On a Quasilinear System Involving Curl

Part 1: Reduced System

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Outline

1. Meissner States
2. Problem in 2 Dimensions
3. Reduced System in 3 Dimensions
4. Proof of Theorem 1
§1. Meissner States
Meissner States

A type II superconductor in an increasing applied magnetic field $H^e$ undergoes phase transition from the Meissner state to the mixed state.

Meissner state is

\[
\begin{aligned}
\text{stable if} & \quad H^e < H_{c_1}, \\
\text{locally stable if} & \quad H_{c_1} < H^e < H_S, \\
\text{unstable if} & \quad H_S < H^e < H_{sh},
\end{aligned}
\]

where $H_{sh}$ is the superheating field, and vortices nucleate in the samples when the applied field reaches $H_{sh}$.
$H_{C_1}$ \textbf{viz} $H_{sh}$

$H_{C_1}$:

- At $H_{C_1}$ superconductors lose global minimality (for Ginzburg-Landau energy), and vortices nucleate in the interior.

- V. Ginzburg and L. Landau, A. Abrikosov, H. Brezis et al., E. Sandier and S. Serfaty, Fanghua Lin et al., Qiang Du et al., S. Jimbo and Y. Morita, and many many more.
At $H_S$ superconductors lose local minimality (for reduced energy), and at $H_{sh}$ vortices nucleate at boundary. These phenomena have not been well-understood yet. See P. de Gennes (1965), J. Matricon, D. St-James (1967), H. Fink (1966), L. Kramer (1967), Chapman et al. (1995-97).
Questions

- How does a Meissner state become unstable?
  Find $H_S$.

- How do vortices nucleate?
  Find $H_{sh}$ and find the location of nucleation.
Ginzburg-Landau theory

Superconductivity is described by order parameter and magnetic potential.

**Ginzburg-Landau energy functional**

\[
\mathcal{G}[\psi, A] = \int_{\Omega} \left\{ \left| \frac{\lambda}{\kappa} \nabla \psi - i A \psi \right|^2 + \frac{1}{2} (1 - |\psi|^2)^2 \right\} dx + \int_{\mathbb{R}^3} |\lambda \text{curl } A - \mathcal{H}^e|^2 dx,
\]

\(\mathcal{H}^e = \gamma \mathcal{H}^e\), div \(\mathcal{H}^e = 0\), curl \(\mathcal{H}^e = 0\) in \(\mathbb{R}^3\).

\(\Omega\): domain in \(\mathbb{R}^3\) occupied by the superconductor,

\(\kappa\): Ginzburg-Landau parameter, \(\kappa = \lambda/\xi\),

\(\lambda\): penetration length, \(\xi\): coherence length.
Ginzburg-Landau system

\[
\begin{cases}
-\nabla^2_{\kappa \lambda^{-1}} A \psi = \kappa^2 \lambda^{-2} (1 - |\psi|^2) \psi & \text{in } \Omega, \\
\lambda^2 \text{curl}^2 A = \lambda \kappa^{-1} \Im (\bar{\psi} \nabla_{\kappa \lambda^{-1}} A \psi) & \text{in } \Omega, \\
\text{curl}^2 A = 0 & \text{in } \Omega^c, \\
(\nabla_{\kappa \lambda^{-1}} A \psi) \cdot \nu = 0, \quad [A_T] = 0, \quad [(\text{curl } A)_T] = 0 & \text{on } \partial \Omega, \\
\lambda \text{curl } A - \mathcal{H}^e \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]

(GL)
\[ \Omega^c = \mathbb{R}^3 \setminus \bar{\Omega}, (\cdot)_T: \text{tangential component at } \partial \Omega. \]

\[ A_T = (\nu \times A) \times \nu. \]

\[ [\cdot]: \text{jump across } \partial \Omega, [B] = B^+ - B^-, \text{where } B^+, B^- \text{ are the outer and inner trace of } B \text{ at } \partial \Omega, \]

\[ \nabla_A \psi = \nabla \psi - iA \psi, \]

\[ \nabla_A^2 \psi = (\nabla - iA)^2 \psi = \Delta \psi - i(2A \cdot \nabla \psi + \psi \text{div } A) - |A|^2 \psi. \]
Meissner solutions

Starting at Meissner state, assume

$$\psi = fe^{i\chi}, \quad \mathcal{A} = A + \frac{\lambda}{\kappa}\nabla \chi, \quad f > 0.$$  

