Homogenized description of multiple Ginzburg-Landau vortices pinned by small holes

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Motivated by discussions with V. Vinokour, Materials ANL:
What to expect in a SC sample with tiny holes?

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Outline of the talk

I Brief review of recent (related) math work on pinning.

II Motivation of our work and formulation of the homogenization problem

III Homogenization of the discrete/continuum problem: Corrector and $\Gamma$-limit

IV Hierarchy of multi-vortices via analysis of the dual problem

Conclusions
I Review: Mathematical models of vortex pinning

Vortices determine EM properties of superconductors that are important for practical applications (e.g., resistance). A key practical issue is to decrease the energy dissipation in superconductors, which occurs due to the motion of vortices. Dissipation can be suppressed by pinning. Specific problem: SC films with periodic array of antidots (holes), optimal design (PRL, 96, 2006).

Goal: What to expect when holes are tiny?, Special scalings?

Brief review of math. models of finitely many pinning sites:
- Lassoued and Mironescu,1999, SGL (no magn. field) discontinuous pinning term for a single inclusion with Dirichlet BC, d-vorticities of degree 1 inside the inclusion.
- Kachmar et. al., 2007-2009 generalized LM for magnetic GL
- Alama &Bronsard, pinning by finitely many normal inclusions (2005) and holes (2006) in magnetic GL. Vortices inside holes and in the bulk of SC (could be infinitely many).
- Dos Santos &Mesiats (2010), SGL, Dirichlet, degree $d$ with finitely many holes, size of holes $\rightarrow 0$ as GL parameter $\rightarrow \infty$, pinning by holes is established.
– Pioneering work by Aftalion, Sandier, Serfaty (2001). Rapidly oscillating continuous pinning term in magnetic GL. An obstacle problem, corresponding to non-multiple vortices was derived.

Infin. many pinning sites, finitely many vortices, no magnetic field: Some composites have essential discontinuities:

– Dos Santos, Mesiats and Mironescu (2010), SGL (no magnetic field) periodic discontinuous pinning term (two values), Dirichlet BC with zero degree. $|u_\varepsilon| \Rightarrow 1$, homogenized solution with no vortices, 2011.

-Dos Santos (2011), SGL degree $d > 0$. Exactly $d$ vortices in the homogenized limit. In dilute limit Renorm. energy (BBH type) is obtained.
While SGL with prescribed Dirichlet data with prescribed degrees works well for finitely many vortices, it may not capture basic physics for large \# of vortices—Sandier & Soret, instead of Abrikosov lattice, vortices concentrate near boundary.

Magnetic field needs to be in the model: Sandier-Serfaty: effective (homogenized) vorticity for large \# of simple vortices (possibly a subdomain). By contrast, pinning sites (prescribed locations) produce multiple vortices in nested subdomains.

Vortex in the hole (a hole vortex) of degree $k$—degree $k$ of the order parameter $u(x)$ on the boundary of the hole.

Mathematics for large \# of vortices even for SGL has not been developed for pinning problems with inclusions (holes)
II Motivation & Formulation of the problem.
Homogenization: large # of pinning sites, large # of vortices & external magnetic field.

Our goal: describe pinning by small holes for large number of vortices and relatively weak magnetic fields such that vortices in the bulk of SC do not appear yet. Find special scaling regimes.

First step: all vortices are pinned by holes (no vortices in the bulk of SC)–potential term amounts to $|u| = 1$ as in harmonic maps.

Next step, use such $S^1$ valued solution for decomposition of energy with potential term (as Alama/Bronsard for finitely many holes).
Consider GL in a perforated domain \( \Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N_\varepsilon} \omega_j^{\varepsilon} \), where \( \Omega \subset \mathbb{R}^2 \) is a bounded simply connected domain, \( \omega_j^{\varepsilon} = B_\rho(a_j^{\varepsilon}) \), and

\[
\kappa^{-1} \ll \rho \ll \varepsilon \quad (1)
\]

(holes radius much greater than vortex core and much less than the spatial period \( \varepsilon \)).

\[
GL_{\kappa}[u, A] = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u - iAu|^2 + \frac{1}{2} \int_{\Omega} |\text{curl}(A) - H|^2 + \frac{\kappa^2}{4} \int_{\Omega_\varepsilon} (|u|^2 - 1)^2 \, dx \quad (2)
\]

Goal: find homogenized (limiting) vorticity of the minimizer \((u_\varepsilon, A_\varepsilon)\) as \( \varepsilon \to 0 \) when vortices are pinned by holes.
By analogy with Serfaty/Sandier for homogeneous SC expect: if $0 < H < C_1 \log \kappa$, then vortices (if exist) are pinned by holes (no vortices in bulk $\Rightarrow$ harmonic map type energy in the bulk).

