Hamilton-Jacobi equations
with discontinuous source terms

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(joint work with Yoshikazu Giga)
0 Research interests

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Key words.
Crystal growth, Hamilton-Jacobi equation, Curvature flow equation, Viscosity solution, Maximum principle (Comparison principle), Large time behavior of solutions, Self-similar solution.

My dissertation also includes

- Improvement of the level set method,
- Eikonal equations ($|\nabla u| = f$) on metric spaces.
1 Introduction

\[
\begin{aligned}
(HJ) \quad \begin{cases}
\partial_t u(x, t) + H(x, \nabla_x u(x, t)) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(x, 0) = u_0(x) \in \text{BUC} & \text{in } \mathbb{R}^n.
\end{cases}
\end{aligned}
\]

Hamiltonian \( H(x, p) \), Discontinuous in \( x \), or l.s.c. (lower semicontinuous) in \( x \).

Typical example. \( \partial_t u - |\nabla u| = c I(x) \) i.e., \( I(x) \)

\[
H(x, p) = -|p| - c I(x) \quad (c > 0)
\]

with \( I(x) := \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases} \)

Goal. To introduce a suitable notion of weak solutions (viscosity solutions) so that (HJ) admits a unique global-in-time solution.
Two dimensional nucleation. ([Schulze-Kohn, ’99])
Crystal growth problem with a source of supply (step source).

Typical example. There is a step source at the origin & the horizontal growth speed of crystals is 1.

\(u(x, t)\), the height of crystals at position \(x \in \mathbb{R}^n\) and time \(t \in (0, T)\).

The equation is
\[
\frac{\partial_t u}{|\nabla u|} = c I(x) = \begin{cases} 
c (x = 0), \\
0 (x \neq 0). 
\end{cases}
\]
Which function is a solution of $\partial_t u - |\nabla u| = cI(x)$?

Let $u_0 \equiv 0$. Then

$$u^c(x, t) = c(t - |x|)_+$$

looks like a solution ("cone-shaped function"). $a_+ = \max\{a, 0\}$.

Indeed $u^c$ is a vis. sol. but not a unique sol. ($\forall \alpha \in [0, c), u^\alpha$ is a sol.)

We must strengthen the definition of solutions!!
More general example.
The step source is distributed at several points.

\[ \partial_t u - |\nabla u| = \sum_{j=1}^{N} c_j I(x - a_j). \]

\( c_j > 0 \) is a supplying rate at \( a_j \in \mathbb{R}^n \) \( (a_i \neq a_j (i \neq j)) \).
Previous work on discontinuous H-J eq.

Our typical equation is $\partial_t u - |\nabla u| = cI(x)$.

- **Pioneer.**
  
  [Ishii, ’85]. $u + H(x, u, \nabla u) = 0$, $\partial_t u + H(x, t, u, \nabla u) = 0$.
  
  ($H$, discontinuous in $t$ and $u$.)

- **Eikonal type.**
  
  [Newcomb-Su, ’95], [Ostrov, ’00], [Deckelnick-Elliott, ’04], [Soravia ’06].
  $H(\nabla u) = n(x)$  ($H$, convex & $n$, discontinuous.)
  
  Other stationary types. [Soravia, ’02], [Briani-Davini, ’05].

- **Hamiltonian measurable in $x$.**
  
  [Camilli-Siconolfi, ’05]. $\partial_t u + H(x, \nabla u) = 0$. ($H$, convex in $p$.)

- **Discontinuous coefficients.**
  
  [Deckelnick-Elliott, ’06], [Chen-Hu, ’08], [De Zan-Soravia, ’10].
  $\partial_t u + f(x, t)h(x, \nabla u) = 0$. ($f$, discontinuous & $h$, continuous.)
2 Definition of solutions

Definition (\(\overline{D}\)-viscosity solution). \( u\): bounded from below.
\( u\): \(\overline{D}\)-vis. supersol. \( \iff \tau + H(\hat{x}, p) \geq 0, \forall (p, \tau) \in \overline{D}^{-} u_*(\hat{x}, \hat{t}) \).