Energy for Meissner states

$$\mathcal{G}[\psi, \mathcal{A}] = \mathcal{E}[f, A]$$

$$= \int_{\Omega} \left\{ \frac{\lambda^2}{\kappa^2} |\nabla f|^2 + |A|^2 |f|^2 + \frac{1}{2} (1 - |f|^2)^2 \right\} dx$$

$$+ \int_{\mathbb{R}^3} |\lambda \text{curl } A - \mathcal{H}^c|^2 dx.$$
Meissner States

Problem in 2 Dimensions

Reduced System in 3 Dimensions

Proof of Theorem 1

Equations for Meissner states

\[
\begin{aligned}
\frac{-\lambda^2}{\kappa^2} \Delta f &= (1 - f^2 - |A|^2)f \quad \text{in } \Omega, \\
\lambda^2 \text{curl}^2 A + f^2 A &= 0 \quad \text{in } \Omega, \\
\text{curl}^2 A &= 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \quad (1) \\
\frac{\partial f}{\partial \nu} &= 0, \quad [A_T] = 0, \quad [(\text{curl } A)_T] = 0 \quad \text{on } \partial \Omega, \\
\lambda \text{curl } A - \mathcal{H}^e &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty.
\end{aligned}
\]

Definition

\((f, A)\) is called a Meissner solution if \(f > 0\) on \(\bar{\Omega}\) and if \(\nu \cdot A = 0\) on \(\partial \Omega\) (which comes from (GL)).
S. J. Chapman (1995): Let $\kappa \to \infty$. Formally

$$\frac{\lambda^2}{\kappa^2} \Delta f \sim 0,$$

so $f^2 \sim 1 - |A|^2$,

and $A$ solves

\[
\begin{cases}
-\lambda^2 \text{curl}^2 A = (1 - |A|^2)A & \text{in } \Omega, \\
\text{curl}^2 A = 0 & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\
[A_T] = 0, & \text{on } \partial \Omega, \\
[(\text{curl } A)_T] = 0 & \text{on } \partial \Omega, \\
\lambda \text{curl } A - \mathcal{H}^e \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]
(2) has a particular solution obtained by first solving

\[ \begin{cases}
-\lambda^2 \text{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} & \text{in } \Omega, \\
\lambda (\text{curl } \mathbf{A})_T = \mathcal{H}_T & \text{on } \partial \Omega.
\end{cases} \]
In the 2 dimensional case, if $\Omega$ simply-connected and $H = \text{curl} \, \mathbf{A}$ satisfies

$$\nabla H = \mathbf{0} \text{ in } \Omega^c, \quad H - \mathcal{H}^e \rightarrow \mathbf{0} \text{ as } |x| \rightarrow \infty,$$

which implies $H \equiv \mathcal{H}^e$ is a constant in $\Omega^c$, (2) is reduced to (3) on $\Omega$. 
Stability

Lemma

(i) A solution of (1) is locally stable (for the reduced functional $E$) provided

$$\inf_{x \in \bar{\Omega}} \{ f^2(x) - |A(x)|^2 \} > \frac{1}{3}.$$ 

(ii) A solution $A$ of (2) (resp. of (3)) satisfying the length condition $|A| < 1/\sqrt{3}$ is locally stable (for the associated energy).

Questions

- **Question 1.** Find the conditions on $\mathcal{H}^\epsilon$ such that (2) has locally stable solutions. Study the behavior of these solutions.

- **Question 2.** Find the conditions on $\mathcal{H}^\epsilon$ such that the locally stable solutions of (1) approach a solution of (2) as $\kappa \to \infty$. 
Reduction System for Magnetic Fields \( H \) in \( \Omega \)

If \( A \) is a solution of (3) and \( \|A\|_{L^\infty(\Omega)} < 1/\sqrt{3} \) then \( H = \lambda \text{curl} A \) is a solution of

\[
\begin{align*}
-\lambda^2 \text{curl} \left[ F(\lambda^2 |\text{curl} H|^2) \text{curl} H \right] &= H \quad \text{in } \Omega, \\
H_T &= \mathcal{H}_T^e \quad \text{on } \partial \Omega,
\end{align*}
\]

(4)

\[
\lambda \|\text{curl} H\|_{L^\infty(\Omega)} < \frac{2}{\sqrt{27}}.
\]
$F$ is determined by

$$v = F(t^2) t \iff t = (1 - v^2)v, \quad 0 \leq t \leq \frac{2}{\sqrt{27}}, \quad 0 \leq v \leq \frac{1}{\sqrt{3}},$$

and $F(0) = 1$.