Increase $H$: $C_1 \log \kappa < H < C_2 \kappa^2$—vortices emerge in the bulk of SC.

For domains with holes need to break $0 < H < C_1 \log \kappa$ into two: no vortices at all, and vortices inside holes but not in the bulk.

Heuristics for scaling. Assume $\int_{cell} h_{ext} \, dx = O(1)$ ($h_{ext}^{\varepsilon} = \sigma \varepsilon^{-2}$).

Effective core of a hole vortex is of order of the hole size $\rho \Rightarrow$ its energy $\log \rho$. Assuming that each hole has a vortex of a finite degree $\Rightarrow \rho = \exp (-\gamma / \varepsilon^2)$.

Much stronger separation than just $\rho \ll \varepsilon$ in (3).
Thus we introduce a strong scale separation for the hole size $\rho$, period $\varepsilon$ and GL-parameter $\kappa$:

$$|\log \kappa^{-1}| \gg |\log \rho| = \gamma \varepsilon^{-2} \quad (3)$$

($|\log \kappa^{-1}| \gg |\log h_{\text{ext}}| = \gamma \varepsilon^{-2}$ -no vortices in the bulk)

Represent minimizer $\hat{u}_{\varepsilon}$: $\hat{u}_{\varepsilon} = u_{\varepsilon} \nu$, where $u_{\varepsilon}$ is minimizer of

$$F_{\varepsilon}(u, A) = \frac{1}{2} \int_{\Omega_{\varepsilon}} |\nabla u - iAu|^2 \, dx + \frac{1}{2} \int_{\Omega} (\text{curl} A - h_{\text{ext}}^\varepsilon)^2 \, dx, \quad \varepsilon > 0, \quad (4)$$

Minimize (4) in $K = \{u \in H^1(\Omega_{\varepsilon}; S^1), \ A \in H^1(\Omega; \mathbb{R}^2)\}$

degrees $d_{j\varepsilon}, \ j = 1, \ldots, N_{\varepsilon}$-unknown integers

Under conditions (3) $\nu$ has no vortices $\Rightarrow$ vorticity density is the same for $\hat{u}_{\varepsilon}$ and $u_{\varepsilon}$—it is sufficient to study minimization of (4).
III Vorticity density and homogenization results

Define the vorticity measure  \( \mu_\varepsilon(x) := \varepsilon^2 \sum_{j=1}^{N_\varepsilon} d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \).
(alt. definition via Jacobian determinant curl\((iu, \nabla_A u) + \text{curl}A\))

\[
\mu(x) := \lim_{\varepsilon \to 0} \mu_\varepsilon(x) \quad \text{(weak limit exists, if bound on } d_j^\varepsilon \text{ established)}
\]

(5)

The limit measure \( \mu \) is defined via the minimizer of the problem

\[
\min \left\{ \frac{1}{2} \int_\Omega (|\nabla f|^2 dx + f^2 + 2\Phi^*(f) + 2\sigma f) dx; \quad f \in H^1_0(\Omega) \right\}, \quad (6)
\]

where \( \Phi^* \) is obtained from homogenization through \( \Gamma \)-limit. Then

\[
\mu = \frac{1}{2\pi} (-\Delta f + f + \sigma), \quad \text{where } (f + \sigma) = h \text{ is homogenized induced magnetic field}.
\]

Observe \( \text{supp}\mu \subset \Omega \) (inner compact). \( \text{supp}\mu \) grows as \( \sigma \uparrow \).

\( \text{meas}(\text{supp}\mu) \to 0 \) as \( \sigma \downarrow \sigma_0 \). \( \sigma_0 \) – 1st critical field when vortices start to appear in holes.
III Homogenization. Coupled discrete/continuum homogenization problem

Minimization of energy (4) is done in two steps. It leads to a coupled discrete/continuum problem with quantization constraints

1. Fix $d_j^\varepsilon$ and get E-L equations

$$\begin{cases} -\Delta h^\varepsilon + h^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \\ h^\varepsilon(x) = h^\varepsilon_{\text{ext}} \text{ on } \partial\Omega, \\ h^\varepsilon(x) = H_j^\varepsilon = \text{unknown const on } \omega_j^\varepsilon, \\ \int_{\partial\omega_j^\varepsilon} \frac{\partial h^\varepsilon}{\partial \nu} \, dx = 2\pi d_j^\varepsilon - \int_{\omega_j^\varepsilon} h^\varepsilon(x) \, dx \end{cases}$$

