Existence of Traveling Wavefront Solutions for the Discrete Nagumo Equation

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In this paper we show that the discrete Nagumo equation
\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \quad n \in \mathbb{Z}, \]
has a traveling wave solution for sufficiently strong coupling \( d \).

The problem is at first simplified into a fixed point problem which can be solved by Brouwer's fixed point theorem. The solutions of the simplified problem are then continued via index-theory to solutions of approximate problems. In the final step it is proven that the solutions of the approximate problems have a limit point which corresponds to a solution of the original problem.

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1. INTRODUCTION

Consider the infinite system of coupled nonlinear differential equations
\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \quad n \in \mathbb{Z}, \quad (1) \]
where \( d \) is a positive real number and \( f \) denotes a Lipschitz continuous function satisfying
\[ f(0) = f(a) = f(1), \quad f(x) < 0, \quad \text{for} \quad 0 < x < a, \]
\[ f(x) > 0, \quad \text{for} \quad a < x < 1, \quad \text{and} \quad \int_0^1 f(x) \, dx > 0. \quad (2) \]

A typical example is the cubic polynomial
\[ f(x) = x(x-a)(1-x), \quad \text{where} \quad 0 < a < \frac{1}{2}. \]

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Equation (1) is interesting for several reasons. First, it is the discrete analogue to the well-known Nagumo equation [9],

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u). \tag{3}$$

The Nagumo equation is used as a model for the spread of genetic traits [2] and for the propagation of nerve pulses in a nerve axon, neglecting recovery [8,10]. The discrete Nagumo equation (1) has been used to derive Eq. (3) [14]. It has also been proposed as a model for conduction in myelinated nerve axons [3]. The continuous Nagumo equation (3) is well studied [7]. It has been shown, for example, that there is a function $U$, with $U(-\infty) = 0$, $U(\infty) = 1$, and a constant $c > 0$ such that

$$u(x, t) := U\left(\frac{x}{\sqrt{D}} + ct\right)$$

is a solution of (3) for all $D > 0$. Such solutions are called traveling wavefronts, or simply, traveling waves.

In the discrete Nagumo equation, a similar result cannot be true for small $d$ [12]. This difference between the continuous and the discrete model has led, for instance, to new insights regarding the formation of spiral waves [13].

There are already some results known for the discrete Nagumo equation. The first results were concerned with threshold properties of the system, in particular, with conditions forcing nonconvergence to zero of solutions as time approaches infinity and with bounds on the speed of propagation of a “wave of excitation” [3,4]. It has been conjectured that there are traveling wave solutions of (1), i.e., solutions of the form

$$u_n(t) = U(n + ct)$$

with $U(-\infty) = 0$ and $U(\infty) = 1$. Numerically, such solutions were found and analyzed for certain cubic polynomials $f$ [6]. In [5], a first attempt was made to prove the existence of traveling wave solutions; unfortunately the proof given in [5] was not conclusive. It is the aim of this paper to give a rigorous proof of the conjectured existence of traveling wave solutions for sufficiently large $d$. The precise statement of the result is:

**Theorem 1.** Suppose $f$ is a Lipschitz continuous function satisfying (2). Then there exists some $d^* > 0$ such that for $d > d^*$ the discrete Nagumo equation (1) admits a solution $u_n(t) = U(n + ct)$, where $c > 0$, $U \in C^1(\mathbb{R}, (0, 1))$, $U(-\infty) = 0$, $U(\infty) = 1$, and $U'(x) > 0$ for all $x \in \mathbb{R}$. 
The proof of Theorem 1 can be broken into four steps.

**Step 1.** The problem is simplified.

Instead of the discrete Nagumo equation we consider

\[
\begin{align*}
\dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n), \\
    u_n &= P(v_n), \quad n \in \mathbb{Z},
\end{align*}
\]

where

\[h(u_n) := u_n - \frac{1}{4}\]

and

\[
P(v_n) := \begin{cases} 
0 & \text{for } v_n < 0 \\
v_n & \text{for } 0 \leq v_n < 1. \\
1 & \text{for } 1 < v_n.
\end{cases}
\]

The nonlinear function \( f \) has been replaced by the affine function \( h \) (see Fig. 1) and then \( u_n \) is forced to take values in the interval \([0, 1]\) via the Lipschitz continuous projection \( P: \mathbb{R} \rightarrow [0, 1] \) (see Fig. 2).

The simplified problem is to find a monotone traveling wave of (4), i.e., a solution of (4) on an interval \([0, \tau]\) which satisfies the conditions

\[
\begin{align*}
\text{(C1)} & \quad u_n(\tau) = u_{n+1}(0), \\
\text{(C2)} & \quad v_n(0) \leq v_{n+1}(0), \\
\text{(C3)} & \quad d(u_{n-1}(0) - 2u_n(0) + u_{n+1}(0)) + h(u_n(0)) > 0 \text{ if } u_n(0) > 0, \\
\text{(C4)} & \quad \lim_{n \to -\infty} v_n(0) = 0, \text{ and } \lim_{n \to \infty} v_n(0) = 1.
\end{align*}
\]

Note that conditions (C1), (C2), and (C4) imply the existence of a function \( U: \mathbb{R} \rightarrow \mathbb{R} \) with \( U(-\infty) = 0, U(\infty) = 1, 0 \leq U \leq 1 \), such that

\[u_n(t) = U(n + ct), \quad \tau = 1/c, \quad \text{for all } n \in \mathbb{Z}, t \in [0, \tau].\]
It will be shown (cf. Lemma 5) that for a monotone traveling wave only finitely many of the values \( u_n(0) \) are different from 0 and 1 (see Fig. 3).

It suffices therefore to consider only a finite number of the equations (4).

**Step 2.** The simplified problem is converted into a fixed point problem.

Note that the initial value problem

\[
\begin{align*}
v_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n), \\
u_n &= P(v_n), \\
v_n(0) &= x_n \quad \text{with} \quad 0 \leq x_n \leq 1 \quad \text{for} \quad n = 0, ..., N,
\end{align*}
\]

where \( u_{-1} \equiv 0 \) and \( u_{N+1} \equiv 1 \), has a unique solution, say

\[
u(x; t) = \{u_n(x; t)\}_{n=0}^N,
\]

which depends continuously on the initial value \( x = \{x_n\}_{n=0}^N \). For an appropriate initial value space \( \mathcal{X} \) we will then consider the following shifted Poincaré map

\[
T: \mathcal{X} \to \mathbb{R}^{N+1},
\]

\[
(Tx)_n := \begin{cases} 
0 & \text{for } n = 0 \\
u_{n-1}(x; t) & \text{for } n = 1, ..., N,
\end{cases}
\]
where $\tau$ is defined implicitly by

$$u_0(x; \tau) = x_1.$$ 

As an appropriate definition of $X$ we will take

$$X := \left\{ \{x_n\}_{n=0}^N \in \mathbb{R}^{N+1} : x_0 = 0, x_1 = -h(0)/d, x_n \leq x_{n+1}, d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0, \right.$$

$$\left. x_n \geq \frac{n}{N}, \text{ for } n = 1, \ldots, N, \text{ where } x_{N+1} := 1 \right\}. \quad (6)$$

Note that $X$ depends on $d$ and $N$. It turns out that for a suitable choice of $d$ and $N$ the map $T$ is well defined and continuous, $X$ is nonempty and convex, and $T$ maps $X$ into $X$.

It follows then by Brouwer's fixed point theorem that $T$ has a fixed point $x$. This fixed point $x$ corresponds to the traveling wave $\{u_n(t)\}_{n=-\infty}^\infty$ of (4), where

$$u_n(t) := \begin{cases} 0 & \text{for } n \leq -1 \\ u_n(x; t) & \text{for } 0 \leq n \leq N \\ 1 & \text{for } n \geq N + 1 \end{cases}$$

and $t \in [0, \tau]$, where $\tau$ is defined implicitly by $u_0(x; \tau) = x_1$.

