STABILITY OF TRAVELING WAVEFRONTS FOR THE DISCRETE NAGUMO EQUATION*

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Abstract. It has been shown that the discrete Nagumo equation

\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \quad n \in \mathbb{Z}, \]

has a traveling wavefront solution for sufficiently strong coupling \( d \). In this paper it is shown that such a traveling wavefront is unique (up to a shift in time) and globally stable.

Key words. traveling waves, lower solution technique, myelinated axon, discrete cells

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1. Introduction. Consider the infinite system of coupled nonlinear differential equations

\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \quad n \in \mathbb{Z} \]

where \( d \) is a positive real number.

A typical example for the nonlinearity \( f \) is the cubic polynomial \( f(x) = x(x-a)(1-x) \), \( 0 < a < \frac{1}{2} \). Equation (1) is the discrete analogue to the well-known Nagumo equation [5]

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u). \]

The discrete Nagumo equation is interesting because it has been used to derive (2) [8] and it has also been proposed as a model for conduction in myelinated nerve axons [1]. The continuous Nagumo equation (2) is well studied [4] and it has been proven that there exist globally stable monotone traveling wavefront solutions.

The analytic approach has been less developed for the discrete than for the continuous Nagumo equation. The first results about the discrete Nagumo equation were concerned with threshold properties, that is, conditions forcing nonconvergence to zero of solutions as time approaches infinity, and bounds on the speed of propagation of a "wave of excitation" [1], [2]. The next results were concerned with wave propagation, that is, with solutions of the form

\[ u_n(t) = U(n + ct). \]

In particular, failure of propagation for small \( d \) and local stability of traveling wavefronts were shown in [6] and [7].

Then traveling wavefronts were analyzed numerically for certain cubic polynomials \( f \) [3]. Only recently has the existence of monotone traveling wavefronts for sufficiently large \( d \) been proved in [9] and [10]. It is the purpose of this paper to prove that such traveling wavefronts are globally stable.

By a traveling wavefront with velocity \( c \), \( c > 0 \), we mean a solution \( \{u_n(t)\}_{n=-\infty}^{\infty} \) of (1) for which there exists \( U \in C^1(\mathbb{R}, (0, 1)) \), \( U(-\infty) = 0 \), \( U(\infty) = 1 \), such that \( u_n(t) = U(n + ct) \) for all \( t \in \mathbb{R} \). If in addition \( U'(z) > 0 \) for all \( z \in \mathbb{R} \), then the wavefront is monotone. The following theorem allows \( f \) to have several zeros in \((0, 1)\) even though

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existence of a monotone traveling wavefront has only been shown for the case where \( f \) has exactly one zero in \((0, 1)\).

**Theorem 1.1.** Suppose \( f \in C^1([0, 1], \mathbb{R}) \), satisfies

1. \( f(0) = f(1) = 0, f'(0) < 0, f'(1) < 0, \)
2. \( f(u) < 0 \) for \( 0 < u < \alpha_0, \)
3. \( f(u) > 0 \) for \( \alpha_1 < u < 1, 0 < \alpha_0 = \alpha_1 < 1, \)

and suppose there exists a monotone traveling wavefront \( \{ v_n(t) \} \), \( v_n(t) = U(n + ct) \). Then for any solution \( \{ u_n(t) \} \) of (1) which satisfies

\[
0 \leq u_n(0) \leq 1 \quad \text{for all integers } n, \quad \text{and}
\lim_{n \to -\infty} u_n(0) < \alpha_0 \leq \alpha_1 < \lim_{n \to \infty} u_n(0),
\]

there exists a constant \( s \) such that

\[
\lim_{t \to \infty} \left( \sup_{n \in \mathbb{Z}} |u_n(t) - U(n + ct - s)| \right) = 0.
\]

The following corollary is a direct consequence of Theorem 1.1.

**Corollary 1.2.** Suppose \( \{ u_n(t) \} \) and \( \{ v_n(t) \} \) are traveling wavefronts of (1). Then there exists a constant \( t_0 \) such that \( \{ u_n(t) \} = \{ v_n(t - t_0) \} \). In particular, there is a unique speed \( c \) for traveling wavefronts.

2. **Proof of the theorem.** We will make use of the following lemma [6, Thms. 4.1, 4.2].

**Lemma 2.1.** If the hypotheses of Theorem 1.1 hold, then there are constants \( z_1, z_2, q_0, \mu_0 \) (the last two positive) such that

\[
U(n + ct - z_1) - q_0 e^{-\mu_0 t} \leq u_n(t) \leq U(n + ct - z_2) + q_0 e^{-\mu_0 t}.
\]

Furthermore, if there are constants \( t_0, z_0, \) and \( \varepsilon \) for which

\[
|u_n(t_0) - U(n + ct_0 - z_0)| < \varepsilon,
\]

then there is a number \( \omega(\varepsilon) \) with \( \lim_{\varepsilon \to 0} \omega(\varepsilon) = 0 \) such that

\[
|u_n(t) - U(n + ct - z_0)| < \omega(\varepsilon) \quad \text{for all } t \equiv t_0.
\]

Lemma 2.1 says that \( \{ u_n(t) \} \) is “more or less” bounded between two shifted wavefronts and if \( \{ u_n(t) \} \) is close to a wavefront \( U \) at some instant then it will remain close to \( U \). The main idea for the following proof is the attempt to replace the constants \( z_1, z_2 \) in (3) by functions \( z_1(t), z_2(t) \) with \( \lim_{t \to \infty} z_1(t) = \lim_{t \to \infty} z_2(t) \).