(usual definition: \( H^*(\hat{x}, p) \))

- \( f^*(z) = \lim_{y \to z} f(y), f_*(z) = \lim_{y \to z} f(y) \). (semincuous envelopes)
- \( D^- f(z) \): subdifferential, \( \overline{D}^- f(z) \): extended subdifferential.
  \( p \in \overline{D}^- f(z) \iff \exists z_m, \exists p_m \in D^- f(z_m) (m = 1, 2, \ldots) \)
  \( \text{s.t. } (z_m, p_m, f(z_m)) \to (z, p, f(z)) \text{ as } m \to \infty. \)

Remark.
- \( D^- \subset \overline{D}^- \).
- \( H(x, p) = -|p| - \sum c_j I(x - a_j) \Longrightarrow H^*(x, p) = -|p|. \)
3 Comparison principles

(\text{CP}) \ u: \overline{D}-\text{subsol.}, \ v: \overline{D}-\text{supersol.}, \ u^{*}(\cdot, 0) \leq v^{*}(\cdot, 0) \text{ in } \mathbb{R}^{n}.
\implies u^{*} \leq v^{*} \text{ in } \mathbb{R}^{n} \times (0, T).

Assumptions on \( H \).

(H_p) \ |H(x, p) - H(x, q)| \leq L|p - q|.
(H_x) \ |H(x, p) - H(y, p)| \leq L(1 + |p|)|x - y| \quad (\forall |p| \geq N).

\textbf{Theorem (CP).} Assume either [A] or [B].

[A] \( H \) satisfies (H_p) and (H_x).
[B] \( H \) satisfies (H_p). \( u \) or \( v \) is Lipschitz in \( \mathbb{R}^{n} \times (0, T) \).

Then (CP) holds.

\textbf{Remark.} \( H(x, p) = -|p| - \sum c_{j}I(x - a_{j}) \) does not satisfy (H_x).
4 Existence results

Unfortunately, $u^c(x, t) = c(t - |x|)_+ \text{ is not a } \overline{D}\text{-supersol.}$

**Definition (Envelope solutions).** $u$: bounded from below. 

$u$: e-supersol. $\iff \exists S \subset \{w \mid w: \overline{D}\text{-supersol.}\} \text{ s.t. } u = \inf_{w \in S} w.$

**Construction of e-sol.**

1. Approximate $H$ by continuous, “good” $H^\varepsilon$ and solve

   \[
   (\varepsilon\text{-HJ}) \begin{cases}
   \partial_t u^\varepsilon + H^\varepsilon(x, \nabla u^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
   u^\varepsilon|_{t=0} = u_0 & \text{in } \mathbb{R}^n.
   \end{cases}
   \]

2. Show $\bar{u} := \inf_{\varepsilon > 0} u^\varepsilon$ is an e-sol. of (HJ).
Main Theorem.
Assume that $H$ is of the form $H(x, p) = H_0(x, p) - r(x)$, where

- $H_0$ is coercive, UC and $\sup \limits_{x \in \mathbb{R}^n} |H_0(x, p)|$ is loc. bdd. in $p$.
- $r$ is bounded and u.s.c.

Then $\forall u_0 \in \text{BUC}, \exists! u$, e-sol. of (HJ). Moreover $u \in \text{BUC}$.

• $H_0$ is coercive. \(\iff\)
\[
\lim \sup_{|p| \to \infty} H_0(x, p) = -\infty \quad \text{or} \quad \lim \inf_{|p| \to \infty} H_0(x, p) = \infty.
\]

Example.
coercive: $-|p|$, $-|p|^2$.
non-coercive: $0$, $-\frac{|p|}{1 + |p|}$. 
Main Theorem.
Assume that $H$ is of the form $H(x, p) = H_0(x, p) - r(x)$, where

- $H_0$ is coercive, UC and $\sup_{x \in \mathbb{R}^n} |H_0(x, p)|$ is loc. bdd. in $p$.
- $r$ is bounded and u.s.c.

Then $\forall u_0 \in \text{BUC}, \exists! u$, e-sol. of (HJ). Moreover $u \in \text{BUC}$.

