Traveling waves in a convolution model with infinite distributed delay and non-monotonicity
Zhaoquan Xu, Peixuan Weng *
School of Mathematics, South China Normal University Guangzhou, 510631, China

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A B S T R A C T

In this paper, we study a population model with nonlocal diffusion and a non-monotonic reaction term with infinite distributed delay. Some existence results of traveling wavefronts for the system with a monotonic reaction term are firstly obtained by the construction of upper-lower solutions and the application of Schauder’s fixed point theorem. Then by constructing a couple of auxiliary equations with monotonicity and using the comparison method, we prove the existence of traveling waves for the system without monotonicity. We also give some discussion on the asymptotic behavior of the traveling waves as $\xi = x + ct \to -\infty$.

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1. Introduction

A traveling wave solution is a spatial translation invariant solution of differential equations with spatial-diffusion. Due to its natural background and applicable potential in physics, biology and epidemics, the theory of traveling wave solutions has been extensively developed in the literature (see [1–3] and the references therein) since the pioneer work of Fisher [4] and Kolmogorov et al. [5]. The studies include the existence, non-existence, minimal wave speed, uniqueness and stability of traveling wave solutions etc., and also involve more extensive dynamical properties such as the asymptotic speed of propagation (see [6–9]). Many types of equations or models are covered in the literature, e.g., nonlinear reaction–diffusion equations with time delay [10–13] or without time delay [14], integral equations and integro-differential equations [7,8,15], lattice differential systems [9,16], etc.

Recently, Al-Omari and Gourley [17] proposed the following mature population model of a single species with age structure:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - \beta u^2 + \alpha e^{-\gamma r} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi dr}} e^{-\frac{(x-y)^2}{4dr}} u(t - r, y) \, dy, \quad t \geq 0, \, x \in \mathbb{R}. \quad (1.1)$$

where $\alpha$, $\beta$, $\gamma$, $r$ are positive constants, and $D > 0$, $d \geq 0$ denote the diffusion rate of the mature and immature population, respectively. By using a perturbation argument together with Fredholm orthogonality theory, they showed that there exists $c^* > 0$ such that for every $c > c^*$, (1.1) admits a traveling wavefront when $d$ is sufficiently small.

In (1.1), $r > 0$ is the time taken from birth to maturity for the species. When taking account of the case that the individuals have no fixed time to be mature, it allows a long term delayed effect on the differential equation. Based on this consideration and the spatial-diffusion of the species, Al-Omari and Gourley [18] further investigated the existence of traveling wavefronts...
for the model with distributed delay:
\[
\frac{du}{dt} = D\frac{\partial^2 u}{\partial x^2} - \beta u^2 + \alpha \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} u(t-s, y) e^{-\gamma s} g(s) \, dy \, ds, \quad t \geq 0, \ x \in R,
\]
where \( g(s) = (s/r^2) e^{-s/r} \) and \( r > 0 \) is sufficiently small.

In [19], Wang and Xu studied the following model:
\[
\frac{du(t, x)}{dt} = D\frac{\partial^2 u(t, x)}{\partial x^2} - ru(t, x) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s, x, y) k(s) f(u(t - s, y)) \, dy \, ds, \quad t \geq 0, \ x \in R,
\]
where
\[
G(t, x, y) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x-y)^2}{4t} \right),
\]
where \( f \) is monotonic and Lipschitz continuous on any compact interval, and \( D > 0, \ r \geq 0 \) represent the diffusive rate and death rate for the individuals, respectively, \( k(s) \) is a probability density function satisfying
\[
\int_{0}^{\infty} k(s) = 1, \quad k(s) \geq 0 \quad \text{for } s \in [0, +\infty).
\]
Weng and Xu considered the posedness of the initial value problem, and also obtained the existence of traveling wavefronts for \( c \geq c^* \).

As we see from above, although the Laplacian operator \( \Delta := \frac{\partial^2}{\partial x^2} \) is common in modeling the diffusion of the species, it is a local operator which suggests that the population at the location \( x \) can only be influenced by the variation of the population near the same location \( x \). However, in many practical problems, individuals can move freely, and the flow of individuals will not be limited to a small area. So, the Laplacian operator may have some shortcomings in expressing the spatial-diffusion for some ecological and epidemiological models. One way to deal with these problems is to replace the Laplacian operator with a convolution diffusion term \( \int_{-\infty}^{\infty} f(x-y) u(t, y) - u(t, x) \, dy \). This implies that there will be \( \int_{-\infty}^{\infty} f(x-y) u(t, y) - u(t, x) \, dy \) individuals from the whole space moving into the location \( x \) at time \( t \). Therefore, one may call it a nonlocal diffusion. Some works on the existence of traveling waves for equations or models with the nonlocal diffusion can be found in [6,20–26].

In the present article, we shall consider a nonlocal diffusion version of (1.3):
\[
\frac{\partial u(t, x)}{\partial t} = D[f(u)(t, x) - u(t, x)] - h(u(t, x)) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s, x, y) k(s) f(u(t - s, y)) \, dy \, ds, \quad t \geq 0, \ x \in R,
\]
where \( D > 0 \) is a constant; \( f, h \) are Lipschitz continuous functions on any compact interval, \( f(0) = h(0) = 0, f(K) = h(K) \) for some \( K > 0 \); \( G(s, x, y) \) and \( k(s) \) are the same as in (1.3); \( f: R \rightarrow R \) is a non-negative function with
\[
J(x) = J(-x) \quad \text{for } x \in R, \quad \int_{-\infty}^{\infty} J(y) \, dy = 1, \quad \int_{-\infty}^{\infty} J(x) e^{-\lambda x} \, dx < +\infty \quad \text{for any } \lambda > 0.
\]
Let
\[
(f * u)(t, x) := \int_{-\infty}^{\infty} f(x-y) u(t, y) \, dy.
\]
Noting that
\[
\int_{R} G(t, x, y) \, dy = 1,
\]
we know that (1.5) has two equilibria \( u = 0 \) and \( u = K \). Throughout this article, we shall assume that (1.5) satisfies the above basic assumptions. Furthermore, we assume that

(H0) there is an \( \sigma \in (0, 1) \) such that
\[
\lim_{u \to 0^+} \sup \{f'(0) - f(u)/u\} u^{-\sigma} < +\infty \quad \text{and} \quad \lim_{u \to 0^+} \sup \{h(u)/u - h'(0)\} u^{-\sigma} < +\infty.
\]

It is clear that condition (H0) holds spontaneously if \( f, h \in C^2([0, K]) \).