Let

\[
w'_n(t) := u_n(t) - U(n + ct - s),
\]

\[
A := \left\{ s \in [z_2, z_1]: \limsup_{t \to \infty} \left[ \sup_{n \in \mathbb{Z}} w'_n(t) \right] \leq 0 \right\}, \quad \text{and}
\]

\[
B := \left\{ s \in [z_2, z_1]: \limsup_{t \to \infty} \left[ \sup_{n \in \mathbb{Z}} -w'_n(t) \right] \leq 0 \right\}.
\]

Note that \( z_2 \in A, z_1 \in B \), and Theorem 1.1 will be proved if we show that \( A \cap B \neq \emptyset \).

By assumption (i) of Theorem 1.1 there exist positive constants \( \mu \) and \( \delta_0 \) such that

(4a) \( f(x) - f(y) \leq -\mu(x - y) \) for all \( x, y \in [0, \delta_0] \), and

(4b) \( f(x) - f(y) \leq -\mu(x - y) \) for all \( x, y \in [1 - \delta_0, 1] \).
Choose $\delta \in (0, \delta_0)$, arbitrary. Since $U(-\infty) = 0$ and $U(\infty) = 1$, we may choose $n_0 \in \mathbb{Z}$ and $n_1 \in \mathbb{Z}$ such that

$$U(n_0 - z_2) \leq \frac{\delta}{2} \quad \text{and} \quad U(n_1 - z_1) \leq 1 - \frac{\delta}{2}.$$

Now let $I(t) := \{ n \in \mathbb{Z} : n_0 - ct \leq n \leq n_1 - ct \},$

$$A_f := \left\{ s \in [z_2, z_1] : \limsup_{t \to \infty} \sup_{n \in I(t)} w_n^s(t) \leq 0 \right\},$$

$$B_f := \left\{ s \in [z_2, z_1] : \limsup_{t \to \infty} \sup_{n \in I(t)} -w_n^s(t) \leq 0 \right\}.$$

**Lemma 2.2.** $A_f = A$ and $B_f = B.$

**Proof.** From the definition of $A_f$, $A$, $B_f$, and $B$ it is clear that $A \subset A_f$ and $B \subset B_f.$ It suffices to show $A_f \subset A$, since the proof of $B_f \subset B$ is identical.

So suppose $s \in A_f$. Then

$$\limsup_{t \to \infty} \sup_{n \in I(t)} w_n^s(t) \leq 0$$

and we have to show that

$$\limsup_{t \to \infty} \sup_{n \in \mathbb{Z}} w_n^s(t) \leq 0,$$

i.e., given $\varepsilon > 0$ there exists $T$ such that

$$w_n^s(t) \leq \varepsilon \quad \text{for all } n \in \mathbb{Z}, \quad t \geq T.$$

Fix $\varepsilon > 0$. Then there exists $T_0$ such that

$$w_n^s(t) \leq \varepsilon \quad \text{for all } n \in I(t), \quad t \geq T_0$$

and

$$u_n(t) \notin [\delta, 1 - \delta] \quad \text{for all } n \notin I(t), \quad t \geq T_0.$$

Let $q \in (0, 1)$ be such that

$$d \left( \frac{1}{q} + \frac{1}{q} - 2 \right) = \frac{1}{2} \mu.$$

We will show that the existence of a number $T_k \geq T_0$, $k \in \mathbb{N}_0$, such that

$$w_n^s(t) \leq \max \{ q^k \delta, \varepsilon \} \quad \text{for all } n \in \mathbb{Z}, \quad t \geq T_k,$$

will imply the existence of $T_{k+1} \geq T_0$ such that

$$w_n^s(t) \leq \max \{ q^{k+1} \delta, \varepsilon \} \quad \text{for all } n \in \mathbb{Z}, \quad t \geq T_{k+1}.$$

Note that (9) only needs to be shown for $n \notin I(t)$. Suppose (8) holds and for some index $n$ we have

$$\max \{ q^{k+1} \delta, \varepsilon \} \leq w_n^s(t) \leq \max \{ q^k \delta, \varepsilon \}, \quad t \geq T_k.$$

Since $u_n$ and $U$ satisfy (1) we obtain

$$w_n^s = d(w_{n-1}^s - 2w_n^s + w_{n+1}^s) + f(u_n) - f(U),$$
which can be rewritten to

\[(12)\quad \dot{w}_n^s = d \left( \frac{w_{n-1}^s + w_{n+1}^s}{w_n^s} - 2 \right) w_n^s + \frac{f(u_n) - f(U)}{w_n^s} w_n^s.\]