The solutions satisfy

$$\text{div } H = 0.$$
§2. Problem in 2 Dimensions
Reduce to scalar equation in 2 dimensions

Consider a cylindrical superconductor with simply-connected cross section $D \subset \mathbb{R}^2$, and $\mathcal{H}^e = h \mathbf{e}_3$, $h$ is a constant, then

$$\mathbf{H} = H \mathbf{e}_3,$$

and $(2) \iff (3) \iff (3')$

$$\begin{cases} -\lambda^2 \text{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2) \mathbf{A} & \text{in } D, \\ \lambda \text{ curl} \mathbf{A} = h & \text{on } \partial D. \end{cases}$$
Under condition $\|A\|_{L^\infty(\Omega)} < 1/\sqrt{3}$, Eq. (3') is equivalent to

$$\left\{ \begin{array}{l} \lambda^2 \text{div} \left[ F(\lambda^2|\nabla H|^2) \nabla H \right] = H \quad \text{in } D, \\
H = h \quad \text{on } \partial D. \end{array} \right. \quad (4')$$
Problem and results in 2 dimensions

**Chapman’s conjecture**

(i) Maximum points of $|A|$ locate at the most negatively curved points of boundary for small $\lambda$.

(ii) This instability leads to the formation of vortices.

(iii) Maximum points of $|A(x)|$ correspond to location of vortices.

**Berestycki-Bonnet-Chapman (1994)**

Under condition $\|A\|_{L^\infty(\Omega)} < 1/\sqrt{3}$, the maximum points of $|A|$ locate at boundary.
Problem in 2 Dimensions

Meissner States

Reduced System in 3 Dimensions

Proof of Theorem 1

Problem and results in 2 dimensions (continuous)


For $\lambda = 1$, the Meissner solution exists for $h \in [0, H_0^*)$ and it converges to a solution of $(3')$ as $\kappa \to \infty$.

—In the 2 D case, thanks to the Gagliardo-Nirenberg inequality, uniform convergence is a consequence of the energy convergence.
Problem and results in 2 dimensions (continuous)

Pan-Kwek JDE (2002)

(i) The maximum points of $|A|$ approach the minimum points of curvature of boundary as $\lambda \to 0$;

(ii) $A$ and $\text{curl} \ A$ exhibit boundary layer behavior.

Beresticki’s problem. Find the location of the maximum points of $A$ for finite $\lambda$. 
§3. Reduced System in 3 Dimensions
Reduced system for magnetic fields $H$ in $\Omega$

\[
\begin{cases}
-\lambda^2 \text{curl} \left[ F(\lambda^2 \text{curl} \, H^2) \text{curl} \, H \right] = H & \text{in } \Omega, \\
H_T = \mathcal{H}_T^e & \text{on } \partial \Omega,
\end{cases}
\]

Existence for small boundary dates.

**R. Monneau (2003)**

Let $\lambda = 1$. If $\Omega$ is a smooth domain homeomorphic to a ball and if $|\mathcal{H}_T^e|$ is sufficiently small, then (4) has a unique solution $H$, and $|\text{curl} \, H|$ attains its maximum on $\partial \Omega$. 
Weak solutions

Can not avoid weak solutions.

(i) Optimal bound of boundary data to allow existence for all small $\lambda$.

(ii) Weak limit of rescaled solutions as $\lambda \to 0$. 
Two approaches for existence

First approach:
Existence for small boundary data $\implies H^2$ regularity under a prior $L^\infty$ bound of $\text{curl} \, H \implies C^{2+\alpha}$ regularity.

Second approach:
establish directly existence and regularity of weak solutions by variational method, without assuming $\text{curl} \, H \in L^\infty(\Omega)$. 
Variational approach

Monneau’s question (2003): why not variational?

**Pan, Cal. Var. PDE 2009**

Existence and regularity obtained using variational method for the modified functional.

Actually we worked out for a more general functional for anisotropic superconductors.
Our conditions

(i) $\Omega$ is a bounded and simply-connected domain in $\mathbb{R}^3$ without holes and with $C^4$ boundary.

(ii) $\mathcal{H}^e_T$ satisfies

\[ \mathcal{H}^e_T \in C^{2+\alpha}(\partial \Omega, \mathbb{R}^3), \]

\[ \| \mathcal{H}^e_T \|_{C^0(\partial \Omega)} < \sqrt{\frac{5}{18}}, \]

\[ \nu \cdot \text{curl } \mathcal{H}^e_T = 0. \]
Theorem 1 (Bates-Pan CMP 2007)

For small $\lambda > 0$, (4) has a unique solution $H^{\lambda} \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$.

