsilon v^\varepsilon\|_{(L^2(G; \mu^\varepsilon))^n} \leq C(\hat{\omega}).$$

Then, along a subsequence,

$$v^\varepsilon \rightharpoonup v^0(x), \quad \nabla_{\mu}^\varepsilon v^\varepsilon \rightharpoonup \nabla^{\text{eff}} v^0(x) + v_1(x, \omega)$$

with $\nabla^{\text{eff}} v^0 \in L^2(G)$ and $v_1 \in L^2(G; L^2_{\text{pot}}(\Omega, \tilde{\mu}))$. 
Lemma

Let

\[ \| v^\varepsilon \|_{L^2(G; \mu^\omega)} < C(\hat{\omega}), \quad \varepsilon \| \nabla^\varepsilon \mu v^\varepsilon \|_{(L^2(G; \mu^\omega))^n} \leq C(\hat{\omega}). \]

Then, along a subsequence,

\[ v^\varepsilon \overset{2}{\rightharpoonup} v^0(x, \omega), \quad \varepsilon \nabla^\varepsilon \mu v^\varepsilon \overset{2}{\rightharpoonup} \partial_\omega v(x, \omega) \]

with \( v \in L^2(G; H^1(\Omega, \tilde{\mu})) \).

Definition

We say that \( u^\varepsilon \) strongly two-scale converge to \( u \) if \( u^\varepsilon \) weakly two-scale converges to \( u \), and

\[ \lim_{\varepsilon \to 0} \int_G v^\varepsilon(x)u^\varepsilon(x)d\mu^\omega(x) = \int_G \int_\Omega v^0(x, \omega)u^0(x, \omega)dxd\tilde{\mu}(\omega) \]

for any weakly two-scale converging sequence \( v^\varepsilon \).
Consider an equation $A^\varepsilon u^\varepsilon + \lambda u^\varepsilon = f^\varepsilon$ in $L^2(\mathbb{R}^n, \mu_\omega)$. The variational formulation reads

$$
\int_{\mathbb{R}^n} a(T_{x/\varepsilon \omega}) \nabla_{\mu^\varepsilon} u^\varepsilon(x) \nabla \varphi(x) d\mu_\omega^\varepsilon(x) + \lambda \int_{\mathbb{R}^n} u^\varepsilon(x) \varphi(x) d\mu_\omega^\varepsilon(x)
$$

$$
= \int_{\mathbb{R}^n} f^\varepsilon(x) \varphi(x) d\mu_\omega^\varepsilon(x).
$$

It is easy to check that $\mathbb{P}$-a.s. the measure $\mu_\omega^\varepsilon$ converges weakly, as $\varepsilon \to 0$, in $\mathbb{R}^n$ to $mdx$, where $m$ is the intensity of the random measure $\mu_\omega$. Without loss of generality we assume that $m = 1$. 
Homogenization

We assume that $\tilde{\mu}$-a.s. the matrix $a(\omega)$ is symmetric and satisfies the uniform ellipticity conditions.

Denote

$$a^{\text{eff}} \xi \cdot \xi = \min \int_{\Omega} a(\omega)(\xi + \nu(\omega)) \cdot (\xi + \nu(\omega)) d\tilde{\mu}(\omega), \quad \nu \in L^2_{\text{pot}}(\Omega, \tilde{\mu}).$$

Assume that $f^\epsilon$ strongly (weakly) converges in $L^2(\mathbb{R}^n, \mu^\epsilon)$ to $f \in L^2(\mathbb{R}^n)$. Consider the following homogenized problem

$$\text{div}(a^{\text{eff}} \nabla u^0) + \lambda u^0 = f$$

or

$$\int_{\mathbb{R}^n} a^{\text{eff}} \nabla u^0(x) \cdot \nabla \varphi(x) dx + \lambda \int_{\mathbb{R}^n} u^0(x) \varphi(x) dx = \int_{\mathbb{R}^n} f(x) \varphi(x) dx$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. 

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Theorem

Assume that $f^\varepsilon$ converges strongly (weakly) in $L^2(\mathbb{R}^n, \mu_\omega)$ to $f \in L^2(\mathbb{R}^n)$, as $\varepsilon \to 0$. Then the solution $u^\varepsilon$ converges strongly (weakly) in $L^2(\mathbb{R}^n, \mu_\omega)$ to the solution $u^0$ of the homogenized problem $\text{P}$-a.s.