The model considered here, not only involve spatial diffusion, but also the infinite distributed delay, which generalizes some existing models. In fact, Eq. (1.5) can be seen as a general form of Eqs. (1.2) and (1.3). If \( h(u) = ru \) and the diffusion kernel
\[
J(x) = \delta(x) + \delta''(x)
\]
with \( \delta \) the Dirac delta function (see [27]), then Eq. (1.5) reduces to the model (1.3). Furthermore if \( h(u) = \beta u^2, k(s) = (\alpha s/r^2) e^{-\gamma s/r} \) (except for a constant factor), \( f(u) = u \), then (1.5) is reduced to Eq. (1.2) as \( D = d \) (this will be the case when the immature population stays with their parents).
This paper is organized as follows. In Section 2, with the monotonicity of the reaction term \( f \), we study the existence of the traveling wavefronts of (1.5). Some existence results are obtained by the upper-lower solutions and the application of Schauder’s fixed point theorem. In Section 3, by using an argument motivated from [10], we construct a couple of associated auxiliary equations with monotonicity to squeeze the system without monotonicity on \( f \). We prove that there is a minimal wave speed \( c^* > 0 \) such that the existence of traveling wave solutions for the system is guaranteed when \( c \geq c^* \). We also give a discussion on asymptotic behavior for the traveling waves as \( \xi = x + ct \to -\infty \). Our results established here seems to be elementary new in the existing literature.

2. Traveling waves with monotonicity

We first give a definition of traveling wave solution (for brief, traveling wave). A traveling wave solution of (1.5) is a solution \( u(t, x) = \varphi(x + ct) \) with \( c > 0 \), where \( c \) is the speed parameter and \( \varphi \in C^1(\mathbb{R}, \mathbb{R}) \) is the wave profile. Moreover, if \( \varphi(\xi) \) is monotone in \( \xi \in \mathbb{R} \), then it is called a traveling wavefront.

In this section, we shall find a traveling wavefront of (1.5) with the asymptotic boundary conditions

\[
\lim_{\xi \to -\infty} \varphi(\xi) = 0, \quad \lim_{\xi \to +\infty} \varphi(\xi) = K.
\]

Substituting \( u(t, x) = \varphi(t + ct) \) into Eq. (1.5) and replacing \( x + ct \) by \( \xi \), we then obtain the wave profile equation

\[
c\varphi'(\xi) = D[(f \ast \varphi)(\xi) - \varphi(\xi)] - h(\varphi(\xi)) + \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} G(s, y)k(s)f(\varphi(\xi - y - cs))dyds,
\]

where \( G(s, y) \) is defined as

\[
G(s, y) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{y^2}{4s}\right).
\]

Let

\[
C_{[0, K]}(\mathbb{R}, \mathbb{R}) = \{ \varphi \in C(\mathbb{R}, \mathbb{R}) | 0 \leq \varphi(\xi) \leq K, \xi \in \mathbb{R} \}.
\]

Define the operator \( Q : C_{[0, K]}(\mathbb{R}, \mathbb{R}) \to C_{[0, K]}(\mathbb{R}, \mathbb{R}) \) by

\[
Q(\varphi)(\xi) = L_0 \varphi(\xi) + DJ \ast \varphi(\xi) - h(\varphi(\xi)) + \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} G(s, y)k(s)f(\varphi(\xi - y - cs))dyds, \quad \xi \in \mathbb{R},
\]

where \( L_0 \) is the Lipschitz constant of \( h \) on \([0, K] \). Then (2.1) is equivalent to

\[
c\varphi'(\xi) = -b\varphi(\xi) + Q(\varphi)(\xi),
\]

where \( b := L_0 + D \). By (2.2) we have

\[
\varphi(\xi) = \frac{1}{c} e^{-\frac{b}{c} \xi} \int_{-\infty}^{\xi} e^{\frac{b}{c} \eta} Q(\varphi)(\eta)d\eta.
\]

Now, we define another operator \( I : C_{[0, K]}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R}) \) by

\[
I(\varphi)(\xi) = \frac{1}{c} e^{-\frac{b}{c} \xi} \int_{-\infty}^{\xi} e^{\frac{b}{c} \eta} Q(\varphi)(\eta)d\eta.
\]

It is clear that the operator \( I \) is well defined and the fixed point of \( I \) satisfies (2.1), and vice versa. Therefore, in order to prove the existence of a traveling wavefront for (1.5), it is sufficient to consider the fixed point of the operator \( I \). We shall achieve the goal by applying Schauder’s fixed point theorem. Thus, we have to construct a proper convex set, which will be defined by a pair of upper and lower solutions.

**Definition 2.1.** \( \bar{\varphi} \in C_{[0, K]}(\mathbb{R}, \mathbb{R}) \) is called an upper solution of (2.1) if \( \bar{\varphi} \) is differentiable on \( R \setminus \Gamma' \) and satisfies

\[
c\bar{\varphi}'(\xi) - D[(f \ast \bar{\varphi})(\xi) - \bar{\varphi}(\xi)] + h(\bar{\varphi}(\xi)) - \int_{-\infty}^{\xi} \int_{-\infty}^{\xi} G(s, y)k(s)f(\bar{\varphi}(\xi - y - cs))dyds \geq 0
\]

for \( R \setminus \Gamma' \), where \( \Gamma' = (\xi_1, \xi_2, \ldots, \xi_n) \) with \( \xi_1 < \xi_2 < \cdots < \xi_n \) is a finite set of points. The lower solution \( \underline{\varphi} \) can be defined by reversing the inequality in (2.5).

Throughout this section, we assume that

(H1) \( f(u) \) is nondecreasing on \([0, K] \) and \( f(u) \neq h(u) \) for all \( u \in (0, K) \);

(H2) \( h'(0) < f'(0), 0 < f(u) \leq f'(0)u \) and \( 0 \leq h'(0)u \leq h(u) \) for all \( u \in (0, K) \).