(i) $\lambda \| \text{curl } H^{\lambda} \|_{L^\infty(\Omega)} < 2/\sqrt{27}$;

(ii) If $\rho_\lambda \leq \frac{c}{\lambda}$ and $\rho_\lambda \to \infty$ as $\lambda \to 0$, then

$$\lim_{\lambda \to 0} \sup_{\text{dist}(x, \partial \Omega) \geq \lambda \rho_\lambda} |H^{\lambda}(x)| = 0.$$

(iii) For $\mu = \| \mathcal{H}_T^e \|_{C^0(\partial \Omega)}$,

$$\lim_{\lambda \to 0} \lambda \| \text{curl } H^{\lambda} \|_{C^0(\partial \Omega)} = [1 - (1 - 2\mu^2)^{1/2}] (1 - 2\mu^2).$$

(iv) If $P^{\lambda}$ is a maximum point of $|\text{curl } H^{\lambda}(x)|$ and if $P^{\lambda} \to P$ for a sequence $\lambda_n \to 0$, then $|\mathcal{H}_T^e(P)| = \| \mathcal{H}_T^e \|_{C^0(\partial \Omega)}$. 
Corollary

In particular if $\mathcal{H} = \mathbf{h}$, a constant vector, and if

$$|\mathbf{h}| < \sqrt{\frac{5}{18}},$$

then $P \in (\partial \Omega)_h \equiv \{x \in \partial \Omega : \mathbf{h} \text{ is tangential to } \partial \Omega \text{ at } x\}$. 
§4. Proof of Theorem 1

Recall: we assume $\Omega$ is a bounded and simply-connected domain in $\mathbb{R}^3$ without holes and with smooth boundary.
4.1. Control \( \nabla B \) by \( \text{div} \ B \), \( \text{curl} \ B \), \( \nu \cdot B \) or \( B_T \).

**Lemma 1.**

\[
\| B \|_{H^{k+1}(\Omega)} \leq C_1(\Omega) \left\{ \| \text{div} \ B \|_{H^k(\Omega)} + \| \text{curl} \ B \|_{H^k(\Omega)} + \left\| \nu \cdot B \right\|_{H^{k+1/2}(\partial \Omega)} + \left\| \nu \times B \right\|_{H^{k+1/2}(\partial \Omega)} \right\},
\]

\[
\| B \|_{C^{k+1+\alpha}(\tilde{\Omega})} \leq C_2(\Omega, k, \alpha) \left\{ \| \text{div} \ B \|_{C^{k+\alpha}(\tilde{\Omega})} + \| \text{curl} \ B \|_{C^{k+\alpha}(\tilde{\Omega})} + \left\| \nu \cdot B \right\|_{C^{k+1+\alpha}(\partial \Omega)} + \left\| \nu \times B \right\|_{C^{k+1+\alpha}(\partial \Omega)} \right\}.
\]
4.2. Boundary condition $A \cdot \nu = 0$.

**Lemma 2.**

If $A, B \in H^2(\Omega, \mathbb{R}^3)$ and if $A_T = B_T$ in $H^{3/2}(\partial \Omega, \mathbb{R}^3)$, then

$$\nu \cdot \text{curl} \, A = \nu \cdot \text{curl} \, B \quad \text{in} \, H^{1/2}(\partial \Omega).$$

If $\mathcal{H}_T^e$ satisfies

$$\nu \cdot \text{curl} \, (\mathcal{H}_T^e) = 0 \quad \text{on} \, \partial \Omega,$$

and if $A \in C^2(\bar{\Omega}, \mathbb{R}^3)$ is a solution of (3) and $\|A\|_{L^\infty(\Omega)} < 1/\sqrt{3}$, then

$$A \cdot \nu = 0 \quad \text{on} \, \partial \Omega. \quad (1.1)$$
4.3. Extension of vector fields.

Lemma 3.

If $\mathcal{H}_T \in H^{3/2}(\partial \Omega, \mathbb{R}^3)$, there exists $H \in H^2(\Omega, \mathbb{R}^3)$ s.t.

$$\nabla H = 0 \quad \text{in} \ \Omega, \quad H_T = \mathcal{H}_T^e \quad \text{on} \ \partial \Omega,$$

(1.2)

$$\|H\|_{H^2(\Omega)} \leq C(\Omega)\|\mathcal{H}_T^e\|_{H^{3/2}(\partial \Omega)}.$$

$H$ can be chosen such that the $L^2$ norm of $\nabla \times H$ is minimal among all vector fields satisfying (1.2).