**Step 3. Continuation of the solution.**

The construction of $T : X \rightarrow \mathbb{R}^{N+1}$ in Step 2 depended on $h$. To indicate this dependence we will write from now on $T_h$ instead of $T$ and $X_h$ instead of $X$. The affinity of $h$, however, was only needed to show that $X_h$ is convex and nonempty (for suitable $d$ and $N$).

It turns out that one can still define a continuous map $T_h : X_h \rightarrow \mathbb{R}^{N+1}$ such that $T_h x \in X$ for all $x \in X_h$ where fixed points of $T_h$ correspond to traveling waves, just as we did in Step 2, as long as $h \in \mathcal{B}_{\text{app}}$, where

$$\mathcal{B}_{\text{app}} := \left\{ h : [0, 1] \rightarrow \mathbb{R} : h \text{ is Lipschitz continuous,} \right.$$

$$h(0) < 0, h(1) > 0, h \text{ has a unique zero in } (0, 1),$$

and \( \int_0^1 h(s) \, ds > 0 \right\}.$$ 

Suppose we want to find a traveling wave of (4) for some $h_1 \in \mathcal{B}_{\text{app}}$. In Step 2 we found a traveling wave for the affine function $h_0 \in \mathcal{B}_{\text{app}}$, where
The idea now is to deform continuously $h_0$ into $h_1$ (see Fig. 4) and to continue the fixed points of $T_{h_0}$ to the fixed points of $T_{h_1}$. We will see in Section 5 that this can be done by using a homotopy invariance theorem from index-theory in an appropriate setup.

**Step 4. Convergence of the approximate solutions.**

Suppose $h^k$ is an approximation of $f$ (see Fig. 5) and $\{u^k_n\}$ is a traveling wave of

$$\dot{v}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h^k(u_n),$$

$$u_n = P(v_n), \quad n \in \mathbb{Z}. $$

One expects that as $h^k$ tends to $f$ the corresponding traveling waves $\{u^k_n\}$ approximate a traveling wave of

$$\dot{v}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n),$$

$$u_n = P(v_n), \quad n \in \mathbb{Z}. $$

(7)
Let $n^k$ be such that

$$u^k_n(0) \leq \frac{1}{2} < u^k_{n+1}(0)$$

and call $n^k$ the center of $\{ u^k_n \}$. It turns out in Section 6 that $\{ u^k_n \}$ shifted by its center, i.e., $\{ u^k_{n+C^k} \}$, will indeed converge to a traveling wave of (7). The proof concludes then by showing that a traveling wave of (7) is in fact a strictly monotone traveling wave of the discrete Nagumo equation.

2. INVARIANCE RESULTS

Let $u(t) = \{ u_n \}_{n=0}^N$ be the unique solution of the initial value problem (5), where $h$ is any element of $\mathcal{B}_{app}$. Since $u(t)$ depends continuously on $x$ and $h$ we will also write $u(x; t)$ or $u(x, h; t)$. In this section we will show that certain properties of initial values are invariant under the flow of $u(t)$.

**Lemma 1.** Suppose $x = \{ x_n \}_{n=0}^N \in \mathbb{R}^{N+1}$ satisfies

(P1) $0 \leq x_0 \leq \ldots \leq x_N \leq 1,$

(P2) $x_n < x_{n+1}$ whenever $0 < x_n < 1$,

(P3) $d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0$ whenever $0 < x_n$, where $x_{N+1} = -1$, and $x_{-1} = 0$.

Furthermore suppose

$x_0 = 0$, $x_1 = -h(0)/d$, and $d > -h(0)$.

Then $u(t)$ satisfies (P1), (P2), and (P3) for all $t \geq 0$.

**Proof.** Let $t_1 := \sup \{ t \geq 0 : u(s) \text{ satisfies (P1), (P2), and (P3) for all } 0 \leq s \leq t \}$. Note that $t_1$ is well defined because $u(0) = x$ satisfies (P1), (P2), and (P3). One has to show that $t_1 = \infty$.

Observe that $u(t)$ satisfies (P3) if and only if $\dot{v}_n(t) > 0$ whenever $v_n(t) > 0$. By assumption $x_1 > 0$ and by (P1) we have $x_n > 0$ for $n = 1, 2, \ldots, N$ and thus by (P3)

$$\dot{v}_n(0) > 0 \quad \text{for } n = 1, 2, \ldots, N.$$

Note that

$$\dot{v}_0(0) = dx_1 + h(0) = 0.$$

One checks that $u_0(t)$ is differentiable at $t = 0$ and $\dot{u}_0(0) = 0$. Therefore $h(u_0(t))$ is differentiable at $t = 0$ and $(d/dt) h(u_0(t))|_{t=0} = 0$. One calculates then that $\dot{v}_0(0) > 0$. Now one can conclude that $\dot{v}_0(t) > 0$ for all sufficiently
small $t > 0$. Using the continuity of $\hat{v}_n$ and $u_n$ one infers that $t_1 > 0$. Suppose that $t_1 < \infty$. Then either

(i) $u_n(t_1) = u_{n+1}(t_1)$ for some $0 < u_n(t_1) < 1$, or

(ii) $\hat{v}_n(t_1) = 0$ for some $u_n(t_1) > 0$.

We will show that both alternatives are impossible. Suppose (i) occurs. Because $u_{N+1} \equiv 1$, one may choose $n$ such that $u_n(t_1) = u_{n+1}(t_1) < u_{n+2}(t_1)$. It follows that \((d/dt)(v_{n+1} - v_n)\) \big|_{t = t_1} = d(u_{n+2}(t_1) - u_{n-1}(t_1)) > 0\), and therefore $u_{n+1}(t_1 - \varepsilon) = v_{n+1}(t_1 - \varepsilon) < v_n(t_1 - \varepsilon) = u_n(t_1 - \varepsilon)$ for sufficiently small $\varepsilon > 0$. This is a contradiction, since $u(t)$ satisfies (P2) for all $0 \leq t \leq t_1$.

Consider $u_n(t)$ on the interval $[0, t_1]$. If there exists $t_0 \in (0, t_1)$ such that $u_n(t_0) = 1$, then $u_n(t) = 1$ for all $t \in [t_0, t_1]$, because $v_n(t)$ is nondecreasing on $[0, t_1]$. Therefore the following three cases may occur:

(a) $u_n(t) = v_n(t)$ for all $t \in [0, t_1]$,

(b) $u_n(t) = \begin{cases} v_n(t) & \text{for all } t \in [0, t_0] \\ 1 & \text{for all } t \in (t_0, t_1] \end{cases}$

(c) $u_n(t) = 1$ for all $t \in [0, t_1]$.