(7)

2. Substitute $h^\varepsilon(d_1^\varepsilon, \ldots, d_N^\varepsilon)$ into functional (4)

$$F^\varepsilon(u, A) = \frac{1}{2} \int_{\Omega^\varepsilon} |\nabla h^\varepsilon|^2 \, dx + \frac{1}{2} \int_{\Omega} (h^\varepsilon - h^\varepsilon_{\text{ext}})^2 \, dx, \quad \varepsilon > 0,$$

(8)

minimization in (8) wrt to integers $d_1^\varepsilon, \ldots, d_N^\varepsilon$

Homogenization objective: Obtain homogenized equation for magnetic field $\bar{h}(x), \ x \in \Omega$ and limiting vorticity density $\mu(x)$ as $\varepsilon \to 0$. 

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Homogenized description of multiple Ginzburg-Landau vortices
Alama/Bronsard: coupled problem for finite \# of holes. Showed that degrees in each hole $\to \infty$ as $h_{\text{ext}} \to \infty$ (integer parts of large numbers).
(also finitely many normal inclusions and BE).

BR work: large number of holes $\varepsilon^{-2} \to \infty$, $h_{\text{ext}} = \sigma \varepsilon^{-2} \to \infty$.
This scaling corresponds to the case when vortices inside holes have finite multiplicity but are not simple as in homogeneous SC.

The key issue: describe hierarchy of multiplicity by vorticity density.
III Main result on homogenized vorticity

THM. Let $d_j^\varepsilon$ be degrees of the minimizer $u_\varepsilon(x)$ of (4) and

$$\Phi(D) := (2p+1)|D| - p^2 - p,$$

when $p \leq |D| < p+1$, $p = 0, 1, 2, \ldots$ \hfill (9)

Then

$$\mu_\varepsilon = \sum d_j^\varepsilon \varepsilon^2 \delta_{a_j^\varepsilon} \rightharpoonup D(x) \text{ (as distributions)},$$

where $D(x)$ minimizes

$$F(D(x)) = \frac{1}{2} \int_\Omega (|\nabla \bar{h}|^2 + ((\bar{h} - \sigma)^2) dx + \pi \gamma \int_\Omega \Phi(D(x))dx,$$

$$\bar{h} = \bar{h}(D(x))$$ is defined via BVP (solve (12), substitute into (11))

$$\begin{cases} -\Delta \bar{h} + \bar{h} = 2\pi D(x) \text{ in } \Omega \\ \bar{h} = \sigma \text{ on } \partial \Omega \end{cases}$$ \hfill (12)

Q? How does $D(x)$ depend on $\sigma$? e.g., find subdomains of multivortices

$D(x) > 1$. Noninteger $D(x)$: (i) same multiplicity but not each hole (e.g., every second hole) (ii) mix of multiplicities $k$ and $k+1$
Key math points

1. \( \Gamma - convergence \) of finite dimensional problems for integer tuples \( \{d_j^\varepsilon\} \) to an infinite dimensional problem for \( D(x) \). \( \text{Liminf} \) is the key difficulty: need convergence of energies to get optimal LB (matching UP), weak \( \varepsilon^2 h^\varepsilon \to \bar{h} \) is not sufficient, corrector \( R^\varepsilon(x) \) is explicitly constructed for strong convergence. In order to find \( \Gamma \)-limit, use the representation:

\[
F^\varepsilon(\varepsilon^2 h^\varepsilon(x)) = F^\varepsilon(h(x) + R^\varepsilon(x)) + o(1) = \bar{F}_0(\bar{h}) + F_1^\varepsilon + o(1), \quad (13)
\]

\( F_1^\varepsilon = \pi \gamma \sum (d_j^\varepsilon)^2 \varepsilon^2 \) energy corrector \( (R^\varepsilon(x) \rightharpoonup 0, \text{no cross term in (13)}) \)

2. Need \( \lim_{\varepsilon \to 0} \) of \( F_1^\varepsilon = \pi \gamma \sum (d_j^\varepsilon)^2 \varepsilon^2 \), but we only have weak convergence of \( \mu_\varepsilon = \sum d_j^\varepsilon \varepsilon^2 \delta_{a_j^\varepsilon} \) to \( D(x) \) as distributions. If \( d_j^\varepsilon = 1, 0 \) all \( j \) (as in homogeneous SC) \( \Rightarrow \) no issue \( (d_j^\varepsilon = (d_j^\varepsilon)^2) \). Introduce analog of \text{Young measures} on a discrete set \( \mathbb{Z} \) (degrees):

3. Use \text{convex duality} to compute homogenized vorticity \( D(x) \).
**Theorem.** Suppose that the radii of the holes are \( \rho = \exp(-\gamma \varepsilon^2) \) and external field is \( h_{\text{ext}} = \sigma \varepsilon^{-2} \).

