Stochastic Variational Inequalities and Random Mechanics

Presentation at CERMICS, Ecole des Ponts Paristech.

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Mathematical interest and connection with applications

- **Risk analysis of failure:** for mechanical structures under random vibrations
Mathematical interest and connection with applications

- Risk analysis of failure: for mechanical structures under random vibrations

- Main application in Earthquake engineering:
Mathematical interest and connection with applications

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- **Main application in Earthquake engineering:**
  - **Collaboration CEA:** Cyril Feau & Laurent Borsoi
    - study the elastic perfectly plastic oscillator excited by a white noise

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Reference:
Karnopp & Scharton, 1966
Feau, 2007
Probabilistic response of an elastic-perfectly-plastic oscillator excitation white noise

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Bensoussan & Turi 2007
Degenerate Dirichlet problems related to the Elasto-Plastic Oscillators, AMO
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- **Risk analysis of failure:** for mechanical structures under random vibrations

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- Risk analysis of failure: for mechanical structures under random vibrations

Main application in Earthquake engineering:

- **Collaboration CEA:** Cyril Feau & Laurent Borsoi
  → study the elastic perfectly plastic oscillator excited by a white noise

- **Reference:**

- **Mathematical Interest:** Connection between this 1D elasto-plastic model and a Stochastic Variational Inequality
  - Bensoussan & Turi 2007 *Degenerate Dirichlet problems related to the Elasto-Plastic Oscillators*, AMO
Outline of the presentation

1. Motivation: An elasto-plastic problem in probabilistic engineering mechanics

2. SVI for the elasto-plastic problem (characterization and computation of the stationary distribution)

3. Short cycles related to the SVI

4. Long cycles related to the SVI

5. Conclusion & open problems
Motivation: An elasto-plastic problem in probabilistic engineering mechanics
Illustrative example: piping system

For a class of structures: A one dimensional (1d) model
  - global behavior of the structure

Seismic excitation (left) / Mass displacement (right): Test vs 1d model results

- nonlinear oscillator with memory
Elastic behavior: start with a linear oscillator ...

$c_0 > 0$ : damping coefficient

$k > 0$ : stiffness

\[
\frac{d w(t)}{d t} : \text{external force white noise}
\]

\[
x(t) : \text{response of the oscillator}
\]
Elastic behavior: start with a linear oscillator ...

\[ c_0 > 0 : \text{damping coefficient} \]
\[ k > 0 : \text{stiffness} \]

\[ "\frac{dw(t)}{dt}" : \text{external force white noise} \]
\[ x(t) : \text{response of the oscillator} \]

**Linear case:** \( x(t) \) solves

\[ \ddot{x}(t) + c_0 \dot{x}(t) + F(t) = \frac{dw(t)}{dt}, \quad F(t) = kx(t) \]
elasto-perfectly-plastic case: $x(t)$ solves

$$\ddot{x}(t) + c_0 \dot{x}(t) + F(t) = \frac{dw(t)}{dt}$$

$|F(t)| \leq kY, \ Y :$ elasto-plastic bound

$\theta_1 :$ first time going into plastic phase
Elastic-perfectly-plastic behavior

**elasto-perfectly-plastic case:** $x(t)$ solves

$$\ddot{x}(t) + c_0 \dot{x}(t) + F(t) = \frac{dw(t)}{dt}, \quad F(t) = k(x(t) - \Delta(t))$$

→ a plastic deformation $\Delta(t)$ occurs in $x(t)$ when $|F(t)| = kY$.