In either case the left derivative of $u_n(t)$ exists for every $t \in (0, t_1]$ and is denoted by $\hat{u}_n(t-\cdot)$. Suppose now case (ii) occurs. Then $\hat{u}_n(t_1-) = 0$. One calculates that

$$\hat{v}_n(t_1-) = d(\hat{u}_{n-1}(t_1-) + \hat{u}_{n+1}(t_1-)), \quad (8)$$

where $\hat{u}_{n-1}(t_1-) \geq 0$ and $\hat{u}_{n+1}(t_1-) \geq 0$. Note that $\hat{v}_n(t_1-) > 0$ would imply $\hat{v}_n(t_1-\varepsilon) < 0$ for sufficiently small positive $\varepsilon$, which is impossible. Therefore $\hat{v}_n(t_1-) = 0$ and by Eq. (8)

$$\hat{u}_{n-1}(t_1-) = 0 \quad \text{and} \quad \hat{u}_{n+1}(t_1-) = 0. \quad (9)$$

Note that $v_{n+1}(t_1) > 1$ would give $\hat{v}_n(t_1) = h(1) > 0$ and hence is not possible. Therefore $\hat{v}_{n+1}(t_1) = \hat{u}_{n+1}(t_1-) = 0$. The arguments which lead to (9) may be repeated with $n$ replaced by $n - 1$. By induction we conclude

$$\hat{u}_j(t_1-) = 0 \quad \text{for } j = 0, 1, \ldots, n + 1. \quad (10)$$

If $v_{n+1}(t_1) > 1$ then $u_{n+1}(t) = 1$ for $t$ sufficiently close to $t_1$ and therefore by (P1)

$$\hat{u}_{n+1}(t_1-) = \cdots = \hat{u}_N(t_1-) = 0. \quad (11)$$
On the other hand, if \( v_{n+1}(t_1) \leq 1 \) then \( v_{n+1}(t_1) = u_{n+1}(t_1 - ) = 0 \) and we may repeat the arguments which lead to (9). In either case we will arrive at (11) which together with (10) gives

\[ \dot{u}_n(t_1 - ) = 0 \quad \text{for } n = 0, 1, \ldots, N. \]  

(12)

Let \( n_1 \) be the maximal \( n \in \{0, 1, \ldots, N\} \) for which \( u_n(t) < 1 \) for all \( 0 \leq t < t_1 \). Then there exists \( t_0 \in [0, t_1) \) such that

\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) \]

for \( n = 0, 1, \ldots, n_1 \), where \( u_{n+1}(t) = 1 \) for \( t \in [t_0, t_1] \). By (12) one has

\[ \dot{u}_n(t_1) = 0 \quad \text{for } n = 0, 1, \ldots, n_1 \]

and by the uniqueness of the initial value problem (5),

\[ u_n(t) = u_n(t_1) \quad \text{for } t \in [t_0, t_1], n = 0, 1, \ldots, n_1. \]

Therefore \( \dot{v}_n(t) = 0 \) for all \( t \in (t_0, t_1) \) and \( n \in \{0, 1, \ldots, n_1\} \) in contradiction to the choice of \( t_1 \).  

Lemma 1 gives rise to introducing the following sets:

**DEFINITION 1.** For \( d > -h(0) \) and \( N \in \mathbb{N} \) define

\[ C(h, d, N) := \{ x \in \mathbb{R}^{N+1} : x_0 = 0, x_1 = -h(0)/d \leq x_2 \leq \cdots \leq x_N \leq 1, \]

and \( x_n \geq n/N \), for \( n = 1, 2, \ldots, N \} \)

\[ O(h, d, N) := \{ x \in \mathbb{R}^{N+1} : d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0, \]

for \( n = 1, 2, \ldots, N \), where \( x_{N+1} := 1 \}).

For shorter notation we will write \( C, O \) instead of \( C(h, d, N), O(h, d, N) \). Note that \( C \) is a closed, bounded, and convex subset of \( \mathbb{R}^{N+1} \) and \( O \) is an open subset of \( \mathbb{R}^{N+1} \). The space we are interested in is \( C \cap O \) for suitable chosen \( h, d, \) and \( N \).

**DEFINITION 2.** Define \( t^*: C \cap O \to (0, \infty) \) by

\[ t^*(x) := \sup \{ t : u_0(x; t) < -h(0)/d \} \]

and define

\[ T: \{ x \in C \cap O : t^*(x) < \infty \} \to \mathbb{R}^{N+1} \]
by
\[
(Tx)_n := \begin{cases} 
0 & \text{for } n = 0 \\
 u_{n-1}(x; t^*) & \text{for } n = 1, \ldots, N.
\end{cases}
\]
The rest of this section is concerned with proving that for sufficiently large \(N\)
\[
T\{x \in C \cap O : t^*(x) < \infty\} \subset C \cap O.
\]

**Lemma 2.** \(T\{x \in C \cap O : t^*(x) < \infty\} \subset O.\)

**Proof.** First consider the case where \(x \in C \cap O\) and \(x\) satisfies (P2) of Lemma 1. Then \(Tx \in O\) by Lemma 1.

Suppose now \(x \in C \cap O\). Let \(n_1\) be the largest \(n \in \{0, 1, \ldots, N\}\) for which \(x_n < 1\) and define for \(\varepsilon > 0\)
\[
x_n^\varepsilon := \begin{cases} 
x_n + \varepsilon & \text{for } n = 0, \ldots, n_1 \\
x_n & \text{for } n = n_1 + 1, \ldots, N.
\end{cases}
\]
Then for all \(\varepsilon\) sufficiently small \(x^\varepsilon \in C \cap O\) and \(x^\varepsilon\) satisfies (P2). Therefore \(u(x^\varepsilon, t^*(x))\) satisfies (P3) by Lemma 1. Since \(x^\varepsilon \to x\) as \(\varepsilon \to 0\) and \(u\) depends continuously on its initial conditions, one concludes that \(y := u(x; t^*)\) satisfies
\[
d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) \geq 0
\]
whenever \(y_n > 0\). By the same arguments as in the proof of Lemma 1(ii), we can lead \(d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) = 0\) to a contradiction. It follows that \(u(x; t^*)\) satisfies (P3) and therefore \(Tx \in O\).

The general case where \(x \in C \cap \overline{O}\) follows also by approximation. Indeed, there exists a sequence \(\{x^k\}\) in \(C \cap O\) such that \(x^k \to x\) as \(k \to \infty\). Since \(u(x^k, t^*(x))\) satisfies (P3) for each \(k\) we may argue as above to conclude that \(u(x, t^*(x))\) satisfies (P3) and therefore \(Tx \in O\).

**Lemma 3.** For every \(h \in \mathcal{R}_{\text{app}}\) there exists a positive \(\delta\) such that
\[
T\{x \in C \cap \overline{O} : t^*(x) < \infty\} \subset C
\]
for \(N \geq 1/\delta\).

**Proof.** Let \(y := Tx\). By the definition of \(t^*\) one has \(y_0 = 0\) and \(y_1 = -h(0)/d\). By the same argument as in Lemma 2 one shows that \(y_1 \leq y_2 \leq \cdots \leq y_n \leq 1\). Therefore, it remains to show the existence of a number \(\delta > 0\) such that \(y_n \geq n/N\) for \(n = 1, \ldots, N\), whenever \(N \geq 1/\delta\). We will construct \(\delta\) explicitly. Let
\[
\delta_1 := -h(0)/d,
\]
\[
\delta_2 := \max \{x - a : a < x \leq 1 \text{ and } b(x) \leq (d/2) \delta_1\},
\]
where \( b(x) := \max_{a \leq s \leq x} h(s) \), \( x \in [a, 1] \), and \( a \in (0, 1) \) is the unique zero of \( h \),

\[
\delta_3 := \min \{ \frac{1}{2} \delta_1, \delta_2 \}, \\
\delta_4 := \delta_3 / 2, \\
\delta_5 := \min \{ h(s) / 4d : a + \delta_3 / 2 \leq s \leq 1 \},
\]

and

\[
\delta_6 := \min \{ d \delta_5 / m_2, -h(0) / dm_1 \} \ d \delta_5,
\]

where \( m_1 := d + \sup_{0 < s \leq 1} |h(s)| \), and \( m_2 := m_1 (4d + \sup_{s \neq t} |(h(s) - h(t)) / (s - t)|) \). Finally, let

\[
\delta := \min \{ \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \}.
\] (13)