If in addition $\mathcal{H}_T \in C^{2+\alpha}(\partial \Omega, \mathbb{R}^3)$, then $H \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ and

$$\|H\|_{C^{2+\alpha}(\bar{\Omega})} \leq C(\Omega, \alpha)\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial \Omega)}.$$
Lemma 3’

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^{k+1}$ boundary, $k \geq 1$, $0 \leq \alpha < 1$, and $B_T \in C^{k+\alpha}(\partial \Omega, \mathbb{R}^3)$ satisfies

$$\nu \cdot B_T = 0 \quad \text{and} \quad \nu \cdot \text{curl} \, B_T = 0 \quad \text{on} \, \partial \Omega.$$

(i) $B_T$ can be extended to $\Omega$ as a curl-free $C^{k+\alpha}$ vector field, namely there exists $\tilde{B} \in C^{k+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ such that

$$\text{curl} \, \tilde{B} = 0 \quad \text{in} \, \Omega, \quad \tilde{B}_T = B_T \quad \text{on} \, \partial \Omega.$$

(ii) If furthermore $\Omega$ is simply-connected, then there exists a harmonic function $\phi \in C^{k+1+\alpha}(\bar{\Omega})$ such that $(\nabla \phi)_T = B_T$. 


4.4. $H^2$ estimates of weak solutions.

Lemma 4.

Assume $\mathcal{H}_T^e \in H^{3/2}(\partial\Omega, \mathbb{R}^3)$. Let $H \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution of (4) and

$$\lambda \|\text{curl } H\|_{L^\infty(\Omega)} \leq M < \frac{2}{\sqrt{27}}.$$ 

Then $H \in H^2(\Omega, \mathbb{R}^3)$, and

$$\|H\|^2_{H^2(\Omega)} \leq C(\Omega, M, \lambda) \left\{ \|\mathcal{H}_T^e\|_{L^1(\partial\Omega)} + \|\mathcal{H}_T^e\|^2_{H^{3/2}(\partial\Omega)} \right\}. $$
4.5. $C^{2+\alpha}$ estimates of weak solutions.

**Proposition 5.**

Assume $0 < \alpha < 1$,

$$\mathcal{H}_T^e \in C^{2+\alpha}(\partial \Omega), \quad \nu \cdot \text{curl} \, \mathcal{H}_T^e = 0 \text{ on } \partial \Omega. \quad (1.3)$$

Let $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution of (4) and

$$\lambda \|\text{curl} \, \mathbf{H}\|_{L^\infty(\Omega)} \leq M < \frac{2}{\sqrt{27}}.$$  

Then $\mathbf{H} \in C^{3+\delta}(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$, and

$$\|\mathbf{H}\|_{C^{2+\alpha}(\bar{\Omega})} \leq C(\Omega, \lambda, \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial \Omega)}, M, \alpha). \quad (1.4)$$
Proof of $C^{2+\alpha}$ estimate in the case $\lambda = 1$.

Step 1. Extend $H^e_T$ over $\tilde{\Omega}$ as in Lemma 3 such that $\text{div} \, H^e = 0$ in $\Omega$, and

$$\|H^e\|_{C^{2+\alpha}(\tilde{\Omega})} \leq C(\Omega, \alpha)\|H^e_T\|_{C^{2+\alpha}(\partial \Omega)}.$$ 

Let $H$ be a weak solution of (4).

$H^2$ estimate + Sobolev imbedding $\Rightarrow H \in C^\delta(\tilde{\Omega}, \mathbb{R}^3)$ for $\delta < \min\{\alpha, \frac{1}{2}\}$.

$\text{div} \, H = 0$ and Lemma 1 $\Rightarrow \exists B \in C^{1+\delta}(\Omega, \mathbb{R}^3)$ s.t.

$$\text{curl} \, B = H, \quad \text{div} \, B = 0 \quad \text{in} \, \Omega, \quad \nu \cdot B = 0 \quad \text{on} \, \partial \Omega,$$

$$\|B\|_{C^{1+\delta}(\tilde{\Omega})} \leq C(\Omega, \delta)\|H\|_{C^\delta(\tilde{\Omega})} \leq C(\Omega, \lambda, \|H^e\|_{H^2(\Omega)}, M, \delta).$$
Step 2

\[ J = \text{curl} \, H \in W^{1,2}(\Omega, \mathbb{R}^3). \text{ From (4)} \]

\[ \text{curl} \left[ F(|J|^2)J + B \right] = 0. \]