$\tau_1$: first time going out of plastic phase

![Diagram showing plastic deformation and infimum conditions](image reference)
Elastic-perfectly-plastic behavior

elasto-perfectly-plastic case: $x(t)$ solves

$$
\ddot{x}(t) + c_0 \dot{x}(t) + F(t) = \frac{dw(t)}{dt}, \quad F(t) = k(x(t) - \Delta(t))
$$

$\Delta(t)$ stops increasing when $x(t)$ stops increasing.
Balance between elastic and plastic

Denote

\[ y(t) = \dot{x}(t), \quad z(t) = x(t) - \Delta(t) \]

then

\[ \ddot{x}(t) + c_0 \dot{x}(t) + k z(t) = \frac{\text{"d}w(t)\text{"}}{dt} \]

becomes

- **elastic** \( |z(t)| < Y \):

  \[
  \begin{cases}
  \dot{y}(t) = -(c_0 y(t) + k z(t)) + \frac{\text{"d}w(t)\text{"}}{dt}, \\
  \dot{z}(t) = y(t)
  \end{cases}
  \]

- **plastic** \( z(t) = Y, y(t) > 0 \) or \( z(t) = -Y, y(t) < 0 \):

  \[
  \begin{cases}
  \dot{y}(t) = -(c_0 y(t) \pm kY) + \frac{\text{"d}w(t)\text{"}}{dt}, \\
  \dot{z}(t) = 0
  \end{cases}
  \]

**Key idea:** Switching between elastic and plastic phases
An example of phases transition

\[
\begin{align*}
1^{st} \text{ elastic phase: } & [0, \theta_1) \\
\dot{y}(t) &= -(c_0 y(t) + k z(t)) + \frac{d w(t)}{d t} \\
\dot{z}(t) &= y(t)
\end{align*}
\]

\[
\begin{align*}
1^{st} \text{ plastic phase: } & [\theta_1, \tau_1) \\
\dot{y}(t) &= -(c_0 y(t) + k Y) + \frac{d w(t)}{d t} \\
\dot{z}(t) &= Y
\end{align*}
\]

\[y(t) := \dot{x}(t)\]

\[\theta_1 := \inf\{t > 0, |z(t)| = Y\}\]

\[\tau_1 := \inf\{t > \theta_1, y(t) = 0\}\]
An example of phases transition

\[
\begin{align*}
\text{2}\text{st} \text{ elastic phase: } & [\tau_1, \theta_2] \\
\dot{y}(t) &= -(c_0 y(t) + k z(t)) + \frac{dw(t)}{dt}, \\
\dot{z}(t) &= y(t)
\end{align*}
\]

\[
\begin{align*}
\text{2}\text{nd} \text{ plastic phase: } & [\theta_2, \tau_2] \\
\dot{y}(t) &= -(c_0 y(t) - k Y) + \frac{dw(t)}{dt}, \\
z(t) &= -Y
\end{align*}
\]

\[
\begin{align*}
\tau_2 &= \inf\{t > \theta_2, y(t) = 0\} \\
\theta_2 &= \inf\{t > \tau_1, |z(t)| = Y\} \\
\tau_1 &= \inf\{t > \theta_1, y(t) = 0\} \\
\theta_1 &= \inf\{t > 0, |z(t)| = Y\}
\end{align*}
\]
An example of the plastic drift of the oscillator

**Figure:** on the top $t, x(t)$ (red) $t, \Delta(t)$ (black : plastic deformation) and at the bottom $t, z(t)$ for $c_0 = 1, k = 1, Y = 1$
Part 2: Stochastic variational inequality for the elasto-plastic problem
Mathematical tool to describe the right dynamic: the stochastic variational inequality

The problem can be reformulated

- without the plastic deformation $\Delta(t)$,
- without the instants of phase transition.

**Theorem (Bensoussan-Turi 2007)**

The process $(y(t), z(t))$ is the unique solution of the stochastic variational inequality (SVI) defined by the following conditions

$$
dy(t) = -(c_0 y(t) + k z(t)) dt + dw(t),
(\Delta z(t) - y(t) dt)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y
$$

References:

SVIs: [Bensoussan-Lions1982].