It is not difficult to verify the following equality:

\[
\int_{-\infty}^{\infty} G(s, y)e^{-\lambda(y+cs)}dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi s}} e^{-\frac{y^2}{4s}} e^{-\lambda(y+cs)}dy = e^{(Ds^2-c\lambda)s}.
\]
The linearization equation of (2.1) at zero solution is
\[ c\phi'(\xi) = D[(f \ast \phi)(\xi) - \phi(\xi)] - h'(0)\phi(\xi) + f'(0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)(\xi - y - c)sdyds. \] (2.7)

Substituting \( \phi(\xi) = e^{\lambda \xi} \) into (2.7), we have from (2.6) that
\[ 0 = c\lambda - D \left[ \int_{-\infty}^{\infty} f(y)e^{-\lambda y}dy - 1 \right] + h'(0) - f'(0) \int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)e^{-\lambda(y+c)}dyds \]
\[ = c\lambda - D \left[ \int_{-\infty}^{\infty} f(y)e^{-\lambda y}dy - 1 \right] + h'(0) - f'(0) \int_{0}^{\infty} k(s)e^{(D\lambda^2 - c\lambda)s}ds \]
\[ =: \triangle(\lambda, c). \]

Since \( f(y) \) is an even function, then \( \triangle(\lambda, c) \) can be rewritten as
\[ \triangle(\lambda, c) = c\lambda - D \left[ \int_{0}^{\infty} f(y)e^{-\lambda y}dy + \int_{0}^{\infty} f(y)e^{\lambda y}dy - 1 \right] + h'(0) - f'(0) \int_{0}^{\infty} k(s)e^{(D\lambda^2 - c\lambda)s}ds. \]

By direct calculations, we have
\[ \triangle(+\infty, c) = -\infty \quad \text{for any given} \quad c > 0, \]
\[ \triangle(\lambda, +\infty) = +\infty \quad \text{for any given} \quad \lambda > 0, \]
\[ \triangle(0, c) = h'(0) - f'(0) < 0 \quad \text{for all} \quad c \geq 0, \]
\[ \frac{\partial \triangle(\lambda, c)}{\partial c} = \lambda + \lambda f'(0) \int_{0}^{\infty} sk(s)e^{(D\lambda^2 - c\lambda)s}ds > 0 \quad \text{for all} \quad \lambda > 0, \]
\[ \frac{\partial \triangle(\lambda, c)}{\partial \lambda} = c + D \int_{0}^{\infty} y^2f(y)e^{-\lambda y}dy - D \int_{0}^{\infty} y^2f(y)e^{\lambda y}dy - (2D\lambda - c)f'(0) \int_{0}^{\infty} sk(s)e^{(D\lambda^2 - c\lambda)s}ds, \]
\[ = -D \int_{0}^{\infty} \frac{y^2}{\lambda} f(y)e^{-\lambda y}dy - D \int_{0}^{\infty} \frac{y^2}{\lambda} f(y)e^{\lambda y}dy - 2Df'(0) \int_{0}^{\infty} sk(s)e^{(D\lambda^2 - c\lambda)s}ds \]
\[ - (2D\lambda - c)^2f'(0) \int_{0}^{\infty} s^2k(s)e^{(D\lambda^2 - c\lambda)s}ds < 0. \]

In view of the above feature of the function \( \triangle(\lambda, c) \), we have the following observation.

**Lemma 2.1.** If \( f'(0) > h'(0) \), then there exists a \( c^* > 0 \) such that
\[ (i) \quad \text{if} \quad c \geq c^*, \quad \text{the function} \quad \triangle(\lambda, c) = 0 \quad \text{has two positive roots} \quad \lambda_a(c), \lambda_b(c) \quad \text{with} \quad \lambda_a(c) \leq \lambda_b(c), \quad \text{the equality holds if and only if} \quad c = c^*; \]
\[ (ii) \quad \text{if} \quad c > c^*, \quad \triangle(\lambda, c) > 0 \quad \text{in} \quad (\lambda_a(c), \lambda_b(c)) \quad \text{and} \quad \triangle(\lambda, c) < 0 \quad \text{in} \quad (-\infty, \lambda_a(c)) \cup (\lambda_b(c), +\infty); \]
\[ (iii) \quad \text{if} \quad c < c^*, \quad \triangle(\lambda, c) < 0 \quad \text{for all} \quad \lambda \geq 0. \]

Now, another assumption will be needed.

(H3) there is a positive constant \( \theta \geq D(1 + \sigma^2)\lambda^2 \) such that \( \int_{0}^{\infty} k(s)e^{\theta y}ds < \infty \), where \( (\lambda_a, c^*) \) is the unique solution of
\[ \Delta(\lambda_a, c^*) = 0, \quad \frac{\partial \Delta}{\partial \lambda}(\lambda_a, c^*) = 0. \]

Assume that \( c > c^* \). For brief, we rewrite \( \lambda_a(c), \lambda_b(c) \) given in Lemma 2.1 as \( \lambda_a, \lambda_b \), respectively. Define two continuous functions:
\[ \overline{\psi}(\xi) = \min \{ K, A \left( e^{\lambda_a \xi} + me^{\alpha \lambda_a \xi} \right) \}, \quad \underline{\psi}(\xi) = \max \{ 0, A \left( e^{\lambda_a \xi} - me^{\alpha \lambda_a \xi} \right) \}, \]
where \( A \) is a given positive number, \( m \geq 1, \frac{1}{\alpha} \left( \frac{\lambda_a}{K} \right)^{\alpha-1} \) is large to be chosen later, \( \alpha \in (1, \min(\frac{\lambda_a}{K}, 1 + \sigma)) \) and \( \sigma \) is defined in (H3).

Let the roots of equations
\[ q(\xi) = A(e^{\lambda_a \xi} + me^{\alpha \lambda_a \xi}) - K = 0 \quad \text{and} \quad p(\xi) = A(e^{\lambda_a \xi} - me^{\alpha \lambda_a \xi}) = 0 \]
be \( \xi^* \) and \( \xi_* \), respectively. Clearly, \( \xi_* = -\frac{\ln m}{(\alpha-1)\lambda_a} \leq 0. \)

**Lemma 2.2.** The following statements hold:
\[ (i) \quad \overline{\psi}(\xi) \leq A(e^{\lambda_a \xi} + me^{\alpha \lambda_a \xi}), \quad \overline{\psi}(\xi) \leq K \quad \text{for all} \quad \xi \in \mathbb{R}; \]
\[ (ii) \quad \underline{\psi}(\xi) \geq 0, \quad \underline{\psi}(\xi) \geq A(e^{\lambda_a \xi} - me^{\alpha \lambda_a \xi}) \quad \text{for all} \quad \xi \in \mathbb{R}; \]
\[ (iii) \quad \overline{\psi}(\xi) \text{ is nondecreasing on} \quad \xi \in \mathbb{R}; \]
\[ (iv) \quad \underline{\psi}(\xi) \leq \overline{\psi}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}; \]
\[ (v) \quad |\overline{\psi}(\xi_1) - \overline{\psi}(\xi_2)| \leq \Delta \alpha \lambda_a K|\xi_1 - \xi_2| \quad \text{for any} \quad \xi_1, \xi_2 \in \mathbb{R}. \]
The conclusion (i)–(iii) can be seen easily from the definition of $\varphi(\xi)$ and $\underline{\varphi}(\xi)$. Next, we prove (iv) and (v).