Then there exists a strictly increasing sequence of critical values \( \sigma_{\text{cr}j} = \sigma_{\text{cr}j}(\gamma, \Omega), j = 1, 2, \ldots \) such that if \( \sigma_{\text{cr}j} < \sigma < \sigma_{\text{cr}(j+1)} \), then the limiting vorticity takes constant values in subsets \( \Omega_k \setminus \Omega_{k+1} \), where \( \Omega_k = \Omega_k(\sigma), k = 0, 1, \ldots, \) are strictly nested sets (vorticity sets) and \( \Omega_0 = \Omega \). Namely, the vorticity \( D(x) = 0 \) in \( \Omega_0 \setminus \Omega_1 \) and \( D(x) = k \) in \( \Omega_k \setminus \Omega_{k+1}, k \leq j - 1 \).

Finally, when \( x \in \Omega_j \), there are two scenarios: (i) if \( \sigma < 2\pi j + (j - 1/2)\gamma \), then \( (j - 1) < D(x) < j \), otherwise (ii) \( D(x) = j \).
Proposition 0. (Ext. field $h_{\text{ext}} = \sigma \varepsilon^{-2}$, $\sigma$ in multiplicity zero interval).

Let $0 < \sigma < \sigma_{1\text{cr}} = \frac{\gamma}{2 \max |f_1|}$, where $f_1$ solves

$$\Delta f_1 - f_1 = 1, \ x \in \Omega, \ f_1 = 0, \ x \in \partial \Omega. \quad (14)$$

Then the vorticity density $D(x) \equiv 0$ in $\Omega$ (no vortices at all).

![Diagram](image)

**Figure:** minimizer of homogenized problem $\bar{h} : \bar{h} - \sigma = f \in (-\gamma/2, 0)$

Cr. field for holes $\sim \log \rho \ll \log \kappa$ (Cr. field for homogeneous SC).
Domains of multiplicity zero, one and mixed: 
\( \sigma_{1cr} < \sigma < \sigma_{2cr} \), 1st scenario, obstacle problem

Define \( \bar{\sigma}_1(\gamma) := 2\pi + \gamma/2 \)

**Proposition 1a.** (Magn. field in multiplicity zero/one interval). Let \( \sigma_{1cr} < \sigma < \bar{\sigma}_1(\gamma) \). Then minimizer of (6) satisfies \(-\gamma/2 \leq f < 0\) (Fig.) and (6) is equivalent to the obstacle problem \((\sigma > 0 \Rightarrow \text{minimizer } f < 0)\)

\[
\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + |f|^2 + 2\sigma f) dx \; ; \; f \geq -\gamma/2 \right\}. \tag{15}
\]

Moreover, the obstacle domain \( \Omega' = \{ x : f(x) = -\gamma/2 \} \) has nonzero measure, \( 0 < D(x) = \sigma/(2\pi) - \gamma/(4\pi) < 1 \) in \( \Omega' \) (holes with multiplicity 0 and 1 coexist) and \( D(x) = 0 \) in \( \Omega \setminus \Omega' \). \( \Omega' \) monotonically expands as \( \sigma \) increases \((f(x, \sigma_1) \leq f(x, \sigma_2), \forall x \in \Omega, \, \sigma_1 \geq \sigma_2, \text{ convexity of } \Phi^* \Rightarrow \text{max principle for EL})
Hierarchy of vortices: domains of multiplicity zero and one, 2nd scenario, no obstacle

1b. \( \sigma_{2cr} := \) maximal \( \sigma \) such that the minimizer \( \bar{f}(x, \sigma) \) of

\[
\frac{1}{2} \int_{\Omega} (|\nabla f|^2 + |f|^2 + 2\sigma f) \, dx + 2\pi \int_{\Omega} (|f| - \gamma/2)^+ \, dx
\]

(16)
satisfies the inequality \( \min_{x \in \Omega} \bar{f}(x, \sigma) \geq -3\gamma/2 \). (\( \bar{f} \downarrow \) as \( \sigma \uparrow \))