The variational inequality: nicely adapted to plastic/elastic transition. → noise effect at the transition from plastic to elastic
Mathematical tool to describe the right dynamic: the stochastic variational inequality

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\]

- $(y(t), z(t))$ reflected diffusion, $\Delta(t)$: reflection process

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- without the plastic deformation $\Delta(t)$,
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**Theorem (Bensoussan-Turi 2007)**

The process $(y(t), z(t))$ is the unique solution of the stochastic variational inequality (SVI) defined by the following conditions

$$
\frac{dy(t)}{dt} = -(c_0 y(t) + kz(t))dt + dw(t),
$$

$$(dz(t) - y(t)dt)(\phi - z(t)) \geq 0, \quad \forall |\phi| \leq Y, \quad |z(t)| \leq Y$$

- $(y(t), z(t))$ reflected diffusion, $\Delta(t)$: reflection process
- Reference:
  - SVIs: [Bensoussan-Lions1982].

- The variational inequality: nicely adapted to plastic/elastic transition.
  - Noise effect at the transition from plastic to elastic
Characterization of the stationary state (balance between elastic and plastic state)

Theorem (Bensoussan-Turi 2007)

\[(y(t), z(t)) \text{ ergodic Markov process}\]

- unique invariant probability distribution \(\nu\) for \((y(t), z(t))\) and \((y(t), z(t)) \xrightarrow{\mathcal{L}} \nu\) (independently of the initial condition).

- elastic domain: \(D := (-\infty, +\infty) \times (-Y, Y)\)
Characterization of the stationary state (balance between elastic and plastic state)

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$(y(t), z(t))$ _ergodic_ Markov process

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- elastic domain: $D := (-\infty, +\infty) \times (-Y, Y)$

- plastic domains: $D^+ := (-\infty, 0) \times \{-Y\}$ and $D^- := (0, +\infty) \times \{Y\}$
Characterization of the stationary state (balance between elastic and plastic state)

**Theorem (Bensoussan-Turi 2007)**

\[(y(t), z(t)) \text{ ergodic Markov process}\]

- **unique invariant probability distribution** \(\nu\) for \((y(t), z(t))\) and \((y(t), z(t)) \overset{\mathcal{L}}{\underset{t \to \infty}{\longrightarrow}} \nu\) (independently of the initial condition).

- elastic domain: \(D := (-\infty, +\infty) \times (-Y, Y)\)
- plastic domains: \(D^+ := (-\infty, 0) \times \{-Y\}\) and \(D^- := (0, +\infty) \times \{Y\}\)
- \(\nu\) has a density denoted by \(m\) is characterized by: \(\forall \varphi\) smooth,

\[
\begin{align*}
\int_D m(y, z) \{ y \varphi_z - (c_0 y + k z) \varphi_y + \frac{1}{2} \varphi_{yy} \} dydz \\
+ \int_{D^+} m(y, Y) \{ -(c_0 y + k Y) \varphi_y(y, Y) + \frac{1}{2} \varphi_{yy}(y, Y) \} dy \\
+ \int_{D^-} m(y, -Y) \{ -(c_0 y - k Y) \varphi_y(y, -Y) + \frac{1}{2} \varphi_{yy}(y, -Y) \} dy &= 0
\end{align*}
\]
Alternative method to the Monte-Carlo simulation (1): Start of my PhD research

From ergodic theory, we know the limiting behavior of \((y(t), z(t))\).

- For all bounded function \(f\) and \(\forall (y_0, z_0) \in \tilde{D},\)

\[
\lim_{t \to \infty} \mathbb{E} f(y_0(t), z_0(t)) = \int_{D} f(y, z)m(y, z)dydz + \int_{D^+} f(Y, y)m(Y, y)dy \\
+ \int_{D^-} f(-Y, y)m(-Y, y)dy.
\]
Alternative method to the Monte-Carlo simulation (1): Start of my PhD research

From ergodic theory, we know the limiting behavior of \((y(t), z(t))\).