- $H_0$ is coercive. $\iff$

\[
\lim_{|p| \to \infty} \sup_{x \in \mathbb{R}^n} H_0(x, p) = -\infty \quad \text{or} \quad \lim_{|p| \to \infty} \inf_{x \in \mathbb{R}^n} H_0(x, p) = \infty.
\]

Proof. Since the Hamiltonian is coercive,

:\text{initial data} \ u_0: \text{Lip.} \implies \text{solution} \ u: \text{Lip.}

Thus we can apply our comparison principle.
(If $u_0: \text{BUC}$, we approximate $u_0$ by $u_0^\delta: \text{Lip.}$) $\Box$
5 Representation formula

Optimal control problem. We study

\[
\begin{aligned}
\text{(HJB)} \quad & \quad \left\{ \begin{array}{l}
\partial_t u - \max_{a \in A} \langle f(x, a), \nabla u \rangle = r(x) \quad \text{in } \mathbb{R}^n \times (0, T), \\
u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^n.
\end{array} \right.
\end{aligned}
\]

Define

\[
v(x, t) := \sup_{\alpha \in \mathcal{A}} C_{x,t}[\alpha]
\]

\[
= \sup_{\alpha \in \mathcal{A}} \left( \int_0^t r(X^\alpha(s)) \, ds + u_0(X^\alpha(t)) \right).
\]

If every function is continuous, then \( v \) is a vis. sol. ([Evans, §10])

How about the case when \( r \) is only u.s.c.?

Then \[v\] is an e-sol. under a suitable controlability assumption on \( A \).
Explicit example 1.
The step source is distributed at several points.

\[
\partial_t u - |\nabla u| = \sum_{j=1}^{N} c_j I(x - a_j).
\]

$c_j > 0$ is a supplying rate at $a_j \in \mathbb{R}^n$ ($a_i \neq a_j (i \neq j)$).
When $u_0 \equiv 0$, the unique e-sol. is

\[
v(x, t) = \max_{j=1}^{N} c_j (t - |x - a_j|)_+.
\]
Explicit example 2.
The step source is concentrated at a general set.

\[ \partial_t u - |\nabla u| = c \chi_S(x). \]

\( c > 0 \) is a supplying rate on a closed subset \( S \subset \mathbb{R}^n \). When \( u_0 \equiv 0 \), the unique e-sol. is

\[ v(x, t) = c(t - \text{dist}(x, S))_. \]
6 Non-coercive Hamiltonian

When $H$ is non-coercive, uniqueness may not hold.

**Example 1.** No horizontal growth.

$$\partial_t u = cI(x) \quad (c > 0).$$

When $u_0 \equiv 0$, $u^c(x, t) = ctI(x)$ looks like a solution.

Indeed $u^c$ is an e-sol. but not a unique e-sol. ($\forall \alpha \in [0, c)$, $u^\alpha$ is a sol.)
Example 2. ([Schulze-Kohn, ’99]) Growth speed is dependent on the gradient of the crystal surface.

\[
\partial_t u - \frac{|\nabla u|}{1 + |\nabla u|} = cI(x) \quad (c > 0).
\]

We are able to prove that \( \forall u_0 \in \text{BUC}, \exists !u, \text{e-sol. if } 0 < c < 1. \)

**Theorem.** Let \( u_0 \equiv 0. \) If \( 0 < c < 1, \)

\[
u^c(x, t) = \begin{cases} 
  c \left( t - \frac{|x|}{1-c} \right) & (|x| \leq (1-c)^2 t), \\
  \{(\sqrt{t} - \sqrt{|x|})_+\}^2 & (|x| \geq (1-c)^2 t)
\end{cases}
\]

is the unique e-sol. of (HJ).
The unique e-sol. is as follows. \((c = 1/2)\)

\[
\partial_t u - \frac{|\nabla u|}{1 + |\nabla u|} = cI(x)
\]

**Remark.** If \(c > 1\), envelope solutions are not unique!!

A well-posedness of this problem depends on the supplying rate \(c\).
7 Conclusion

• We introduced a new notion of viscosity solutions and proved that there exists a unique global-in-time envelope solution of the initial-value problem for

\[ \partial_t u + H_0(x, \nabla u) = r(x) \]

with a continuous, coercive \( H_0 \) and an upper semicontinuous \( r \).

• We obtained a representation formula of the unique solution as a value function of the optimal control problem.

• When \( H_0 \) is non-coercive, uniqueness of solutions may not hold.