\( \Omega \) simply-connected, there exists \( \varphi \) s.t.

\[ F(|J|^2)J + B = \nabla \varphi. \]

(1.3) and Lemma 2 \( \implies \) \( \nu \cdot J = 0 \) on \( \partial \Omega \). Since \( \nu \cdot B = 0 \) on \( \partial \Omega \),

\[ \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega. \]
Step 3: $C^{2+\delta}$ regularity of $\varphi$

$$
J = \frac{\nabla \varphi - B}{F(|J|^2)} = (1 - |\nabla \varphi - B|^2)(\nabla \varphi - B).
$$

$$
\text{div } J = 0 \implies \begin{cases} 
\text{div } [(1 - |\nabla \varphi - B|^2)(\nabla \varphi - B)] = 0 \quad \text{in } \Omega, \\
\frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\end{cases}
$$

(1.5)

$$
|J| \leq M < \frac{2}{\sqrt{27}} \implies |\nabla \varphi - B| = F(|J|^2)|J| \leq b(M) < \frac{1}{\sqrt{3}}.
$$

Elliptic estimate $\implies$

$$
\|\varphi\|_{C^{2+\delta}(\bar{\Omega})} \leq C(\Omega, \lambda, \|\mathcal{H}^e\|_{H^2(\Omega)}, M, \delta).
$$
Step 4. $C^{2+\delta}$ estimates of $H$

$B$ and $\nabla \phi \in C^{1+\delta}(\bar{\Omega}, \mathbb{R}^3) \implies$

$$J = (1 - |\nabla \phi - B|^2)(\nabla \phi - B) \in C^{1+\delta}(\bar{\Omega}, \mathbb{R}^3).$$

Applying Lemma 1 to

$$\text{curl } H = J, \quad \text{div } H = 0 \quad \text{in } \Omega, \quad H_T = H^e_T \quad \text{on } \partial \Omega, \quad (1.6)$$

$$\implies H \in C^{2+\delta}(\bar{\Omega}, \mathbb{R}^3), \text{ and}$$

$$\|H\|_{C^{2+\delta}(\bar{\Omega})} \leq C(\Omega, \delta)\{\|J\|_{C^{1+\delta}(\bar{\Omega})} + \|H^e_T\|_{C^{2+\delta}(\partial \Omega)}\} \leq C(\Omega, \lambda, \|H^e\|_{H^2(\Omega)}, \|H^e_T\|_{C^{2+\delta}(\partial \Omega)}, M, \delta).$$
Step 5. Global $C^{2+\alpha}$ estimates of $H$

Applying Lemma 1 to (1.6) $\implies B \in C^{2+\delta}(\Omega, \mathbb{R}^3)$.

Write (1.5) as a linear equation for $\varphi$ with $C^{1+\delta}$ coefficients

$$\implies \varphi \in C^{3+\delta}(\Omega) \implies J \in C^{2+\delta}(\Omega, \mathbb{R}^3) \implies H \in C^{3+\delta}(\Omega, \mathbb{R}^3).$$

—If $\alpha < 1/2$ we already have $H \in C^{2+\alpha}(\Omega, \mathbb{R}^3)$.

—If $\frac{1}{2} \leq \alpha < 1$, applying Lemma 1 to (1.6) again,

$$J \in C^{2+\delta}(\Omega, \mathbb{R}^3) \subset C^{1+\alpha}(\Omega, \mathbb{R}^3) \implies H \in C^{2+\alpha}(\Omega, \mathbb{R}^3).$$
4.6. Classification of solutions in $\mathbb{R}^3$

**Lemma 6**

(i) If $A$ is a weak solution of

$$-\nabla^2 A = (1 - |A|^2)A \quad \text{in } \mathbb{R}^3.$$ 

and $\|A\|_{L^\infty(\mathbb{R}^3)} < 1$, then $A \equiv 0$.

(ii) If $H$ is a weak solution of

$$-\nabla \left( F(|\nabla H|^2) \nabla H \right) = H \quad \text{in } \mathbb{R}^3.$$ 

and $\|
abla H\|_{L^\infty(\mathbb{R}^3)} < \frac{2}{\sqrt{27}}$, then $H \equiv 0$. 