- For all bounded function \(f\) and \(\forall (y_0, z_0) \in \tilde{D}\),
  \[
  \lim_{t \to \infty} \mathbb{E}f(y^{y_0}(t), z^{z_0}(t)) = \int_{D} f(y, z)m(y, z)dydz + \int_{D^+} f(Y, y)m(Y, y)dy + \int_{D^-} f(-Y, y)m(-Y, y)dy.
  \]

- But, it is also well known that
  \[
  \lim_{t \to \infty} \mathbb{E}f(y^{y_0}(t), z^{z_0}(t)) = \lim_{\lambda \to 0} \lambda \int_{0}^{\infty} e^{-\lambda t} \mathbb{E}f(y^{y_0}(t), z^{z_0}(t))dt.
  \]
Alternative method to the Monte-Carlo simulation (2)

Denote $u_\lambda(y_0, z_0; f) = \mathbb{E} \left[ \int_0^\infty \exp(-\lambda t)f(y_{y_0}(t), z_{z_0}(t))dt \right]$. 

Alternative method to the Monte-Carlo simulation (2)

Denote $u_\lambda(y_0, z_0; f) = E \left[ \int_0^\infty \exp(-\lambda t) f(y^{y_0}(t), z^{z_0}(t)) dt \right].$

Equivalent characterization of the asymptotic limit:

\[
\begin{align*}
\lambda u_\lambda + Au_\lambda &= f(y, z) \quad \text{in } D \\
\lambda u_\lambda + B_+ u_\lambda &= f(y, Y) \quad \text{in } D^+ \\
\lambda u_\lambda + B_- u_\lambda &= f(y, -Y) \quad \text{in } D^-
\end{align*}
\]

Nonlocal problem: $y \to u_\lambda(y, \pm Y; f)$ are continuous.

$\forall (y_0, z_0) \in \bar{D}$

\[
\lim_{\lambda \to 0} \lambda u_\lambda(y_0, z_0; f) = \int_D f(y, z)m(y, z)dydz
\]
\[
+ \int_{D^+} f(y, Y)m(y, Y)dy + \int_{D^-} f(y, -Y)m(y, -Y)dy
\]
Alternative method to the Monte-Carlo simulation (2)

- Denote \( u_\lambda(y_0, z_0; f) = \mathbb{E} \left[ \int_0^\infty \exp(-\lambda t)f(y_{t_0}(t), z_{t_0}(t))dt \right] \).

- Equivalent characterization of the asymptotic limit:
  \[
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  \lambda u_\lambda + Au_\lambda &= f(y, z) \quad \text{in } D \\
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  \lambda u_\lambda + B_- u_\lambda &= f(y, -Y) \quad \text{in } D^-
  \end{align*}
  \]

- **Nonlocal problem:** \( y \rightarrow u_\lambda(y, \pm Y; f) \) are continuous.

- **Publications:**
  \[\text{Bensoussan, Mertz, Pironneau, Turi 2009, SIAM Journal on Numerical Analysis, Volume 47 Issue 5}\]

  This result is fundamental for the numerical resolution of \( m \): alternative method to Monte-Carlo, that requires simulations for long durations.
Alternative method to the Monte-Carlo simulation (3)

Superposition of three local problems:

\[
\begin{align*}
\lambda \nu_\lambda + A\nu_\lambda &= f \quad \text{in } D, \\
\lambda \nu_\lambda + B_+ \nu_\lambda &= f_+ \quad \text{in } D^+, \\
\lambda \nu_\lambda + B_- \nu_\lambda &= f_- \quad \text{in } D^-, \\
\end{align*}
\]

with \( \nu_\lambda(0^+, Y) = 0, \nu_\lambda(0^-, -Y) = 0, \)
Alternative method to the Monte-Carlo simulation (3)