Suppose that \( N > \frac{1}{\delta} \). We will show that

\[
y_n \geq n / N \quad \text{for} \quad n = 1, 2, \ldots, N.
\]

Suppose \( y_n \leq a \), where \( a \) denotes the unique zero of \( h \). Then

\[
y_{n+1} - y_n \geq y_n - y_{n-1} - h(y_n) / d \geq y_n - y_{n-1},
\]

because \( y \in O(h, d, N) \) and \( h(x) \leq 0 \) for \( x < a \). We conclude by induction that

\[
y_{i+1} - y_i \geq y_1 - y_0 = -h(0) / d = \delta_1,
\]

i.e.,

\[
y_i \geq i \delta \geq i / N \quad \text{for} \quad i = 1, \ldots, n + 1.
\]

If \( y_{N-1} \leq a \) we are done. Otherwise there is an index \( n_0 \) such that

\[
y_{n_0 - 1} \leq a \quad \text{and} \quad y_{n_0} > a.
\]

It remains to check that

\[
y_n \geq n / N \quad \text{for} \quad n = n_0 + 1, \ldots, N.
\]

Either \( y_{n_0} \leq a + \delta_2 \) in which case

\[
y_{n_0 + 1} - y_{n_0} \geq (y_{n_0} - y_{n_0 - 1}) - (1/d) h(y_{n_0}) \geq \delta_1 - \frac{1}{2} \delta_1 = \frac{1}{2} \delta_1,
\]

or \( y_{n_0} > a + \delta_2 \). In either case, it suffices to show that

\[
y_n \geq n / N \quad \text{for} \quad y_n \geq a + \delta_3.
\]
Suppose the contrary and let $n_1$ be the smallest index for which
\[ y_{n_1} \geq a + \delta_3 \quad \text{and} \quad y_{n_1} < n_1/N. \]
Then
\[ (n_1 - 2)/N \leq x_{n_1 - 2} \leq x_{n_1 - 1} \leq y_{n_1} < n_1/N \]
and
\[ x_{n_1 - 1} \geq a + \delta_3/2. \]
Since
\[ 1/N \leq \delta_5 \leq h(x_{n_1 - 1})/4d, \]
it follows that
\[ x_{n_1 - 1} - x_{n_1 - 2} < \frac{n_1}{N} - \frac{n_1 - 2}{N} = \frac{n_1 - 2}{N} \leq \frac{h(x_{n_1 - 1})}{2d}. \]
(14)
Suppose $0 < u_n(0) < 1$ and $0 < u_n(t) < 1$. Then using
\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) \]
one estimates that
\[ |\dot{u}_n(t) - \dot{u}_n(0)| \leq tm_2. \]
This implies
\[ \dot{u}_n(t) \geq \dot{u}_n(0) - tm_2 \]
and thus
\[ \dot{u}_n(t) \geq \frac{1}{2} \dot{u}_n(0) \quad \text{for} \quad 0 \leq t \leq \dot{u}_n(0)/2m_2. \]
This gives
\[ u_n(t) - u_n(0) = \int_0^t \dot{u}_n(s) \, ds \geq t \frac{1}{2} \dot{u}_n(0) \]
(15)
for $0 \leq t \leq \dot{u}_n(0)/2m_2$. One estimates
\[ \dot{u}_{n-1}(0) = d(u_{n-2}(0) - 2u_{n-1}(0) + u_n(0)) + h(u_{n-1}(0)) \]
\[ \quad \geq h(u_{n-1}(0)) - d(u_{n-1}(0) - u_{n-2}(0)) \]
\[ = h(x_{n-1}) - d(x_{n-1} - x_{n-2}) \]
\[ \quad \geq h(x_{n-1})/2, \quad \text{by} \ (14). \]
Using this together with (15) for \( n = n_1 - 1 \) we obtain

\[
\begin{align*}
    u_{n_1-1}(t) - x_{n_1-1} &\geq t \frac{h(x_{n_1-1})}{4} \quad \text{for} \quad 0 \leq t \leq \frac{h(x_{n_1-1})}{4m_2}. \\
\end{align*}
\]  

(16)

Using

\[
\begin{align*}
    - \frac{h(0)}{d} = u_0(t^*) - u_0(0) &= \int_0^{t^*} \dot{u}_0(s) \, ds \leq t^* \sup_s \dot{u}_0(s)
\end{align*}
\]

and \( \dot{u}_0 = d(u_1 - 2u_0) + h(u_0) \), one estimates that \( t^* \geq -h(0)/dm_1 \). We infer now from (16) that

\[
\begin{align*}
    u_{n_1-1}(t^*) - x_{n_1-1} &\geq \delta_6
\end{align*}
\]

and thus

\[
\begin{align*}
    y_{n_1} &\geq \delta_6 + x_{n_1-1} \geq n_1/N.
\end{align*}
\]

This is a contradiction to \( y_{n_1} < n_1/N \). 

3. A PRIORI ESTIMATES AND PROPERTIES

In this section we will prove a priori estimates and properties of fixed points of \( T \), where \( T \) is defined in Definition 2. Recall from the introduction, that fixed points of \( T \) correspond to traveling waves of (4). We will denote by \( a(h) \) or \( a \) the unique zero of \( h \in \mathcal{B}_{app} \) in \((0, 1)\).

**Lemma 4.** Suppose \( x \) is a fixed point of \( T \), and let \( \tau := t^*(x) \). Let

\[
\begin{align*}
    e(h) := \min_{a/4 \leq s \leq a/2} \frac{-h(s)}{d}, \quad m(h) := \min \left\{ \frac{a(h)}{4}, e(h) \right\},
\end{align*}
\]

and

\[
\begin{align*}
    M(h) := \max_{0 \leq s \leq 1} \left\{ 2d + h(s) \right\}.
\end{align*}
\]

Then \( \tau \geq \tau_0(h) := m(h)/M(h) > 0 \).

**Proof.** There are two cases. First suppose that there exists an integer \( n \) such that \( a/4 \leq x_n \leq a/2 \). Since \( x = Tx \in O \) by Lemma 2, we have \( d(x_n - x_{n-1}) + h(x_n) > 0 \), and thus \( x_n + 1 - x_n > -h(x_n)/d + x_n - x_{n-1} \). This implies \( x_{n+1} - x_n \geq e(h) \).
If there is no index $n$ such that $a/4 \leq x_n \leq a/2$, then there exists $n$ such that $x_{n+1} - x_n > a/4$. In either case there exists an integer $n$ such that

$$x_n \leq a/2, \quad \text{and} \quad x_{n+1} - x_n \geq m(h). \quad (17)$$

Since $u_n(t) \leq M(h)$ for all $t \geq 0$ we obtain

$$m(h) \leq u_{n+1}(0) - u_n(0) = u_n(\tau) - u_n(0) = \int_0^\tau \dot{u}_n(t) \, dt \leq \tau M(h),$$

i.e.,

$$\tau \geq m(h)/M(h).$$

**Lemma 5.** Let

$$\varepsilon_0 := \frac{1}{2} \min\{a, 1 - a\}$$

and for $x = \{x_n\}_0^N$ let $(x, \varepsilon)$ be the number of $x_n$'s in $x$ such that $\varepsilon < x_n < 1 - \varepsilon$. Then for all $\varepsilon \in [0, \varepsilon_0]$ there exists a bound $S(\varepsilon, h)$ independent of $N$ such that

$$N(x, \varepsilon) \leq S(\varepsilon, h)$$

for all $x \in \{x \in \bigcup_{n=0}^N C \cap \overline{O} : Tx = x\}.$

**Proof.** We pick any $h \in A_{\text{app}}, \varepsilon \in [0, \varepsilon_0]$ and suppose $x$ is a fixed point of $T$. We construct $S(\varepsilon, h)$ in four steps.