4.7. Classification of solutions in $\mathbb{R}^+_3$

Consider

\[
\begin{cases}
-\text{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2) \mathbf{A} & \text{in } \mathbb{R}^+_3, \\
(\text{curl} \mathbf{A})^T = \mathbf{h} & \text{on } \partial \mathbb{R}^+_3.
\end{cases}
\]  

(1.7)

Let $\mathbf{h} = (h_1, h_2, 0)$, and assume

\[|\mathbf{h}| < \sqrt{\frac{5}{18}}.\]
Lemma 7

(1.7) has a smooth solution \( A \) satisfying

\[
\| A \|_{L^\infty(\mathbb{R}_+^3)} = \| A \|_{L^\infty(\partial\mathbb{R}_+^3)} < \frac{1}{\sqrt{3}},
\]

\[
\| \text{curl } A \|_{L^\infty(\mathbb{R}_+^3)} = \| \text{curl } A \|_{L^\infty(\partial\mathbb{R}_+^3)} = |h|,
\]

\[
\| \text{curl}^2 A \|_{L^\infty(\mathbb{R}_+^3)} = \| \text{curl}^2 A \|^2_{L^\infty(\partial\mathbb{R}_+^3)} = M(|h|) < \frac{2}{\sqrt{27}},
\]

where

\[
M(|h|) = [1 - (1 - 2|h|^2)^{1/2}](1 - 2|h|^2).
\]
Consider the equation

\[
\begin{cases}
-\text{curl } [F(|\text{curl } H|^2)\text{curl } H] = H \text{ in } \mathbb{R}_+^3, \\
H_T = h \text{ on } \partial \mathbb{R}_+^3.
\end{cases}
\] (1.8)

(1.8) has a unique weak solution satisfying

\[\|\text{curl } H\|_{L^\infty(\mathbb{R}_+^3)} < \frac{2}{\sqrt{27}}.\]
4.8. Existence of solutions in a bounded domain

\( \mathcal{H}_T^e \) has been extended to \( \bar{\Omega} \). Consider

\[
\begin{cases}
-\lambda^2 \text{curl} \left[ F(\lambda^2 |\text{curl} H|^2) \text{curl} H \right] = H & \text{in } \Omega, \\
H_T = \mu \mathcal{H}_T^e & \text{on } \partial \Omega.
\end{cases}
\] (1.9)
Lemma 8
There is a constant $\mu^* = \mu^*(\mathcal{H}^e_T, \lambda) > 0$ such that:

(i) For all $0 \leq \mu < \mu^*$, (1.9) has a unique solution $\mathbf{H}_\mu$ satisfying (M), and $|\text{curl} \mathbf{H}_\mu(x)|$ reaches its maximum only on $\partial \Omega$.

(ii) $\lim_{\mu \to \mu^*} \lambda \|\text{curl} \mathbf{H}_\mu\|_{L^\infty(\Omega)} = \frac{2}{\sqrt{27}}$.

Proof. Implicit Function Theorem + $C^{2+\alpha}$ estimates.
Lemma 9

\[ \liminf_{\lambda \to 0} \mu^*(\mathcal{H}^e_T, \lambda) \geq \sqrt{\frac{5}{18}} \left( \| \mathcal{H}^e_T \|_{C^0(\partial \Omega)} \right)^{-1}. \]

Proof. Blow-up + $C^{2+\alpha}$ estimates + Classification of solutions in $\mathbb{R}^3$. \qed
**Theorem 10**

For all $\lambda > 0$ small:

(4) has a unique solution $\mathbf{H} \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ satisfying

$$\lambda \|\text{curl} \mathbf{H}\|_{L^\infty(\Omega)} < \frac{2}{\sqrt{27}}, \quad \|\mathbf{H}\|_{C^{2+\alpha}(\bar{\Omega})} \leq C(\Omega, \mathcal{H}_T^e, \lambda).$$

(3) has a unique solution $\mathbf{A} \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ satisfying

$$\|\mathbf{A}\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{3}}, \quad \|\mathbf{A}\|_{C^{2+\alpha}(\bar{\Omega})} \leq C(\Omega, \mathcal{H}_T^e, \lambda).$$
4.9. Complete the Proof of Theorem 1

Theorem 10 $\implies$ there exists a solution $H^\lambda$ satisfying (M).

From the conditions on $|H^e_T|$ and Lemma 9, we can choose $c_0 > 1$ and $\lambda^* > 0$ small such that

$$\mu^*(H^e_T, \lambda) > c_0 > 1 \quad \text{for all } 0 < \lambda \leq \lambda^*.$$

Then we show

$$\sup_{0 < \lambda \leq \lambda^*} \lambda \|\text{curl } H^\lambda\|_{L^\infty(\Omega)} < \frac{2}{\sqrt{27}}. \quad (1.10)$$
(1.10) + Interior blow-up + $C^{2+\alpha}$ estimates + classification of solutions in $\mathbb{R}^3 \implies$ decay (ii).