Superposition of three local problems:

\[
\begin{aligned}
\lambda v_\lambda + Av_\lambda &= f \quad \text{in } D, \\
\lambda v_\lambda + B_+ v_\lambda &= f_+ \quad \text{in } D^+, \\
\lambda v_\lambda + B_- v_\lambda &= f_- \quad \text{in } D^-, \\
\end{aligned}
\]

with \( v_\lambda(0^+, Y) = 0, v_\lambda(0^-, -Y) = 0, \)

\[
\begin{aligned}
\lambda \pi^+_\lambda + A\pi^+_\lambda &= 0 \quad \text{in } D, \\
\lambda \pi^+_\lambda + B_+ \pi^+_\lambda &= 0 \quad \text{in } D^+, \\
\lambda \pi^+_\lambda + B_- \pi^+_\lambda &= 0 \quad \text{in } D^-, \\
\end{aligned}
\]

with \( \pi^+(0^+, Y) = 1, \pi^+(0^-, -Y) = 0, \)
Alternative method to the Monte-Carlo simulation (3)

Superposition of three local problems:

\[
\begin{aligned}
\lambda v_\lambda + Av_\lambda &= f \quad \text{in } D, \\
\lambda v_\lambda + B_+ v_\lambda &= f_+ \quad \text{in } D^+, \\
\lambda v_\lambda + B_- v_\lambda &= f_- \quad \text{in } D^-,
\end{aligned}
\]

with \( v_\lambda(0^+, Y) = 0, \, v_\lambda(0^-, -Y) = 0, \)

\[
\begin{aligned}
\lambda \pi_\lambda^+ + A \pi_\lambda^+ &= 0 \quad \text{in } D, \\
\lambda \pi_\lambda^+ + B_+ \pi_\lambda^+ &= 0 \quad \text{in } D^+, \\
\lambda \pi_\lambda^+ + B_- \pi_\lambda^+ &= 0 \quad \text{in } D^-,
\end{aligned}
\]

with \( \pi^+(0^+, Y) = 1, \pi^+(0^-, -Y) = 0, \)

\[
\begin{aligned}
\lambda \pi_\lambda^- + A \pi_\lambda^- &= 0 \quad \text{in } D, \\
\lambda \pi_\lambda^- + B_+ \pi_\lambda^- &= 0 \quad \text{in } D^+, \\
\lambda \pi_\lambda^- + B_- \pi_\lambda^- &= 0 \quad \text{in } D^-,
\end{aligned}
\]

with \( \pi^+(0^+, Y) = 0, \pi^-(0^-, -Y) = 1. \)
We look for $p_+$ and $p_-$:

\[ v_\lambda + p_+ \pi^+_\lambda + p_- \pi^-_\lambda \]  

continuous in $(0, \pm Y)$
Alternative method to the Monte-Carlo simulation (4)

- We look for $p_+$ and $p_-$:

\[ v_\lambda + p_+ \pi^+_\lambda + p_- \pi^-_\lambda \quad \text{continuous in } (0, \pm Y) \]

- Finally, we solve the following linear system:

\[
\Pi := \begin{pmatrix}
\pi^+ (0^+, Y) - \pi^+(0^-, Y) & \pi^- (0^+, Y) - \pi^- (0^-, Y) \\
\pi^- (-0^+, -Y) - \pi^- (0^-, -Y) & \pi^- (0^+, -Y) - \pi^- (0^-, -Y)
\end{pmatrix}
\]

then

\[
\Pi \begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \begin{pmatrix}
v_\lambda (0^-, Y) - v_\lambda (0^+, Y) \\
v_\lambda (0^-, -Y) - v_\lambda (0^+, -Y)
\end{pmatrix}
\]
Numerical result Vs Monte Carlo method:

\[ Y = 1 \]

- **left:** plot of \( m \) with the deterministic method,
- **right:** plot of \( m \) with the Monte Carlo method,

\( T = 10, \ MC = 10^7 \)(number of trajectories),

\(-1 \leq z \leq 1, \ -7 \leq y \leq 7.\)
Numerical result Vs Monte Carlo method: \( Y = 2 \)

- **left:** plot of \( m \) with the deterministic method,
- **right:** plot of \( m \) with the Monte Carlo method,

\[ T = 10, \ MC = 10^7 \text{(number of trajectories),} \]

\[ -2 \leq z \leq 2, -7 \leq y \leq 7. \]
Part 3: Short cycles related to the stochastic variational inequality
Short cycle behavior