**Step 1.** Let

$$p(\varepsilon, h) := \max_{\varepsilon < s \leq a/2} \frac{d}{-h(s)}$$

and suppose

$$\varepsilon < x_i \leq x_{i+1} \leq \cdots \leq x_j \leq a/2.$$

Then

$$1 \geq x_{j+1} - x_i = \sum_{n=i}^j (x_{n+1} - x_n) \geq \sum_{n=i}^j (x_n - x_{n-1}) - \frac{1}{d} \sum_{n=i}^j h(x_n)$$

$$\geq - \frac{1}{d} \sum_{n=i}^j h(x_n) \geq \frac{j+1-i}{p(\varepsilon, h)}.$$ 

Hence

$$j+1-i \leq p(\varepsilon, h),$$

i.e., the number of $x_n$'s with $\varepsilon < x_n \leq a/2$ is bounded above by $p(\varepsilon, h)$. 
Step 2. In the proof of Lemma 4 (cf. (17)) it is shown that there exists an integer \( n_0 \) such that

\[ x_{n_0} \leq a/2, \quad \text{and} \quad x_{n_0 + 1} - x_{n_0} \geq m(h), \]

where \( m(h) \) is defined in Lemma 4. Since

\[ x_{n + 1} - x_n > x_n - x_{n - 1} - (1/d) \ h(x_n) \geq x_n - x_{n - 1} \]

for all \( 0 \leq x_n \leq a \), we obtain

\[ x_{n + 1} - x_n \geq m(h) \quad \text{for all} \quad a/2 \leq x_n \leq a. \]

We conclude that the number of \( x_n \)'s with \( a/2 \leq x_n \leq a \) is bounded above by \( 1 + a/2m(h) \).

Step 3. Let

\[ b(x, h) := \max_{a \leq s \leq x} \frac{h(s)}{d}, \]

\[ \sigma_1(h) := \max \{ x - a : a \leq x \leq 1, b(x, h) \leq m(h)/2 \}, \]

and let

\[ \sigma_2(h) := \min \{ \sigma_1(h), m(h)/2 \}. \]

Note that \( \sigma_2(h) > 0 \). There is an integer \( n_0 \), such that

\[ x_{n_0 - 1} \leq a \quad \text{and} \quad x_{n_0} > a. \]

Either \( x_{n_0} \leq a + \sigma_1(h) \), in which case

\[ x_{n_0 + 1} - x_{n_0} > (x_{n_0} - x_{n_0 - 1}) - (1/d) \ h(x_{n_0}) \geq m(h) - \frac{1}{2}m(h) = m(h)/2 \]

or \( x_{n_0} > a + \sigma_1(h) \). In both cases there is at most one \( x_n \) such that \( a \leq x_n \leq a + \sigma_2(h) \).

Step 4. Suppose \( a + \sigma_2(h) \leq x_n \leq x_{n + 1} < 1 - \varepsilon \). Then for \( \tau := t^*(x) \)

\[ x_{n + 1} - x_{n - 1} \geq x_{n + 1} - x_n = u_n(\tau) - u_n(0) = \int_0^\tau \dot{u}_n(s) \ ds. \]  

For \( 0 \leq s \leq \tau \) we estimate \( \dot{u}_n(s) \) as follows:

\[ \dot{u}_n = d(u_{n - 1} - 2u_n + u_{n + 1}) + h(u_n) \]

\[ \geq h(u_n) - d(u_n - u_{n - 1}) \]

\[ \geq \min_{x_n \leq s \leq x_{n + 1}} h(s) - d(x_{n + 1} - x_{n - 1}). \]
Using this estimate in (18), one obtains

\[ x_{n+1} - x_{n-1} \geq \frac{\tau}{d \tau + 1} \left( \min_{x_n \leq s \leq x_{n+1}} h(s) \right). \]

Since

\[ \frac{\tau}{d \tau + 1} \geq \frac{\tau_0}{d \tau_0 + 1}, \quad \text{for} \quad \tau \geq \tau_0, \]

one concludes by Lemma 4 that

\[ x_{n+1} - x_{n-1} \geq \sigma_3 := \frac{\tau_0(h)}{d \tau_0(h) + 1} \left( \min_{a + \sigma_2 \leq s \leq 1 - \varepsilon} h(s) \right). \]

Therefore, if one assumes that

\[ a + \sigma_2 \leq x_i \leq \ldots \leq x_{j+1} < 1 - \varepsilon, \]

then

\[ 2 \geq \sum_{n=i}^{j} (x_{n+1} - x_{n-1}) \geq (j + 1 - i) \sigma_3(h). \]

This shows that the number of \( x_n \)'s such that

\[ a + \sigma_2 \leq x_n < 1 - \varepsilon, \]

is bounded above by \( 2/\sigma_3(h) + 1 \). Summarizing the results in Steps 1, 2, 3, and 4, we infer

\[ \#(x, \varepsilon) \leq p(\varepsilon, h) + \left( 1 + \frac{a(h)}{2m(h)} \right) + 1 + \left( \frac{2}{\sigma_3(h)} + 1 \right) := S(\varepsilon, h). \]

In Definition 2 we defined \( T \) for all \( x \in C(h, d, N) \cap O(h, d, N) \) for which \( t^*(x) < \infty \). The following two lemmas will be used to show that for \( d \) sufficiently large \( t^*(x) < \infty \) for all \( x \in C \cap O \).

**Lemma 6.** Let \( x \in C(h, d, N) \cap O(h, d, N) \), \( \dot{u}(t) = \ddot{u}(x, t) \), and \( D > 0 \). Suppose that for all \( n \in \{1, 2, \ldots, N\} \), \( 0 < u_n(t) < 1 \) implies \( \dot{u}_n(t) < D \). Then for all \( k \in \mathbb{N} \) for which \( d \geq k^2(D + \sup |h|) \), \( u_n(t) - u_{n-1}(t) \leq 2/k \) for \( n = 1, 2, \ldots, N \).

**Proof.** For a shorter notation let

\[ A_n := u_n(t) - u_{n-1}(t). \]
For $0 < u_n(t) < 1$ one has
\[ \dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) \]
and therefore
\[ A_{n+1} - A_n = (1/d)(\dot{u}_n(t) - h(u_n(t))). \]
Using the assumption $\dot{u}_n(t) < D$ and $d > k^2(D + \sup |h|)$ one obtains
\[ |A_{n+1} - A_n| \leq 1/k^2 \quad \text{if} \quad 0 < u_n < 1. \]
If $u_n(t) = 0$, then $n = 0$, $t = 0$, and $u_i(0) = -h(0)/d$. Therefore,
\[ |A_{n+1} - A_n| = -h(0)/d < 1/k^2. \]
If $u_n(t) = 1$, then $A_{n+1} = 0$ and one calculates that
\[ |A_{n+1} - A_n| = \Delta_n < h(1)/d < 1/k^2 \]
by using $d(A_{n+1} - A_n) + h(u_n) > 0$. In either case one has
\[ |A_{n+1} - A_n| \leq 1/k^2. \]
Now assume contrary to the conclusion that there exists $n_0$ such that
\[ A_{n_0} > 2/k. \]
Then
\[ A_{n_0 + m} = A_{n_0} - \sum_{i=1}^{m} (A_{n_0 + (i-1)} - A_{n_0 + i}) > 2/k - m/k^2 \]
and thus
\[ A_{n_0 + m} > 1/k \quad \text{for} \quad m = 0, 1, \ldots, k. \]
This implies that $n_0 + k \leq N$ because $A_{N+1} = 0$. Then
\[ 1 \geq u_{n_0 + k}(t) - u_{n_0 - 1}(t) = \sum_{m=0}^{k} A_{n_0 + m} \geq \frac{k + 1}{k}, \]
which is a contradiction. 

