Aircraft routing: complexity and algorithms

MPRO - Axel Parmentier

CERMICS

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Table of contents

1 Problem statement
   - Aircraft and crew schedule problems
   - Aircraft routing problem statement

2 Equigraph problem
   - Aircraft routing as a graph cover problem
   - Equigraph properties

3 Polynomially solvable when fleet size is fixed

4 NP-completeness

5 Conclusion
Applications of Operations Research to Air Transport

Airline company management

- Aircraft and crew scheduling
  - Schedule design
  - Fleeting
  - Aircraft routing
  - Crew pairing
  - Matriculation
  - Crew scheduling

Revenue management

- Yield management
  - Fare classes
  - Overbooking
  - Varying prices
  - Go/No shows
  - etc.

Traffic management

- Air traffic management
- Airport management
The four successive aircraft and crew schedule problems

1. Schedule planning

**Inputs:**
OD time dependent demand estimations, previous schedule, etc.

**Outputs:**
A schedule
The four successive aircraft and crew schedule problems

1. Schedule planning
2. Fleet assignment

Fleet assignment

*Inputs:*
A schedule, flights cost (depending on demand and airplane type), fleet sizes

*Outputs:*
A fleeting
The four successive aircraft and crew schedule problems

1. Schedule planning
2. Fleet assignment
3. Aircraft routing

Aircraft routing

Inputs:
A one-fleet schedule, maintenance constraints, border conditions

Outputs:
A feasible routing
The four successive aircraft and crew schedule problems

1. Schedule planning
2. Fleet assignment
3. Aircraft routing
4. Crew pairing

Crew pairing

Inputs:
A one-fleet schedule, a routing, crew working rules

Outputs:
A feasible pairing
Feasible string and feasible routing

Day 1

Day 2

Day 3

Day 4

Paris

NY - Base

Flight
Feasible string and feasible routing

- Cover constraint
Feasible string and feasible routing

- Cover constraint
- Maintenance constraint
- Initial and final conditions
Aircraft routing problem

**Instance:**
- Horizon $H$, Time discretization $T$
- Set of airports $A$, Set of bases $B \subseteq A$
- Set of flights $F \subseteq (A \times [T] \times [H])^2$
- Maintenance constraint $D$
- Initial and final conditions $S^o_a$, $T^o_a$

**Question:**
- Does a feasible routing exist?
From aircraft routing to graph cover
Equigraph problem

Aircraft routing as a graph cover problem

Equigraph definition

An acyclic directed graph is an *equigraph* if its vertices can be partitioned in three sets:

- **Sources** $v \in S$ satisfying $\delta^-(v) = \emptyset$ and $\delta^+(v) \neq \emptyset$.
- **Internal vertices** $v \in I$ satisfying $\delta^-(v) = \delta^+(v) > 0$.
- **Sinks** $v \in S$ satisfying $\delta^-(v) \neq \emptyset$ and $\delta^+(v) = \emptyset$. 

![Equigraph Definition Diagram](image-url)
Directed cuts and nights

A set $C \subseteq A$ is a **directed cut** if there exists a set $U \subseteq V$ such that $C = \delta^-(U)$ and $\delta^+(U) = \emptyset$.

A collection of $d$ directed cuts $N_i = \delta^-(U_i)$ is a collection of **nights** if $T = U_d \subseteq U_{d-1} \subseteq \ldots \subseteq U_1$.
Equigraph routing problem

**Instance:**
- An equigraph \(((S, I, T), A)\)
- Night sets \(N_d\) and maintenance night sets \(M_d \subseteq N_d\)
- Maintenance constraint \(D\)
- Initial and final constraints \(S_s^o, T_t^o\)

**Question:**
- Does a feasible routing exist?

**Theorem:**
Aircraft routing problem and equigraph routing problem are equivalent.
Equigraph cover problem and greedy algorithm

**Instance:**
- An equigraph
  $((S, I, T), A)$

**Solution:**
- An arc cover of $G$ by arc-disjoint $S - T$ paths.

**Greedy algorithm lemma**

If $P$ is a path from $S$ to $T$, then $G \setminus P$ is still an equigraph.
Equigraph characterization by directed cuts

A directed cut \( C = \delta^-(U) \) is *terminal* if \( T \subseteq U \).

**Characterization**

\( G \) is an equigraph \( \iff \) All terminal directed cuts have the same cardinal \( |S| \)

**Proof.**

\( \Rightarrow \) Let \( P \) be a source sink path, then \( |P \cap C| = 1 \)

\( \Leftarrow \) Consider a vertex \( v \) such that \( \delta^+(v) \neq \delta^-(v) \) and maximal directed cuts \( \delta^-(U) \) and \( \delta^-(U \cup \{v\}) \)

\( \square \)
Aircraft routing problem complexity

Theorem

Aircraft routing problem is polynomial when fleet size $k$ is fixed. It can be solved in $O(|F|D^k)$. 