Short cycle: path, solution of the SVI, starting from a point \((y, z) \in D\) and which contains
- only one phase evolving in \(D\) (elastic domain)
- and only one phase evolving in \(D^+\) or \(D^-\) (plastic domains).

\[
y(t) := \dot{x}(t)
\]

\[
\tau := \inf\{t > 0, |z(t)| = Y, y(t) = 0\}
\]

\[
-y(0), z(0)
\]

\[
-Y, Y
\]

\[
y(t), z(t)
\]

Short cycle
Short cycle behavior

- Short cycles: a new analytic characterization to the invariant measure of \((y(t), z(t))\).

- **Key finding:** connection between local problems and nonlocal problems.
  → interpretation of local problems in terms of trajectory of \((y(t), z(t))\).
Definition and analysis of short cycles

Let $f$ be a bounded function on $D$, define $v(y, z; f)$ the solution of

\[ \begin{align*}
Av &= f \text{ in } D, \\
B_+ v &= f \text{ in } D^+, \\
B_- v &= f \text{ in } D^-
\end{align*} \]  

(P$_v$)

with the local boundary conditions

\[ v(0^+, Y) = 0 \text{ and } v(0^-, -Y) = 0 \]

We call $v(y, z; f)$ a short cycle.
Definition and analysis of short cycles

Let \( f \) be a bounded function on \( D \), define \( v(y, z; f) \) the solution of

\[
Av = f \text{ in } D, \quad B_+ v = f \text{ in } D^+, \quad B_- v = f \text{ in } D^-
\]

\( (P_v) \)

with the local boundary conditions

\[
v(0^+, Y) = 0 \text{ and } v(0^-, -Y) = 0
\]

We call \( v(y, z; f) \) a short cycle.

Theorem (Analysis of short cycles, A. Bensoussan, L.M.)

There exists a unique solution to \( (P_v) \) of the form

\[
v(y, z; f) = \varphi^+(y; f)1_{\{y>0\}} + \varphi^-(y; f)1_{\{y<0\}} + w(y, z; f)
\]

where

- \( w \) is a bounded function
- \( B_+ \varphi^+ = f(y, Y), \quad y > 0, \quad \varphi^+(0^+; f) = 0 \)
- \( B_- \varphi^- = f(y, -Y), \quad y < 0, \quad \varphi^-(0^-; f) = 0 \).
New ergodic theorem: Statement of the result

Theorem (New ergodic theorem, A. Bensoussan, L. M.)

A new characterization of the invariant distribution:

\[ \nu(f) = \frac{\nu(0^-, Y; f) + \nu(0^+, -Y; f)}{2\nu(0^-, Y; 1)} \]

A new characterization of the invariant distribution:

\[ \nu(f) = \frac{\nu(0^-, Y; f) + \nu(0^+, -Y; f)}{2\nu(0^-, Y; 1)} \]

Expansion of \( u_\lambda \)

\[ u_\lambda(y, z; f) = u(y, z; f) + \frac{\nu(f)}{\lambda} + o\left(\frac{1}{\lambda}\right) \]

where

\[ Au = f - \nu(f) \text{ in } D \quad B_+u = f - \nu(f) \text{ in } D^+ \quad B_-u = f - \nu(f) \text{ in } D^- \]

with the non local boundary condition given by the fact that \( y \to u(y, \pm Y; f) \) are continuous.
Probabilistic interpretation of the new characterization

Figure: $Av = f, B_+ v = f, B_- v = f$ and $v(0^+, -Y) = 0, v(0^-, Y) = 0$

$y(t) := \dot{x}(t)$

$\tau := \inf\{t > 0, |z(t)| = Y, y(t) = 0\}$

A new characterization of the invariant distribution:

$$\nu(f) = \frac{v(0^-, Y; f) + v(0^+, -Y; f)}{2v(0^-, Y; 1)}$$
Probabilistic interpretation of the new characterization