Proposition 1b. If \( \max\{2\pi + \gamma/2, \sigma_{1cr}(\gamma, \Omega)\} < \sigma < \sigma_{2cr}(\gamma, \Omega) \), minimizer of (??) satisfies \(-3\gamma/2 < f \) (no obstacle domain, previous obstacle \( f = -\gamma/2 \) is overcomed), and (6) \( \iff \) (16).
Proposition 2a. Let $\sigma_{2\text{cr}} < \sigma < \bar{\sigma}_2(\gamma) := 4\pi + 3\gamma/2$. Then minimizer of (6) satisfies $-3\gamma/2 \leq f < 0$, and (6) is equivalent to the obstacle problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + |f|^2 + 2\sigma f) dx + 2\pi \int_{\Omega} (|f| - \gamma/2)_+ dx; \ f \geq -3\gamma/2 \right\}$$

Obstacle domain $\Omega'' = \{x : f(x) = -3\gamma/2\}$ has nonzero measure, $1 < D(x) = \sigma/(2\pi) - 3\gamma/(4\pi) < 2$ in $\Omega''$ (holes with multiplicity 1 and 2 coexist) and $D(x) = 1$ in $\Omega' \setminus \Omega''$, $D(x) = 0$ in $\Omega \setminus \Omega'$. 
Hierarchiy of vortices: multiplicity 0, 1 and 2

\[ \sigma_{3cr} := \text{maximal } \sigma \text{ such that the minimizer } f(x, \sigma) \text{ of } \]

\[ \frac{1}{2} \int_{\Omega} (|\nabla f|^2 + |f|^2 + 2\sigma |f|) \, dx + 2\pi \int_{\Omega} [(|f| - \gamma/2)_+ + (|f| - 3\gamma/2)_+] \, dx \quad (18) \]

satisfies the inequality \( \min_{x \in \Omega} f(x, \sigma) \geq -\frac{5\gamma}{2} \).

**Proposition 2b.** Let \( \max\{\sigma_{2cr}, 4\pi + 3\gamma/2\} < \sigma < \bar{\sigma}_{3cr} \). Then minimizer of (6) satisfies \( -\frac{5\gamma}{2} \leq \min_{x \in \Omega} f(x, \sigma) < -\frac{3\gamma}{2} \), and (6) is equivalent to minimization of (18) (no obstacle)

As \( \sigma \) grows nested domains of higher multiplicity appear in the same way.

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Different scaling ⇒ $D(x) \equiv const$ in $\Omega$, no pretty picture

Recall homogenized energy

$$\bar{F}(D(x)) = \frac{1}{2} \int_{\Omega} (|\nabla \bar{h}|^2 dx + (\bar{h} - \sigma)^2) \, dx + \pi \gamma \int_{\Omega} \Phi(D(x)) \, dx$$

$$= \bar{F}_0(\bar{h}(x)) + \bar{F}_1(D(x)), \quad (19)$$

1. Large holes: $F_1(D(x)) = 0$. Minimizer of $\bar{F}_0(\bar{h}(x))$ is $\bar{h} \equiv \sigma$ (=scaled $h_{ext}$). ⇒ constant uniform vorticity $D(x) \equiv \frac{\sigma}{2\pi}$.

2. Very small holes $\bar{F}_0(\bar{h}(x)) \gg F_1(D(x))$, $D(x) \equiv 0, x \in \Omega$ No vortices at all. Inside the sample uniform magnetic field given by

$$- \Delta \bar{h} + \bar{h} = 0, \quad x \in \Omega, \quad \bar{h} = \sigma, \quad x \in \partial \Omega \quad (20)$$

(unless much larger magnetic fields are considered, here we consider weak magnetic fields which do not create bulk vortices).

Nothing interesting in both cases.
Conclusions

- Pinning of a large # of vortices by a large # of periodic small holes in a SC thin film was considered. In a typical situation vortices are uniformly distributed in the domain. However, for special scaling relations vortices of increasingly large multiplicity form nested inner subdomains. These subdomains are described by a sequence of variational (obstacle) problems.

- By contrast, in homogeneous SC the locations are not fixed and it is energetically profitable to have multiplicity 1 for all vortices, which are typically uniformly distributed but may concentrate in an inner subdomain for weak magnetic fields.

- **Main mathematical issues:** the unknown degrees $d_j^\varepsilon$ must be integer (quantization), which required development of a specific $\Gamma$-convergence techniques wrt to convergence of vorticity measures for the coupled discrete/continuum minimization problem. $L_{\text{liminf}}$ is the key difficulty. Why? Need convergence of energies to get *optimal* LB (matching UP), weak $\varepsilon^2 h^\varepsilon \rightharpoonup \tilde{h}$ is not sufficient, **corrector** is needed for strong convergence. Relevant analog of **Young measures** introduced to handle quantization.