For (iv), it is sufficient to prove \( \max_{\xi \in (-\infty, \xi_1)} p(\xi) \leq K \). Let
\[
p'(\xi) = \Lambda \left( \lambda_a e^{\alpha_0 \xi} - \alpha \lambda_a m e^{\alpha \xi} \right) = 0.
\] (2.8)

Clearly, the solution of (2.8) is $\xi = \xi_2 := -\frac{\ln m \alpha}{(\alpha - 1) \lambda_\alpha}$. Therefore, we have the conclusion that
\[p(\xi) \text{ is increasing for } \xi \in (-\infty, \xi_1) \text{ and decreasing for } \xi \in (\xi_1, \xi_*).\]

Since $m \geq \frac{1}{\sqrt[\alpha]{1 + \frac{1}{\lambda}}}$, then
\[
\max_{\xi \in (-\infty, \xi_1)} p(\xi) = p(\xi_1) \leq \Lambda e^{\alpha \xi_1} = \Lambda e^{-\frac{\ln m \alpha}{(\alpha - 1) \lambda_\alpha}} \leq K.
\]

For any $\xi_1, \xi_2 \in R$, assuming that $\xi_1 < \xi_2$. If $\xi^* \leq \xi_1 < \xi_2$, then we have
\[
|\overline{\varphi}(\xi_1) - \overline{\varphi}(\xi_2)| = |K - K| = 0 \leq \Lambda \alpha \lambda \alpha K |\xi_1 - \xi_2|.
\]

If $\xi_1 < \xi^* \leq \xi_2$, then we have
\[
|\overline{\varphi}(\xi_1) - \overline{\varphi}(\xi_2)| = \left| \Lambda \left( e^{\alpha \xi_1} + me^{\alpha \xi_1} \right) - K \right| = \left| \Lambda \left( e^{\alpha \xi_1} + me^{\alpha \xi_1} \right) - \Lambda \left( e^{\alpha \xi_1} + me^{\alpha \xi_1} \right) \right| = \left| e^{\alpha \xi_1} + me^{\alpha \xi_1} \right| |\xi_1 - \xi_1| = \Lambda \alpha \lambda \alpha K |\xi_1 - \xi_1| \leq \Lambda \alpha \lambda \alpha K |\xi_1 - \xi_2|.
\]

where $\xi_1 \in (\xi_1, \xi^*)$. Similarly, if $\xi_1 < \xi_2 \leq \xi^*$, then we have
\[
|\overline{\varphi}(\xi_1) - \overline{\varphi}(\xi_2)| = \left| \Lambda \left( e^{\alpha \xi_1} + me^{\alpha \xi_1} \right) - \Lambda \left( e^{\alpha \xi_2} + me^{\alpha \xi_2} \right) \right| = \left| \Lambda \left( e^{\alpha \xi_1} + me^{\alpha \xi_1} \right) - \Lambda \left( e^{\alpha \xi_2} + me^{\alpha \xi_2} \right) \right| |\xi_1 - \xi_2| = \Lambda \alpha \lambda \alpha K |\xi_1 - \xi_2|,
\]

where $\xi_2 \in (\xi_1, \xi_2)$. Summarily, we get the conclusion (v). The proof is complete.

By using (2.6) and (H3), we can obtain
\[
\int_0^\infty \int_{-\infty}^\infty G(s, y)k(s)e^{-\frac{1}{1+\sigma}\lambda_\alpha(y+c)s}dyds = \int_0^\infty k(s)e^{D(1+\sigma)^2 \lambda_\alpha^2 s - c(1+\sigma)\lambda_\alpha s}ds \leq \int_0^\infty k(s)e^{D(1+\sigma)^2 \lambda_\alpha^2 s}ds \leq \int_0^\infty k(s)e^{\theta s}ds < \infty. \quad (2.9)
\]

We introduce two notations for the following use of convenience:
\[
G_{1+\sigma} := \int_0^\infty \int_{-\infty}^\infty G(s, y)k(s)e^{-\frac{1}{1+\sigma}\lambda_\alpha(y+c)s}dyds, \quad \vartheta_k := \int_0^\infty k(s)e^{\theta s}ds.
\]

Now, we can verify the upper solution and lower solution.

**Lemma 2.3.** Assume that (H2)–(H3) hold, then for every $\alpha \in (1, \min \left\{ \frac{1}{\alpha_0}, 1 + \sigma \right\})$, there exists a large enough number $\Theta(\alpha, c) \geq 1$ such that for any $m > \Theta(\alpha, c)$, the functions $\overline{\varphi}(\xi), \underline{\varphi}(\xi)$ are upper and lower solutions of (2.1), respectively.

**Proof.** For $\xi > \xi^*$, we have $\overline{\varphi}(\xi) = K$ and
\[
c(\overline{\varphi}'(\xi) - D [\varphi(\xi) - \overline{\varphi}(\xi)]) + h(\overline{\varphi}(\xi)) - \int_0^\infty \int_{-\infty}^\infty G(s, y)k(s)f(\overline{\varphi}(\xi - y - c)s)dyds
\]
\[
= -D \int_{-\infty}^\infty J(y) \overline{\varphi}(\xi - y) dy - K + h(K) - \int_{-\infty}^\infty \int_{-\infty}^\infty G(s, y)k(s)f(\overline{\varphi}(\xi - y - c)s)dyds
\]
\[
\geq -D \left[ K \int_{-\infty}^\infty J(y) dy - K \right] + h(K) - f(K) \int_{-\infty}^\infty \int_{-\infty}^\infty G(s, y)k(s)dyds = 0.
\]