**Lemma 7.** There exists a number $d_1$ which depends only on $\sup |h|$, $\sup_{s \neq t} |(h(s) - h(t))/(s - t)|$, and $\int_0^1 h(s) \, ds$, such that for all $x \in C(h, d, N) \cap O(h, d, N)$, $t \in [0, t^*)$, $d > d_1$, the following holds:
\[ \sup_n \dot{u}_n(t) \geq \frac{1}{2} \int_0^1 h(s) \, ds. \]
The sup here is taken over all $n$ for which $0 < u_n < 1$. 
Proof. Let \( n_1 > 0 \) be such that \( u_{n_1}(t) < 1 \). In the following \( t \) is fixed and we write \( u_n \) instead of \( u_n(t) \). Then

\[
d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) = \dot{u}_n \quad \text{for } n = 0, \ldots, n_1.
\]

This implies

\[
h(u_n)(u_{n+1} - u_n) + d(u_{n-1}u_{n+1} - 2u_nu_{n+1} + u_n^2
- u_{n-1}u_n + 2u_n^2 - u_nu_{n+1}) = \dot{u}_n(u_{n+1} - u_n).
\]

and

\[
h(u_n)(u_n - u_{n-1}) + d(u_{n-1}u_n - 2u_n^2 + u_nu_{n+1}
- u_{n-1}^2 + 2u_{n-1}u_n - u_{n-1}u_{n+1}) = \dot{u}_n(u_n - u_{n-1}).
\]

Adding the two equations gives

\[
h(u_n)(u_{n+1} - u_n) + h(u_n)(u_n - u_{n-1}) + d(u_{n+1}^2 - u_{n-1}^2 + 2u_{n-1}u_n - 2u_nu_{n+1})
= \dot{u}_n(u_{n+1} - u_{n-1}) \leq \left( \max_{0 \leq n \leq n_1} \dot{u}_n \right) (u_{n+1} - u_{n-1}).
\]

Adding over \( n \) from 0 to \( n_1 \) gives

\[
\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) + d((u_{n+1} - u_{n-1})^2 - u_0^2)
\leq \left( \max_{0 \leq n \leq n_1} \dot{u}_n \right) (u_{n+1} - u_0)
\]

and therefore

\[
\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) \leq du_0^2 + 2 \max_{0 \leq n \leq n_1} \dot{u}_n.
\]

Recall that \( u_0 \leq -h(0)/d \) and thus

\[
\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) \leq h(0)^2/d + 2 \max_{0 \leq n \leq n_1} \dot{u}_n. \quad (19)
\]

Assume now the contrary of the claim, i.e., suppose

\[
\dot{u}_n < D := \frac{1}{2} \int_0^1 h(s) \, ds \quad \text{for } n = 0, 1, \ldots, n_1.
\]
Denote by \( n_1(d) \) the largest \( n \in \{0, \ldots, N\} \) such that \( u_n < 1 \). Then there exists

a number \( d_1 \) by Lemma 6 such that

\[
d_1 > \frac{2h(0)^2}{\int_0^1 h(s) \, ds},
\]

and

\[
\left| \sum_{n = 0}^{n_1(d)} h(u_n)(u_{n+1} - u_n) - \int_0^1 h(s) \, ds \right| < \frac{1}{4} \int_0^1 h(s) \, ds,
\]

for \( d > d_1 \). Note that \( d_1 \) depends only on \( d \).

This gives a contradiction in (19) for \( d > d_1 \).

4. Solution of the Simplified Problem

Let \( d_2 := \max\{8, d_1\} \), where \( d_1 \) is chosen according to Lemma 7. In this section it will be shown that the simplified problem has a solution for \( d > d_2 \). Let \( \delta \) be defined by Eq. (13) so that the conclusion of Lemma 3 holds. For the rest of this section \( d \) is any number larger than \( d_2 \) and \( N \) an integer larger than \( 1/\delta \). For shorter notation let

\[
C_0 := C(h_0, d, N) \quad \text{and} \quad O_0 := O(h_0, d, N),
\]

where \( h_0(x) = x - \frac{1}{3} \). All \( u_n \)'s are nondecreasing by Lemma 1 and the proof of Lemma 2. Therefore one concludes from Lemma 7 that for \( t \in (0, t^*) \)

\[
N \geq \sum_{n = 0}^N u_n(t) - \sum_{n = 0}^N u_n(0) = \int_0^t \sum_{n = 0}^N \dot{u}_n(s-) \, ds \geq t \frac{1}{2} \int_0^1 h_0(s) \, ds
\]

and thus

\[
t^*(x) \leq 2N \int_0^1 h_0(s) \, ds =: M^*.
\]

(20)

Denote by \( T_0 \) the map \( T = T(h_0) \) which has been defined in Definition 2. Since \( t^*(x) \leq M^* \), the map \( T_0 \) is defined on \( C_0 \cap \mathcal{O}_0 \). The map \( T_0 \) has a fixed point by Brouwer's fixed point theorem if the following holds:
(i) \( C_0 \cap O_0 \) is a closed, bounded, and convex subset of \( \mathbb{R}^{N+1} \).
(ii) \( T_0(C_0 \cap O_0) \subset C_0 \cap O_0 \).
(iii) \( T_0 \) is continuous.
(iv) \( C_0 \cap O_0 \) is nonempty.

The proof of (i) is straightforward and (ii) follows from Lemmas 2 and 3. In order to show that \( T_0 \) is continuous define

\[ r: \overline{C_0 \cap O_0} \times [0, M^*] \to \mathbb{R} \]

by \( r(x, t) := u_0(x; t) \), where \( M^* \) is defined in (20) and define

\[ s: \overline{C_0 \cap O_0} \times [0, M^*] \to \mathbb{R} \]

by \( s(x, t) := -h(0)/d \). Then \( r \) and \( s \) are continuous functions. Let

\[ G(t^*) := \{(x, t) : x \in C_0 \cap O_0, t = t^*(x)\} \]

be the graph of \( t^* \). Then

\[ G(t^*) \]

is closed in \( \overline{C_0 \cap O_0} \times [0, M^*] \) and therefore compact. Since the graph of \( t^* \) is compact, it follows that \( t^* \) is continuous. Inspection of the definition of \( T_0 \) shows then that \( T_0 \) is continuous. We now show (iv) by constructing an \( x \in C_0 \cap O_0 \). In this construction let \( a = \frac{1}{4} \) and thus \( h_0(x) = x - a \).

\textbf{Step 1.} Let \( x_0 := 0 \) and define inductively

\[ x_{n+1} := x_n + \frac{(n+1) a}{d} \quad \text{for} \quad n = 0, 1, \ldots, n_0, \]

where \( n_0 \) is determined by the condition

\[ x_{n_0} \leqslant a \quad \text{and} \quad x_{n_0+1} > a. \]

Then for \( n = 1, \ldots, n_0 \) one has

\[ d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n > 0. \]

\textbf{Step 2.} Since

\[ \frac{x_{n_0}}{a} < \frac{x_{n_0+1}}{a} \iff \frac{n_0(n_0+1) a}{2d} \leqslant a < \frac{(n_0+1)(n_0+2) a}{2d}, \]

it follows that

\[ (n_0 + 1) a/d \leqslant 2a/n_0 \quad \text{and} \quad 2d < (n_0 + 1)(n_0 + 2). \]
which implies

\[ n_0 > \sqrt{2d - 2} > 2. \]