Theorem

Aircraft routing problem is NP-complete in the general case.
Equigraph routing problem – Equigraph coloring problem

Instance:
- An equigraph \(((S, I, T), A)\)
- Night sets \(N_d\) and maintenance night sets \(M_d \subseteq N_d\)
- Maintenance constraint \(D\)
- Initial and final conditions \(S^o, T^o\)

Question:
- Does a feasible routing exist?
Equigraph routing problem – Equigraph coloring problem

Instance:
- An equigraph \(((S, I, T), A)\)
- Night sets \(N_d\) and maintenance night sets \(M_d \subseteq N_d\)
- Maintenance constraint \(D\)
- Initial and final constraints \(S_s^o, T_t^o\)

Question:
- Does a feasible coloring exist?

Theorem:
- Aircraft routing \(\Leftrightarrow\) Equigraph coloring
Proposition: feasible coloring characterization

A coloring is feasible if and only if it satisfies the following properties:

1. **D colors:** $c : A \rightarrow [D]$.

2. **Border conditions:**
   - if $s$ is a source, then $|c^{-1}(d) \cap \delta^+(s)| = S^d_s$ for all $d \in [D]$,
   - if $t$ is a terminal, then $|c^{-1}([d]) \cap \delta^-(t)| \geq \sum_{o=1}^{d} S^d_t$

3. **Color changes happen at nights.**
   - if $a \in B$ is the successor of a maintenance night arc, then $c(a) = 1$
   - if $\delta^-(v) \subseteq (\cup_o N^o) \setminus B$, then $|\delta^-(v) \cap c^{-1}(d)| = |\delta^+(v) \cap c^{-1}(d + 1)|$ for $d \in [D - 1]$
   - if $v \in A \setminus \cup_o N^o$, then $|\delta^-(v) \cap c^{-1}(d)| = |\delta^+(v) \cap c^{-1}(d)|$ for $d \in [D]$
Polynomially solvable when fleet size is fixed

**D-graph**

![Diagram of D-graph with non-maintenance and maintenance night arcs, and labels o = 1, 2, 3.]
**D-graph path**

A path in the $D$-graph is a feasible path in the original graph.
Polynomially solvable when fleet size is fixed

Simplified $D$-graph

![Diagram of a simplified $D$-graph with nodes and edges labeled $o = 1$, $o = 2$, and $o = 3$.]
Directed ordering

An ordering is a *Directed ordering* if

\[(v_i \rightarrow v_j) \Rightarrow v_i \prec v_j\]  \hspace{1cm} (1)
Pebbling game

Game goal:
Move $k$ identical pebbles from the source configuration to the sink configurations set using legal moves.

Vertices are enumerated using a level-respecting ordering. For each vertex $u$, one legal move is done:

1. Pebble initially on $u_{d_u}$ can be moved to $v_{d_v}$ if $(u_{d_u}, v_{d_v})$ is in $G_D$

2. Exactly one copy of each $a \in \delta^-(v)$ is traversed by a pebble.

Lemma

Pebbling game $\Leftrightarrow$ Equigraph coloring can be won is feasible

Initial position
Final position
Unmoved pebble
Polynomially solvable when fleet size is fixed

Pebble configurations

Directed ordering: a set of induced subgraphs $G_i = (V_i, A_i)$ where $V_i = \{v_j | j \geq i\}$

Number of pebbles on $v$

- Before playing $v_i$: $|\delta^-(G)(v)| - |\delta^-(G_i)(v)|$
- After playing $v_i$: $|\delta^-(G)(v)| - |\delta^-(G_{i+1})(v)|$
Algorithm complexity

The configuration graph $G_C$ is defined as follows:

- Vertices $V_C$: Pebble configurations
- Arcs $A_C$: Legal moves. $|A_C| \leq |V| \cdot D^k$

A path from the source configuration to the sink configurations set gives a solution to the pebbling game

**Theorem**

Aircraft routing problem is polynomial when $k$ is fixed. Pebbling game algorithm gives a solution in $O(|F| \cdot D^k)$

**Proof.**

A path finding algorithm is linear in the number of arcs in an acyclic directed graph.
NP-completeness

Theorem
Aircraft routing is NP-complete

Proof.
Reduction from the two commodity arc disjoint paths on acyclic directed graph.

\( S_1 \) \( S_2 \) \( s_1 \) \( s_2 \) \( t_1 \) \( t_2 \) \( T_1 \) \( T_2 \)

- Arcs of initial graph \( H \)
- Initial and terminal arcs
- Balance arcs to obtain an equigraph
Research actually performed

- Generalized framework (network graph / state graph)
  - Crew pairing
  - Aircraft routing with delay
- Integer linear program with fewer variables
- Statistical treatment on delay
- Simultaneous aircraft routing and crew pairing