For $\xi < \xi^*$, we have $\overline{\varphi}(\xi) = \Lambda (e^{\alpha \xi} + me^{\alpha \xi})$. By Lemma 2.1, we get that $\Delta(\alpha \lambda_\alpha, c) > 0$ because of $\lambda_a < \alpha \lambda_a < \lambda_b$. Combining (2.9) with the following facts, that $\overline{\varphi}(\xi) \leq \Lambda (e^{\alpha \xi} + me^{\alpha \xi})$ for $\xi \in R, 0 < f(u) \leq f'(0)u$ and $0 \leq h'(0)u \leq h(u)$
for \( u \in (0, K) \), we have
\[
c(\psi)'(\xi) - D \int \varphi(\xi) + h(\varphi(\xi)) - \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds
\]
\[
= c \left( \lambda a e^{\alpha y} + \alpha \lambda a e^{\alpha \lambda y} \right) - D \left( \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds \right)
\]
\[
+ h(\varphi(\xi)) - \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds
\]
\[
= \Lambda e^{\lambda y} \Delta(\lambda a, c) + \Lambda me^{\alpha \lambda y} \Delta(\alpha \lambda a, c) + D \Lambda e^{\lambda y} \int_{-\infty}^{\infty} J(y)e^{-\lambda a y}dy - h'(0) \Lambda e^{\lambda y}
\]
\[
+ f'(0) \Lambda e^{\lambda y} \int_0^{\infty} k(s)e^{(D_2 - m \lambda a)y}ds + D \Lambda me^{\alpha \lambda y} \int_{-\infty}^{\infty} J(y)e^{-\alpha \lambda y}dy - h'(0) \Lambda me^{\alpha \lambda y}
\]
\[
+ f'(0) \Lambda me^{\alpha \lambda y} \int_0^{\infty} k(s)e^{(D_2 - m \lambda a)y}ds - D \int_{-\infty}^{\infty} J(y)\varphi(\xi - y)dy + h(\varphi(\xi))
\]
\[
- \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds
\]
\[
\geq 0.
\]
Therefore, \( \varphi(\xi) \) is an upper solution of (2.1).

Next, we shall prove that \( \underline{\varphi}(\xi) \) is a lower solution of (2.1). For \( \xi > \xi_s \), we have \( \underline{\varphi}(\xi) = 0 \) and
\[
c(\underline{\varphi})'(\xi) - D \left( \int \varphi(\xi) - \varphi(\xi) \right) + h(\varphi(\xi)) - \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds
\]
\[
= -D \int_{-\infty}^{\infty} J(y)\varphi(\xi - y)dy - \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds \leq 0.
\]
Under the assumption (H2), there exists a number \( M > 0 \) such that \( f'(0)u - f(u) \leq Mu^{1+\sigma} \) and \( h(u) - h'(0)u \leq Mu^{1+\sigma} \) for all \( u \in (0, K) \). If we choose
\[
m \geq \max \left\{ 1, \frac{1}{\alpha} \left( \frac{\Lambda}{K} \right)^{2-1}, \frac{MA^{\sigma}}{\Delta(\alpha \lambda a, c)} \right\} =: \Theta(\alpha, c),
\]
then for \( \xi < \xi_s \leq 0 \), we have \( \underline{\varphi}(\xi) = \Lambda(e^{\lambda y} - me^{\alpha \lambda y}) \) and
\[
c(\underline{\varphi})'(\xi) - D \left( \int \varphi(\xi) - \varphi(\xi) \right) + h(\varphi(\xi)) - \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds
\]
\[
= c \Lambda (\lambda a e^{\lambda y} - \alpha \lambda a e^{\alpha \lambda y}) - D \left( \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds \right)
\]
\[
+ h(\varphi(\xi)) - \int_0^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi - y - c))dyds
\]
\[
= \Lambda e^{\lambda y} \Delta(\lambda a, c) - \Lambda me^{\alpha \lambda y} \Delta(\alpha \lambda a, c) + D \Lambda e^{\lambda y} \int_{-\infty}^{\infty} J(y)e^{-\lambda a y}dy - h'(0) \Lambda e^{\lambda y}
\]
Assume that \( \phi(\xi) \) is nondecreasing for all \( \xi \in \mathbb{R} \); and \( \phi(\xi) \leq \varphi(\xi) \leq \varphi(\xi) \) for all \( \xi \in \mathbb{R} \); and \( |\varphi(\xi) - \varphi(\xi)| \leq \max \left\{ \lambda \alpha \lambda_0 \xi_0, \frac{2bK}{c} \right\} |\xi_1 - \xi_2| \) for \( \xi_1, \xi_2 \in \mathbb{R} \).

Therefore, \( \varphi(\xi) \) is a lower solution of (2.1). The proof is complete.

Let \( b, c, Q \) and \( I \) be defined in (2.2)-(2.4). For \( 0 < \mu < \frac{b}{c} \), let
\[
X_\mu = \{ \psi \in C(\mathbb{R}, \mathbb{R}) \mid \sup_{\xi \in \mathbb{R}} |\psi(\xi)| e^{-\mu |\xi|} < \infty \}
\]
with a norm
\[
|\psi|_\mu = \sup_{\xi \in \mathbb{R}} |\psi(\xi)| e^{-\mu |\xi|}.
\]

Then it is easily seen that \( (X_\mu, |.|_\mu) \) is a Banach space.

Define a set
\[
\Omega = \left\{ \varphi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}) \mid \begin{array}{l}
1. \varphi(\xi) \text{ is nondecreasing for all } \xi \in \mathbb{R};
2. \varphi(\xi) \leq \varphi(\xi) \leq \varphi(\xi) \text{ for all } \xi \in \mathbb{R};
3. |\varphi(\xi) - \varphi(\xi)| \leq \max \left\{ \lambda \alpha \lambda_0 \xi_0, \frac{2bK}{c} \right\} |\xi_1 - \xi_2| \text{ for } \xi_1, \xi_2 \in \mathbb{R}.
\end{array} \right\}
\]

By Lemma 2.2, it is clear that \( \varphi(\xi) \in \Omega \). Therefore, \( \Omega \) is nonempty. Furthermore, it is easy to verify that \( \Omega \) is convex and compact in \( X_\mu \).

The proof of the following lemma is easy, and we omit it.