Here we used that \( d > 8 \). Hence

\[ x_{n_0 + 1} = x_n + (n_0 + 1) \frac{a}{d} \leq a + 2a/n_0 < 2a \]

which proves \( x_{n_0 + 1} < \frac{1}{2} \). Let

\[ x_{n+1} := x_n + (n_0 + 1) \frac{a}{d} \quad \text{for} \quad n = n_0 + 1, \ldots, n_1, \]

where \( n_1 \) is determined by the condition

\[ x_{n_1} \leq \frac{1}{2} \quad \text{and} \quad x_{n_1 + 1} > \frac{1}{2}. \]

It could be that \( n_0 = n_1 \) in which case no new \( x_n \)'s are constructed. So suppose that \( n_1 \geq n_0 + 1 \). Then for \( n = n_0 + 1, \ldots, n_1 \) one calculates that

\[ d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - a > 0. \]

**Step 3.** Next let

\[ x_{n+1} := x_n + (n_1 + n_0 + 1 - n) \frac{a}{d} \quad \text{for} \quad n = n_1 + 1, \ldots, n_0 + n_1. \]

Then for \( n = n_1 + 1, \ldots, n_0 + n_1 \) one calculates that

\[ d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - 2a > 0. \]

Since

\[ x_{n_0 + n_1 + 1} = x_{n_1 + 1} + \left( \sum_{n=1}^{n_0} n \right) \frac{a}{d} = x_{n_1 + 1} + x_{n_0} \]

and

\[ x_{n_1 + 1} = x_n + (n_0 + 1) \frac{a}{d} \leq \frac{1}{2} + x_{n_0 + 1} - x_{n_0} < 1 - x_{n_0}, \]

one concludes

\[ x_{n_0 + n_1 + 1} < 1. \]

**Step 4.** Define

\[ x_{n+1} := x_n + a/d \quad \text{for} \quad n = n_0 + n_1 + 1, \ldots, n_2, \]

where \( n_2 \) is determined by the condition

\[ 1 - a/d \leq x_{n_2 + 1} < 1. \]
Then for \( n = n_0 + n_1 + 1, \ldots, n_2 \) we obtain

\[
d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - a > 0.
\]

**Step 5.** Finally let

\[
x_n := 1 \quad \text{for all} \quad n = n_2 + 2, \ldots, N.
\]

Then for \( n = n_2 + 1, \ldots, N \) we have

\[
d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) > x_n - 2a > 0. \]

5. SOLUTION OF THE APPROXIMATE PROBLEM

Let \( h_1 \in \mathcal{B}_{\mathrm{app}} \). We will now construct a homotopy \( h_\lambda \) that deforms \( h_0 \) continuously into \( h_1 \) in the set \( \mathcal{B}_{\mathrm{app}} \). Denote by \( a_0 \) the unique zero of \( h_0 \) and by \( a_1 \) the unique zero of \( h_1 \). Let

\[
h(\lambda, x) := \begin{cases} \left( 1 - \frac{\lambda}{a_0} \right) h_0(x) + \frac{\lambda}{a_0} g(a_0, x) & \text{for } 0 \leq \lambda \leq a_0 \\ g(\lambda, x) & \text{for } a_0 \leq \lambda \leq a_1 \\ \left( \frac{1 - \lambda}{1 - a_1} \right) g(a_1, x) + \frac{\lambda - a_1}{1 - a_1} h_1(x) & \text{for } a_1 \leq \lambda \leq 1, \end{cases}
\]

where \( g(\lambda, x) : [a_0, a_1] \times [0, 1] \to \mathbb{R} \) is defined by

\[
g(\lambda, x) := \max \left\{ -\frac{(1 - \lambda)^2}{4\lambda}, x - \lambda \right\}.
\]

A calculation shows that

\[
\int_0^1 g(\lambda, x) \, dx = \frac{(1 - \lambda)^2}{4} + \frac{(1 - \lambda)^4}{32\lambda^2} > 0 \quad \text{for } a_0 \leq \lambda \leq a_1.
\]

Then \( h_\lambda := h(\lambda, \cdot) \) is a homotopy that deforms \( h_0(x) \) continuously into \( h_1(x) \) in the set \( \mathcal{B}_{\mathrm{app}} \). Let

\[
C_\lambda := C(h_\lambda, d, N), \quad O_\lambda := (h_\lambda, d, N),
\]

\[
\mathcal{C} := \{ (\lambda, x) : \lambda \in [0, 1], x \in C_\lambda \},
\]

and

\[
\mathcal{O} := \{ (\lambda, x) : \lambda \in [0, 1], x \in O_\lambda \}.
\]
By Lemma 7 there exists a constant $d_3 > d_2$ such that

$$\sup_n \dot{u}_n(t) \geq \frac{1}{2} \int_0^1 h_\lambda(s) \, ds \quad \text{for all } (\lambda, x) \in \overline{C} \cap \overline{C}.$$

We will now specify the parameters $d$ and $N$. Let $d$ be any number greater than $d_3$. An inspection of the definition of $\delta$ in (13) shows that $\delta$ depends continuously on $h$. One can choose therefore a $\delta$ such that the conclusion of Lemma 2 holds for all $h_\lambda$. Then let $N$ be some integer greater than $1/\delta$. Next, we define

$$I: \overline{C} \cap \overline{C} \to \mathbb{R}^{N+1}$$

similarly to Definition 2. First let

$$t^*: \overline{C} \cap \overline{C} \to (0, \infty)$$

be defined by

$$t^*(\lambda, x) := \sup \{ t : u_0(x, h_\lambda; t) < -h_\lambda(0)/d \}.$$

Analogous to (20) one has

$$t^*(\lambda, x) \leq 2N / \left( \min_{0 \leq i \leq 1} \int_0^1 h_\lambda(s) \, ds \right) =: M^*.$$

One can show that $t^*$ is continuous just as we did in Section 4. Let $T(\lambda, x)$ be defined by

$$(T(\lambda, x))_n := \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x, h_\lambda; t^*(\lambda, x)) & \text{for } n = 1, \ldots, N. \end{cases}$$

Then $T$ is continuous. We will use the following notation. If $X \subset [0, 1] \times \mathbb{R}^{N+1}$ then let

$$X_\lambda := \{ x : (\lambda, x) \in X \}$$

be the "slice" of $X$ at $\lambda$. Furthermore we denote by $F_\lambda: X_\lambda \to \mathbb{R}^{N+1}$ the restriction of $F: X \to \mathbb{R}^{N+1}$ to the slice $X_\lambda$. We will use the following general homotopy invariance theorem from index-theory (cf. [1]).

**Theorem 2.** Let $A$ be a nonempty compact interval, let $A$ be a retract of some Banach space, and let $U$ be an open subset of $A \times A$. Suppose $F: \overline{U} \to A$ is a compact map such that $F(\lambda, x) \neq x$ for every $(\lambda, x) \in \partial U$. Then $i(F_\lambda, U_\lambda, A)$ is well defined and independent of $\lambda \in A$. 