Figure: \( \dot{A}v = f, B_\pm v = f \) and \( \nu(0^+, -Y) = 0, \nu(0^-, Y) = 0 \)

\[
y(t) := \dot{x}(t)
\]

\[
\tau := \inf\{t > 0, |z(t)| = Y, y(t) = 0\}
\]

A new characterization of the invariant distribution:

\[
\nu(f) = \frac{\nu(0^-, Y; f) + \nu(0^+, -Y; f)}{2\nu(0^-, Y; 1)}
\]

means “formally”

\[
\nu(f) = \frac{1}{2} \mathbb{E}_{(0^-, Y)} \left( \int_0^\tau f(y(s), z(s))ds \right) + \frac{1}{2} \mathbb{E}_{(0^+, -Y)} \left( \int_0^\tau f(y(s), z(s))ds \right)
\]
Part 4: Long cycles related to the stochastic variational inequality
Long cycle behavior

- Independent sequences in the trajectory.
Long cycle behavior

- Independent sequences in the trajectory.

- Long cycle: path, solution of the SVI, starting and ending in one of the two points of \{(0, Y), (0, -Y)\}, knowing that the trajectory had a stop by the other point.
Long cycle behavior

- Independent sequences in the trajectory.

- Long cycle: path, solution of the SVI, starting and ending in one of the two points of \{(0, Y), (0, -Y)\}, knowing that the trajectory had a stop by the other point.

- Long cycles help to characterize the plastic behavior.
Definition and analysis of long cycles

Define

\[
\begin{align*}
t_0 &= \inf\{t > 0, \quad y(t) = 0, \quad |z(t)| = Y\}, \\
\delta &= \text{sign}(z(t_0)), \\
s_0 &= \inf\{t > t_0, \quad y(t) = 0, \quad z(t) = -\delta Y\}.
\end{align*}
\]
Definition and analysis of long cycles

- Define

\[ t_1 = \inf\{ t > s_0, \ y(t) = 0, \ z(t) = \delta Y \} , \]

\[ s_0 = \inf\{ t > t_0, \ y(t) = 0, \ z(t) = -\delta Y \} \]

\[ t_0 = \inf\{ t > 0, \ y(t) = 0, |z(t)| = Y \} \]

\[ \delta = \text{sign}(z(t_0)) \]

Long cycle

\[ t_1 = \inf\{ t > 0, \ y(t) = 0, \ z(t) = \delta Y \} \]
Definition and analysis of long cycles

Then in a recurrent manner, knowing \( t_n \), we can define for \( n \geq 0 \)

\[
\begin{align*}
  t_{n+1} &= \inf\{t > s_n, \quad y(t) = 0, \quad z(t) = \delta Y \}, \\
  s_{n+1} &= \inf\{t > t_{n+1}, \quad y(t) = 0, \quad z(t) = -\delta Y \}.
\end{align*}
\]

Accordingly to these settings, we can define the \( n \)-th long cycle as the piece of trajectory enclosed by \( [t_n, t_{n+1}) \).
Definition and analysis of long cycles

Theorem (Long cycle behavior, A. Bensoussan, L.M.)

In this context, we have

\[
\lim_{t \to +\infty} \frac{\sigma^2(x(t))}{t} = \frac{\mathbb{E}(\int_{t_0}^{t_1} y(s)ds)^2}{\mathbb{E}(t_1 - t_0)}
\]

Key idea: PDEs related to long cycles and connection with short cycles.

submitted: [Bensoussan, Mertz 2011] Behavior of the plastic deformation of an elasto-perfectly-plastic oscillator with noise
PDEs related to Long cycles (type one way)

\[ t_0 = \inf \{ t > 0, y(t) = 0, \left| z(t) \right| = Y \} \]
\[ \delta = \text{sign}(z(t_0)) \]

\[ \bar{v}(0, -Y) = \mathbb{E}_{(0, -Y)} \left( \int_{t_0}^{s_0} f(y(s), z(s))ds \right) \]

where nonlocal problem: \( y \rightarrow \bar{v}(y, -Y) \) is continuous
PDEs related to Long cycles (type return)

\[ t_0 = \inf \{ t > 0, y(t) = 0, |z(t)| = Y \} \]

\[ \delta = \text{sign}(z(t_0)) \]

\[ t_1 = \inf \{ t > 0, y(t) = 0, z(t) = \delta Y \} \]

\[ \delta = \text{sign}(z(t_0)) \]