**Lemma 2.4.** Assume that \( H_1 \) holds, then

(i) \( Q(\psi(\xi)) \leq Q(\phi(\xi)), I(\psi(\xi)) \leq I(\phi(\xi)) \) for any \( \psi, \phi \in C_{[0,K]}(\mathbb{R}, \mathbb{R}) \) with \( 0 \leq \psi(\xi) \leq \phi(\xi), \xi \in \mathbb{R} \);
(ii) \(0 \leq Q(\psi)(\xi) \leq bK, 0 \leq I(\psi)(\xi) \leq K\) for any \(\psi \in C_{[0,K]}(\mathbb{R}, \mathbb{R})\);
(iii) \(Q(\psi)(\xi), I(\psi)(\xi)\) is nondecreasing if \(\psi \in C_{[0,K]}(\mathbb{R}, \mathbb{R})\) is nondecreasing in \(\xi \in \mathbb{R}\).

**Lemma 2.5.** Assume that \((H_1)\) holds, then \(I(\Omega) \subset \Omega\).

**Proof.** For any \(\varphi \in \Omega \subset C_{[0,K]}(\mathbb{R}, \mathbb{R})\) and any \(u, v \in \mathbb{R}\), assuming that \(u \leq v\), since \(0 \leq Q(\varphi)(\xi) \leq bK\), then we have

\[
|I(\varphi)(u) - I(\varphi)(v)| = \frac{1}{c} \left| e^{-\frac{b}{2}v} \int_{-\infty}^{u} e^{\frac{b}{2}y} Q(\varphi)(y) dy - e^{-\frac{b}{2}u} \int_{-\infty}^{v} e^{\frac{b}{2}y} Q(\varphi)(y) dy \right|
\]

\[
\leq \frac{1}{c} \left\{ e^{-\frac{b}{2}u} \int_{-\infty}^{u} e^{\frac{b}{2}y} Q(\varphi)(y) dy - e^{-\frac{b}{2}u} \int_{-\infty}^{v} e^{\frac{b}{2}y} Q(\varphi)(y) dy \right\}
\]

\[
+ \left| e^{-\frac{b}{2}v} \int_{-\infty}^{u} e^{\frac{b}{2}y} Q(\varphi)(y) dy - e^{-\frac{b}{2}v} \int_{-\infty}^{v} e^{\frac{b}{2}y} Q(\varphi)(y) dy \right|
\]

\[
\leq \frac{1}{c} \left\{ e^{-\frac{b}{2}u} - e^{-\frac{b}{2}v} \right\} \int_{-\infty}^{u} e^{\frac{b}{2}y} Q(\varphi)(y) dy + e^{-\frac{b}{2}v} \int_{u}^{v} e^{\frac{b}{2}y} Q(\varphi)(y) dy
\]

\[
\leq \frac{1}{c} \frac{b}{c} |u - v| \int_{-\infty}^{u} e^{\frac{b}{2}(y-u)} Q(\varphi)(y) dy + \int_{u}^{v} e^{\frac{b}{2}(y-v)} Q(\varphi)(y) dy
\]

\[
\leq \frac{1}{c} \left[ bK |u - v| + \int_{u}^{v} Q(\varphi)(y) dy \right]
\]

\[
\leq \frac{2bk}{c} |u - v|,
\]

which also implies that \(I(\varphi)(\xi)\) is continuous for \(\xi \in \mathbb{R}\).

Since \(\varphi(\xi) \in C_{[0,K]}(\mathbb{R}, \mathbb{R})\) is a lower solution of (2.1) and the set \(\mathcal{I}\) in the **Definition 2.1** contains only a single point \(\xi_*\), we have

\[
Q(\varphi)(\xi) \geq c\varphi'(\xi) + b\varphi(\xi), \quad \xi \in \mathbb{R} \setminus \{\xi_*\}.
\]

Therefore,

\[
I(\varphi)(\xi) = \int_{-\infty}^{\xi} e^{\frac{b}{2}(y-\xi)} Q(\varphi)(y) dy
\]

\[
\geq \frac{1}{c} \left\{ \left( \int_{-\infty}^{\xi_*} + \int_{\xi_*}^{\xi} \right) e^{\frac{b}{2}(y-\xi)} \left[ c\varphi'(y) + b\varphi(y) \right] dy \right\}
\]

\[
= \varphi(\xi), \quad \text{for} \ \xi \in \mathbb{R} \setminus \{\xi_*\}.
\]

By the continuity of \(I(\varphi)(\xi)\), we get that

\[
I(\varphi)(\xi) \geq \varphi(\xi), \quad \xi \in \mathbb{R}.
\]

Similarly, we can get that

\[
I(\overline{\varphi})(\xi) \leq \overline{\varphi}(\xi), \quad \xi \in \mathbb{R}.
\]

Then for any \(\varphi \in \Omega\), by **Lemma 2.4**, we have

\[
\varphi(\xi) \leq I(\varphi)(\xi) \leq I(\varphi)(\xi) \leq I(\overline{\varphi})(\xi) \leq \overline{\varphi}(\xi) \quad \text{for} \ \xi \in \mathbb{R}.
\]

By the above argument, we have \(I(\Omega) \subset \Omega\). The proof is complete. \(\square\)

**Lemma 2.6.** Assume that \((H_3)\) holds. The operator \(I : C_{[0,K]}(\mathbb{R}, \mathbb{R}) \to X_\mu\) is continuous with respect to the norm \(|.|_\mu\) in \(X_\mu\).

**Proof.** First of all, we claim that \(Q : C_{[0,K]}(\mathbb{R}, \mathbb{R}) \to X_\mu\) is continuous with respect to the norm \(|.|_\mu\) in \(X_\mu\). For any \(\varphi_1, \varphi_2 \in \Omega\), we have

\[
|Q(\varphi_1)(\xi) - Q(\varphi_2)(\xi)| e^{-\mu|\xi|} = |L_{h} (\varphi_1(\xi) - \varphi_2(\xi)) + D [J * \varphi_1(\xi) - J * \varphi_2(\xi)] - [h(\varphi_1(\xi)) - h(\varphi_2(\xi))] + \int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y) k(s) [f (\varphi_1(\xi - y - cs)) - f (\varphi_2(\xi - y - cs))] dy ds| e^{-\mu|\xi|}
\]

\[
\leq |L_{h} |\varphi_1(\xi) - \varphi_2(\xi)|| + D \int_{-\infty}^{\infty} f(y) |\varphi_1(\xi - y) - \varphi_2(\xi - y)| dy + L_{h} |\varphi_1(\xi) - \varphi_2(\xi)|
\]
where \( L_h \) and \( L_f \) are the Lipschitz constants of \( h \) and \( f \) on \([0, K]\), respectively. Noting that \( f(x) \) is an even function and \( \int_{-\infty}^{\infty} f(y) e^{\mu y} dy < \infty \), then we have

\[
\int_{-\infty}^{\infty} f(y) e^{\mu y} dy = \int_{-\infty}^{0} f(y) e^{-\mu y} dy + \int_{0}^{\infty} f(y) e^{\mu y} dy = 2 \int_{-\infty}^{0} f(y) e^{-\mu y} dy < \infty.
\]