In order to apply this theorem let
\[ \Phi: [0, 1] \times \mathbb{R}^{N+1} \to [0, 1] \times \mathbb{R}^{N+1} \]
be defined by \( \Phi(\lambda, x) = (\lambda, y) \), where \( y = \{ y_n \}_{n=0}^N \) is given by
\[
y_n := \begin{cases} 
\frac{h_n(0)}{h_0(0)} x_n & \text{for } n = 1 \\
x_n & \text{for } n \neq 1.
\end{cases}
\]
Then \( \Phi \) is an homeomorphism. Let \( U := \Phi^{-1}(C \cap \partial) \) and \( A := C_0 \). Note that \( A \) is a retract of \( \mathbb{R}^{N+1} \), because \( C_0 \) is closed and convex. Define
\[ F: U \to A \text{ by } F(\lambda, x) := \Phi^{-1}(T \circ \Phi(\lambda, x)). \]
It follows from Lemmas 2 and 3 that
\[ T(\lambda, x) \in C_0 \cap O_0 \quad \text{for } (\lambda, x) \in C \cap \partial. \]
In fact, by the same arguments as in the proof of Lemma 1(ii), it follows that
\[ T(\lambda, x) \in C_0 \cap O_0 \quad \text{for } (\lambda, x) \in \overline{C} \cap \overline{\partial}. \]
Therefore \( F(U) \subseteq A \). Since \( \partial \) is an open set, \( U = ([0, 1] \times A) \cap \Phi^{-1}(\partial) \) is an open subset of \( [0, 1] \times A \) (in the relative topology). Now one can apply Theorem 2 and conclude that \( i(F_\lambda, U_\lambda, A) \) is well defined and independent of \( \lambda \in [0, 1] \). Since \( \Phi_0(x) \equiv x \) it follows that \( F_0 \equiv T_0 \) and \( U_0 = C_0 \cap O_0 \). Therefore \( U_0 \) is convex and one calculates that
\[ i(F_0, U_0, A) = 1 \]
via the homotopy \((1 - \tau) x^0 + \tau F_0\), where \( x^0 \in U_0 \). Therefore
\[ i(F_1, U_1, A) = 1 \]
and by the solution property of the index, there exists an \( x \in U_1 \) such that \( F_1 x = x \). Therefore \( \Phi_1(x) \) is a fixed point of \( T_1 \). As in Section 4 the fixed point of \( T_1 \) corresponds to a traveling wave solution.

6. Convergence of Approximate Solutions

Let \( \{ h_k \} \) be a sequence in \( \mathcal{A}_{\text{app}} \) that converges to \( f \) in the norm defined by
\[
\| h \| := \sup |h| + \sup_{s \neq t} \left| \frac{h(s) - h(t)}{s - t} \right|.
\]
We have seen in Section 5 that for every \( h_k \in \mathscr{R}_{\text{app}} \) there exists a number \( d_3(h_k) \) such that

\[
\begin{align*}
\dot{u}_n &= d(u_{n_1} - 2u_n + u_{n+1}) + h_k(u_n), \\
u_n &= p(v_n), & n \in \mathbb{Z}
\end{align*}
\]

has a traveling wave solution for \( d > d_3(h_k) \). Recall that \( d_3 \) was defined according to Lemma 7 and depended only on

\[
\sup_{s \neq t} \left| \frac{h_k(s) - h_k(t)}{s - t} \right|, \quad \text{and} \quad \int_0^1 h_k(s) \, ds.
\]

Therefore there exists a number \( d^* < \infty \) such that (21) has a traveling wave solution for all \( d > d^* \). Let \( d \) be any number greater than \( d^* \). Then for every \( k \in \mathbb{N} \) there exists a traveling wave solution, say \( u^k, v^k \), of (21). Let \( x^k = \{x^k_n\}_{n=-\infty}^{\infty} \in l^\infty \) be defined by \( x^k_n := u^k_n(0) \). Then there exists an integer \( n_k \) such that

\[ x^k_{n_k} \leq \frac{1}{2} < x^k_{n_k + 1}. \]

Let \( y^k = \{y^k_n\}_{n=-\infty}^{\infty} \) be defined by

\[ y^k_n := x^k_{n+n_k} \quad \text{for} \quad n \in \mathbb{Z}. \]

In other words the sequence \( x^k \) is shifted to the sequence \( y^k \) in order that

\[ y^k_0 \leq \frac{1}{2} < y^k_1. \]

An inspection of \( S(\varepsilon, h) \) in Lemma 5, cf. (3), shows that \( \lim_{k \to \infty} S(\varepsilon, h^k) < \infty \) for all \( \varepsilon > 0 \). Therefore the sequence \( \{y^k\} \) has a convergent subsequence in \( l^\infty \), i.e., we may assume without loss of generality that

\[ \lim_{k \to \infty} y^k = y \quad \text{for some} \quad y \in l^\infty \text{ with } \lim_{n \to -\infty} y_n = 0 \text{ and } \lim_{n \to \infty} y_n = 1. \]

(22)

It follows from Lemmas 4 and 7 that

\[ 0 < \inf_{k} \tau_0(h_k) \leq \tau_k \leq \sup_{k} \frac{2}{\int_0^1 h_k(s) \, ds} < \infty, \quad \text{where} \quad \tau_k := \tau^*(x^k). \]

The first inequality follows from Lemma 4 and for the third note that

\[ 1 = \sum_n (u_{n+1}(0) - u_n(0)) = \sum_n (u_n(\tau) - u_n(0)) 
\]

\[ = \sum_n \int_0^\tau \dot{u}_n(s) \, ds 
\]

\[ = \int_0^\tau \sum_n \dot{u}_n(s) \, ds 
\]

\[ \geq \tau \inf_{0 \leq t \leq \tau} (\sup_n \dot{u}_n(t)). \]
Therefore we may assume without loss of generality that
\[ \lim_{k \to \infty} \tau_k = \tau \quad \text{for some } 0 < \tau < \infty. \] (23)

We will show now that the solution \( \{u_n\} \) of the initial value problem
\[
\begin{align*}
\dot{u}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \\
v_n(0) &= y_n, \quad n \in \mathbb{Z}.
\end{align*}
\]
is a traveling wave with velocity \( c = 1/\tau \), where \( \tau \) is determined in (23) and \( y \) is determined in (22).

First note that \( 0 < u_n(t) < 1 \) for all \( n \in \mathbb{Z} \) and all \( t \geq 0 \) because \( \{u_n\}, \ y_n \equiv 0 \) is a lower solution and \( \{\tilde{u}_n\}, \ \tilde{u}_n \equiv 1 \) is an upper solution of \( \{u_n\} \). Therefore \( \{u_n\}, \ \{v_n\} \), where \( v_n = u_n \), is the unique solution of
\[
\begin{align*}
\dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \\
v_n(0) &= y_n, \quad n \in \mathbb{Z}.
\end{align*}
\]
(24)

Because of the continuous dependence of the solution of (24) on its initial condition \( y \) and on the function \( f \), we conclude that \( u_n(t) \) is nondecreasing and \( u_n(0) = u_{n-1}(0) \). It remains to be shown that \( \dot{u}_n(t) > 0 \) for all \( n \in \mathbb{Z} \) and all \( t \in \mathbb{R} \). We already know that
\[ \dot{u}_j(s) \geq 0 \quad \text{for all } j \in \mathbb{Z} \text{ and all } s \in \mathbb{R}. \] (25)

Suppose that \( \dot{u}_n(t) = 0 \) for some \( n \in \mathbb{Z} \) and some \( t \in \mathbb{R} \). Then \( \tilde{u}_n(t) \) exists and
\[ \tilde{u}_n(t) = d(\tilde{u}_{n-1}(t) + \tilde{u}_{n+1}(t)). \] (26)

Using (25) we conclude \( \tilde{u}_n(t) \geq 0 \). Since \( \tilde{u}_n(t) > 0 \) would lead to \( \dot{u}_n(t - \varepsilon) < 0 \) for sufficiently small \( \varepsilon > 0 \) therefore contradicting (25), we infer \( \tilde{u}_n(t) = 0 \). It follows then from (25) and (26) that \( \dot{u}_{n-1}(t) = 0 \) and \( \dot{u}_{n+1}(t) = 0 \). Hence \( \dot{u}_n(t) = 0 \) for all \( n \in \mathbb{Z} \) by induction, which implies that the wave \( \{u_n\} \) has zero speed in contradiction to \( c = 1/\tau > 0 \). This completes the proof of Theorem 1.

REFERENCES

1. H. AMANN, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, Siam Rev. 18, No. 4 (1976).