(1.10)+ Boundary blow-up +$C^{2+\alpha}$ estimate + classification of solutions in $\mathbb{R}_+^3 \implies$ asymptotic estimate (iii).
More precisely, let $x^\lambda \in \partial \Omega$ and $x^\lambda \to x^0$ as $\lambda \to 0$. In the local coordinates near $x^0$, the rescaled vector field $\tilde{H}_\lambda(y)$ converges in $C_{loc}^{2+\alpha}(\mathbb{R}_+^3, \mathbb{R}^3)$ to the unique solution of the equation

$$
\begin{cases}
-\text{curl} \left( F(|\text{curl} H|^2)\text{curl} H \right) = H & \text{in } \mathbb{R}_+^3, \\
H_T = \tilde{h} & \text{on } \partial \mathbb{R}_+^3,
\end{cases}
$$

where $\tilde{h} = \mathcal{H}^e_T(x^0)$. 

Lemma 7 \[\implies \|\text{curl } H\|_{L^\infty(\mathbb{R}^3_+)} = M(\|\tilde{h}\|).\]

The function $M$ is increasing. It follows that

$$
\lim_{\lambda \to 0} \max_{x \in \bar{\Omega}} \lambda |\text{curl } H^\lambda(x)| = M(\|H^e_T\|_{C^0(\partial\Omega)}).
$$

Let $P^\lambda$ be a maximum point of $|\text{curl } H^\lambda(x)|$ and assume

$$
P^\lambda \to P
$$

for a sequence $\lambda_n \to 0$. Then $P \in \partial\Omega(H^e_T)$. \qed
 Remarks.

- The condition

\[ \| \mathcal{H}_T^e \|_{C^0(\partial \Omega)} < \sqrt{\frac{5}{18}} \]

is the optimal bound. V. Galaiko 1966, L. Kramer 1967:

\[ H_S(\mathbb{R}^3) = \sqrt{\frac{5}{18}}. \]

- The additional condition

\[ \nu \cdot \text{curl} \mathcal{H}_T^e = 0 \]

is satisfied in physics. This condition and the topological condition on the domain are removed in Lieberman-Pan, Proc. Royal Soc. Edinburgh 2011.

Assume $\Omega$ is bounded with $\partial\Omega \in C^{2+\alpha}$, $\mathcal{H}_T^\varphi \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$, and $\mathbf{H}$ is a weak solution of (4) satisfying

$$\lambda \|\text{curl} \mathbf{H}\|_{L^\infty(\Omega)} \leq M < \frac{2}{\sqrt{27}}.$$

Then $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and

$$\|\mathbf{H}\|_{C^{2+\alpha}(\overline{\Omega})} \leq C(\Omega, R, \lambda, M, \lambda^{-1} M, \alpha, \|\mathcal{H}_T^\varphi\|_{C^{2+\alpha}(\partial\Omega)}).$$
Proof of Theorem 1′

To prove, we establish local estimates. So we need local estimates of vector fields. Write

\[ \Omega[r] = \{ x \in B(x_0, r) : g(x) > 0 \}, \]

\[ \Sigma[r] = \{ x \in B(x_0, r) : g(x) = 0 \}. \]

Eq. (1.5) now is replaced by

\[ \begin{aligned}
\text{div} \left[ (1 - |\nabla \varphi - \mathbf{B}|^2)(\nabla \varphi - \mathbf{B}) \right] &= 0 \quad \text{in } \Omega[3R/4], \\
(1 - |\nabla \varphi - \mathbf{B}|^2) \frac{\partial \varphi}{\partial \nu} &= \lambda \nu \cdot \text{curl } \mathcal{H}^\varphi \quad \text{on } \Sigma[3R/4].
\end{aligned} \]
Further remarks

- Equations for anisotropic superconductors.
  - Existence and regularity without assuming the bound on \( \text{curl} \mathbf{H} \): Pan, Calc. Var. PDE 2009.
  - Asymptotic behavior: Pan, JMP 2011.
  - Singular limit: under investigation. (1.5) is replaced by an oblique derivative problem for degenerate elliptic equation. Guan-Sawyer ......
Related works: Yin Hongmin, J. Aramaki etc ......

Equations on exterior domains, unstable solutions: partial result by Xiang Xingfei.
Thank You!