Moreover, there is a \( \mu \in (0, \frac{c}{2}) \) such that \( D\mu^2 + c\mu \leq \theta \) and then we have from (H\textsubscript{3}) that

\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y) k(s) e^{\mu(y+cs)} dy ds \leq \int_{0}^{\infty} \int_{-\infty}^{\infty} k(s) e^{(c\mu + D\mu^2)s} \frac{1}{\sqrt{4\pi s}} e^{-\frac{(y-2D\mu^2)^2}{4s}} dy ds
\]

\[
= \int_{0}^{\infty} k(s) e^{(D\mu^2 + c\mu)s} ds
\]

\[
\leq \int_{0}^{\infty} k(s) e^{\beta s} ds < \infty.
\]

Let

\[
\eta := 2L_h + D \int_{-\infty}^{\infty} f(y) e^{\mu y} dy + L_f \int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y) k(s) e^{\mu(y+cs)} dy ds.
\]

Then we have

\[
|Q(\varphi_1) - Q(\varphi_2)| \leq \eta |\varphi_1 - \varphi_2|_\mu.
\]

This implies \( Q : C_{[0,K]}(\mathbb{R}, \mathbb{R}) \to X_\mu \) is continuous with respect to the norm \( |.|_\mu \) in \( X_\mu \).

Next, we shall show that the operator \( I \) is also continuous with respect to the norm \( |.|_\mu \) in \( X_\mu \). For \( \xi > 0 \), we have

\[
|I(\varphi_1)(\xi) - I(\varphi_2)(\xi)| e^{-\mu |\xi|} = \left| \frac{1}{c} e^{\frac{c}{2}\xi} \int_{-\infty}^{\xi} e^{\frac{c}{2}y} Q(\varphi_1)(y) dy - \frac{1}{c} e^{\frac{c}{2}\xi} \int_{-\infty}^{\xi} e^{\frac{c}{2}y} Q(\varphi_2)(y) dy \right| e^{-\mu |\xi|}
\]

\[
\leq \frac{1}{c} e^{\frac{c}{2}\xi} e^{-\mu |\xi|} \int_{-\infty}^{\xi} e^{\frac{c}{2}y} |Q(\varphi_1)(y) - Q(\varphi_2)(y)| dy
\]

\[
\leq \frac{1}{c} e^{\frac{c}{2}\xi} e^{-\mu |\xi|} \left[ \int_{-\infty}^{\xi} e^{\frac{c}{2}y} e^{\mu |\xi|} dy \right] |Q(\varphi_1) - Q(\varphi_2)|_\mu
\]

\[
= \frac{1}{c} e^{-(\xi + \frac{c}{2}\xi)} \left[ \int_{-\infty}^{\xi} e^{\frac{c}{2}y} dy + \int_{0}^{\xi} e^{\frac{c}{2}(y+\mu)|\xi|} dy \right] |Q(\varphi_1) - Q(\varphi_2)|_\mu
\]

\[
= \frac{1}{c} e^{-(\xi + \frac{c}{2}\xi)} \left[ \frac{c}{b - c\mu} + \frac{c}{b + c\mu} (e^{\frac{c}{2}(\xi + \mu)|\xi|} - 1) \right] |Q(\varphi_1) - Q(\varphi_2)|_\mu
\]

\[
\leq \frac{2b}{b^2 - c^2\mu^2} |Q(\varphi_1) - Q(\varphi_2)|_\mu.
\]

For \( \xi \leq 0 \), we have

\[
|I(\varphi_1)(\xi) - I(\varphi_2)(\xi)| e^{-\mu |\xi|} \leq \frac{1}{c} e^{\frac{c}{2}\xi} e^{-\mu |\xi|} \left[ \int_{-\infty}^{\xi} e^{\frac{c}{2}y} e^{\mu |\xi|} dy \right] |Q(\varphi_1) - Q(\varphi_2)|_\mu
\]

\[
= \frac{1}{c} e^{\mu |\xi| - \frac{c}{2}\xi} \int_{-\infty}^{\xi} e^{\frac{c}{2}y} dy |Q(\varphi_1) - Q(\varphi_2)|_\mu
\]

\[
\leq \frac{1}{b - c\mu} |Q(\varphi_1) - Q(\varphi_2)|_\mu.
\]

Therefore, \( I \) is also continuous with respect to the norm \( |.|_\mu \) in \( X_\mu \) since \( Q \) is continuous. The proof is complete. \( \square \)

Now, we state and prove the main result of this section.
Theorem 2.1. Assume that $(H_0)-(H_2)$ hold. If $c^* > 0$ is defined in Lemma 2.1, then for every $c \geq c^*$, Eq. (1.5) has a traveling wavefront $u(t, x) = \varphi(x + ct)$ satisfying $\varphi(-\infty) = 0$, $\varphi(\infty) = K$ and

$$\lim_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_d(c)\xi} = \Lambda,$$

where $\lambda_d(c) > 0$ is the smallest root of the equation $\Delta(\lambda, c) = 0$.

**Proof.** If $c > c^*$, by Lemmas 2.5 and 2.6 and the Schauder’s fixed point theorem, we know that the operator $I$ admits a fixed point $\varphi \in \Omega$ which is a solution of (2.1) and

$$\varphi(\xi) = \frac{1}{c} e^{-\frac{1}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{1}{c}y} Q(\varphi)(y)dy.$$

Since $\varphi(\xi)$ is nondecreasing and bounded, the limit of $\varphi(\xi)$ as $\xi \to \pm \infty$ exists. It is easy to see that $\varphi(-\infty) = 0$. Let $\varphi(\pm \infty) = \omega$. By using L’Hôpital’s rule, we get that

$$\omega = \lim_{\xi \to +\infty} \varphi(\xi) = \frac{1}{c} \lim_{\xi \to +\infty} \int_{-\infty}^{\xi} e^{\frac{1}{c}y} Q(\varphi)(y)dy \frac{e^{\frac{1}{c}y}}{e^{\frac{1}{c}y}} = \frac{1}{c} \lim_{\xi \to +\infty} \frac{Q(\varphi)(\xi)}{\frac{b}{c}} = \omega - \frac{h(\omega) - f(\omega)}{b}.$$