Long cycle

\[ A \bar{v} = f, \quad B_+ \bar{v} = f, \quad B_- \bar{v} = f \text{ and } \bar{v}(0^+, -Y) = 0 \]

where nonlocal problem: \( y \rightarrow \bar{v}(y, Y) \) is continuous

\[ \bar{v}(0, Y) = \mathbb{E}_{(0,Y)} \left( \int_{s_0}^{t_1} f(y(s), z(s)) ds \right) \]
In this section, we provide computational results which confirm our theoretical results.

\[
\sigma^2(x(t)) \frac{t}{\sigma^2(x(t))}, \ t = 500
\]

\[
\mathbb{E}(\int_{t_0}^{t_1} y(s) ds)^2 \frac{\mathbb{E}(t_1 - t_0)}{\mathbb{E}(t_1 - t_0)}
\]

\[
\mathbb{E}(t_1 - t_0)
\]

<table>
<thead>
<tr>
<th>c_0 = 1, k = 1</th>
<th>Relative error %</th>
</tr>
</thead>
</table>
| \begin{array}{cccc}
0.1 & 0.807 \pm 0.031 & 0.834 \pm 0.069 & 6.61 \pm 0.11 & 3.2 \\
0.2 & 0.649 \pm 0.026 & 0.624 \pm 0.047 & 8.74 \pm 0.13 & 3.8 \\
0.3 & 0.493 \pm 0.020 & 0.464 \pm 0.034 & 10.45 \pm 0.16 & 5.8 \\
0.4 & 0.361 \pm 0.014 & 0.355 \pm 0.026 & 12.12 \pm 0.18 & 1.7 \\
0.5 & 0.266 \pm 0.011 & 0.257 \pm 0.019 & 13.80 \pm 0.21 & 3.3 \\
0.6 & 0.195 \pm 0.008 & 0.198 \pm 0.014 & 16.15 \pm 0.26 & 1.5 \\
0.7 & 0.137 \pm 0.005 & 0.149 \pm 0.011 & 18.84 \pm 0.31 & 8 \\
0.8 & 0.103 \pm 0.004 & 0.112 \pm 0.008 & 22.80 \pm 0.39 & 8 \\
0.9 & 0.071 \pm 0.003 & 0.086 \pm 0.006 & 26.79 \pm 0.47 & 15 \\
\end{array}
\]

Table: Monte-Carlo simulations \( t = 500 \), \( \delta t = 10^{-4} \) and \( MC = 5000 \).
Part 5: Conclusion and open problems
to summarize:

- 1: Presentation of a stochastic variational inequality modeling an elasto-plastic problem with noise
to summarize:

1: Presentation of a stochastic variational inequality modeling an elasto-plastic problem with noise

2: Numerical analysis of the invariant distribution related to the solution
to summarize:

1: Presentation of a stochastic variational inequality modeling an elasto-plastic problem with noise

2: Numerical analysis of the invariant distribution related to the solution

3: New characterization of the invariant distribution by short cycles
to summarize:

1: Presentation of a stochastic variational inequality modeling an elasto-plastic problem with noise

2: Numerical analysis of the invariant distribution related to the solution

3: New characterization of the invariant distribution by short cycles

4: Characterization of the plastic drift by long cycles
Now, we wish to extend our work to the following:

- Short and long cycles in the case of the elastic-plastic problem excited with a filtered noise?
  → previous arguments can not be applied.
Now, we wish to extend our work to the following:

- Ergodicity for an elasto-plastic oscillator with bilinear force?

- Morally, seems to be more ergodic, but much more challenging mathematically.
Fin.

Merci de votre attention.