From (7), (8), and (10), we deduce

\[(13)\quad d \left( \frac{w_{n-1}^s + w_{n+1}^s}{w_n^s} - 2 \right) \leq \frac{1}{2} \mu\]

and from (4a), (4b), and (6) we get

\[(14)\quad \frac{f(u_n) - f(U)}{w_n^s} \leq -\mu.\]

Finally, from (12), (13), and (14) we estimate

\[(15)\quad \dot{w}_n^s \leq -\frac{1}{2} \mu w_n^s.\]

With Gronwall's lemma we deduce (9) where \(T_{k+1}\) is determined by

\[\exp\left(-\frac{1}{2} \mu (T_{k+1} - T_k)\right) = q.\]

The claim (5) then follows by induction. □

Because of Lemma 2.2 it now suffices to show that \(A_f \cap B_f \neq \emptyset\). To reach a contradiction suppose \(A_f \cap B_f = \emptyset\).

We consider \(A_f\) and \(B_f\) as topological subspaces of the interval \([z_2, z_1]\). It is easy to check that \(B_f\) is closed and therefore we may pick \(s \in \partial B_f \setminus A_f\). Since \(s \in \partial B_f\) there exist sequences \(\{t_k\}_{k=1}^\infty, \lim_{k \to \infty} t_k = \infty, \{n_k\}_{k=1}^\infty, n_k \in I(t_k)\), such that

\[(16)\quad \lim_{k \to \infty} w_{n_k}^s(t_k) = 0.\]

Since \(s \notin A_f \cap B_f = A \cap B\) there exists \(\epsilon > 0\) such that

\[(17)\quad \sup_{n \in Z} |w_n^s(t_k)| \geq \epsilon\]

in view of Lemma 2.1.

It follows from the definition of \(I(t)\) that there exists \(T_0\) such that

\[(18)\quad |w_n^s(t)| < \delta \quad \text{for all } n \notin I(t), \quad t \geq T_0.\]

Either

\[(19)\quad \lim_{k \to \infty} w_{m_k}^s(t_k) = 0 \quad \text{implies} \quad \lim_{k \to \infty} (|w_{m_k-1}^s(t_k)| + |w_{m_k+1}^s(t_k)|) = 0\]

holds for all sequences \(\{m_k\}_{k=1}^\infty, m_k \in I(t_k)\),

or (19) does not hold.

If (19) holds, then it follows from (16) and by induction that \(\lim_{k \to \infty} \sup_{n \in I(t_k)} |w_n^s(t_k)| = 0\). Together with (18) this implies that there exists \(K \in \mathbb{N}\) such that \(\sup_{n \in Z} |w_n^s(t_k)| < \delta\) for all \(k \geq K\). Since \(\delta\) may be chosen to be less than \(\epsilon\), this contradicts (17).

Therefore (19) does not hold and hence there exist a subsequence of \(\{t_k\}\), which we also denote by \(\{t_k\}\), a sequence \(\{m_k\}, m_k \in I(t_k)\), and \(\varepsilon_0 > 0\) such that

\[(20)\quad \lim_{k \to \infty} w_{m_k}^s(t_k) = 0 \quad \text{and} \quad |w_{m_k-1}^s(t_k)| + |w_{m_k+1}^s(t_k)| \geq \varepsilon_0.\]
It follows from (20) and \( s \in \partial B \subset B \) that there exists \( K \in \mathbb{N} \) such that

\[
-2w_{m_k}^s(t_k) > \frac{\epsilon_0}{4},
\]

\[
w_{m_k-1}^s(t_k) + w_{m_k+1}^s(t_k) > \frac{\epsilon_0}{2},
\]

and

\[
f(u_{m_k}(t_k)) - f(U(m_k - ct_k - s)) > \frac{-d\epsilon_0}{8}
\]

holds for all \( k \geq K \). Therefore (11) implies

\[
\dot{w}_{m_k}^s(t_k) > \frac{d\epsilon_0}{8} \quad \text{for all } k \geq K.
\]

It also follows from (11) that there exists \( M > 0 \) such that

\[
\dot{w}_n(t) < M \quad \text{for all } n \in \mathbb{N} \text{ and } t \geq 0,
\]

which implies together with (21) that there exists \( h > 0 \) such that

\[
\dot{w}_{m_k}(t) > \frac{d\epsilon_0}{16} \quad \text{for all } t \in [t_k - h, t_k], \quad k \geq K.
\]

Therefore \(-w_{m_k}^s(t_k - h) > (h\epsilon_0/16) - w_{m_k}^s(t_k)\) for all \( k \geq K \) which implies together with (20) that

\[
\lim_{t \to \infty} \sup_{m \in \mathbb{N}} -w_n^s(t) \geq \frac{h\epsilon_0}{16} > 0,
\]

in contradiction to \( s \in B \).

REFERENCES