Therefore, $h(\omega) - f(\omega) = 0$. Since $f(u) \neq h(u)$ for all $u \in (0, K)$, then

$$\lim_{\xi \to +\infty} \varphi(\xi) = \omega = K. \quad (2.11)$$

Furthermore, notice that $\Lambda(e^{\lambda_d(c)\xi} - me^{a\lambda_d(c)\xi}) \leq \varphi(\xi) \leq \Lambda(e^{\lambda_d(c)\xi} + me^{a\lambda_d(c)\xi})$, $\xi \in \mathbb{R}$, then we have

$$\lim_{\xi \to -\infty} |\varphi(\xi)e^{-\lambda_d(c)\xi} - \Lambda| = \lim_{\xi \to -\infty} \Lambda me^{(a-1)\lambda_d(c)\xi} = 0,$$

which implies that $\lim_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_d(c)\xi} = \Lambda$.

In the case of $c = c^*$, let $c_k > c^*$ with $\lim_{k \to \infty} c_k = c^*$, then the Eq. (2.1) with $c = c_k$ admits a nondecreasing solution $\varphi_k(\xi) \in \Omega$ satisfying $\varphi_k(-\infty) = 0$, $\varphi_k(\infty) = K$ and $\varphi_k(\xi)e^{-\lambda_d(c_k)\xi} = \Lambda$. Without loss of generality, we may assume that $\varphi_k(0) = \frac{K}{2}$. Since $\varphi_k(\xi)$ is a fixed point of $I$ in $\Omega$ when $c = c_k$, then

$$\varphi_k(\xi) = \frac{1}{c_k} e^{-\frac{1}{c_k}\xi} \int_{-\infty}^{\xi} e^{\frac{1}{c_k}y} Q(\varphi_k)(y)dy, \quad (2.12)$$

and $\varphi_k(\xi)$ is uniformly bounded and equi-continuous on $R$. By applying the Arzéla–Ascoli theorem, there exists a subsequence of functions $\varphi_k(\xi)$ that uniformly converges to $\varphi^*(\xi)$ on any bound subset of $R$ as $n \to \infty$. Clearly, $\varphi^*(\xi)$ is also nondecreasing and $\varphi^*(0) = \frac{K}{2}$. By using the dominated convergence theorem to (2.12), we obtain that

$$\varphi^*(\xi) = \frac{1}{c} e^{-\frac{1}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{1}{c}y} Q(\varphi^*)(y)dy,$$

which implies that $\varphi^*(\xi)$ is a fixed point of the operator $I$ in $\Omega$ when $c = c^*$. Noting that $\lambda_d(c)$ is continuous on $c$ with $c \geq c^*$, similar to the above argument, we can easily show that $\varphi^*(-\infty) = 0$, $\varphi^*(+\infty) = K$ and $\lim_{\xi \to -\infty} \varphi^*(\xi)e^{-\lambda_d(c^*)\xi} = \Lambda$. The proof is complete. \qquad \Box

**Remark 2.2.** We know that any translation $\phi(\xi) = \varphi(\xi + s)$ for a traveling wavefront $\varphi(\xi)$ of (1.5) is still a traveling wavefront of (1.5). Therefore, if we have a traveling wavefront $\varphi(\xi)$ such that $\lim_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_d(c)\xi} = 1$, then, we have a traveling wavefront $\phi(\xi) = \varphi(\xi + s)$ such that

$$\lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_d(c)\xi} = \lim_{\xi \to -\infty} \varphi(\xi + s)e^{-\lambda_d(c)(\xi+s)}e^{\lambda_d(c)s} = \Lambda,$$

where we choose $s$ such that $e^{\lambda_d(c)s} = \Lambda$.

**Remark 2.3.** We say that the $c^*$ is the minimal wave speed in the sense that (1.5) has no traveling wavefront with $c \in (0, c^*)$. In fact, the linearization of (2.1) at zero solution is

$$c\varphi'(\xi) = D([\varphi(\xi) - \psi(\xi)] - h'(0)\varphi(\xi) + f'(0)\int_{-\infty}^{\infty} G(s, y)k(s)\varphi(\xi - y - cs)dy)dy,$$

and the function $\Delta(\lambda, c)$ is obtained by substituting $e^{\frac{1}{c}\xi}$ in (2.13). For $0 < c < c^*$, we know from (iii) of Lemma 2.1 that $\Delta(\lambda, c) < 0$ for any $\lambda \geq 0$. Thus (2.1) can not have a solution $\varphi(\xi)$ that satisfies $\lim_{\xi \to -\infty} \varphi(\xi) = 0$. 

3. Traveling waves without monotonicity

In Section 2, we have made the assumption that \( f \) is nondecreasing on \([0, K]\) in order to obtain the existence of a traveling wavefront. However, we know that few reaction functions satisfy this condition in real models. So, it is natural to ask whether we still can establish the existence of a traveling wave solution when \( f \) is non-monotonic. This is the purpose of this section.

In this section, we still assume that \((H_3)\) holds.

In what follows, we assume that

\[(H_1')\] there exists \( K^* \geq K \), such that \( h(K^*) \geq f(u) \) for all \( u \in (0, K^*) \) and \( h(K^*) > h(u) \) for all \( u \in [0, K^*) \);

\[(H_2')\] \( h'(0) < f'(0) \), \( h(u) < f'(0)u \), \( 0 < f(u) \leq f'(0)u \) and \( h'(0)u \leq h(u) \) for all \( u \in (0, K^*) \).

If \((H_2')\) holds, then \( K_* = \inf \{ u \mid h(u) = \inf_{s \in [0, K^*]} \{ f(s) \mid f(s) \leq h(s) \} \} \) is well defined, \( 0 < K_* \leq K \leq K^* \) and \( f(u) > h(u) \) for all \( u \in (0, K_*) \). We also make the following assumption.

\[(H_3')\] \( K_* < K < K^* \), \( h(u) \) is strictly increasing on \([K_*, K^*]\) and

\[
h(u) < f(u) < 2h(K) - h(u) \quad \text{for} \quad u \in [K_*, K),
\]

\[
h(u) > f(u) > 2h(K) - h(u) \quad \text{for} \quad u \in (K, K^*].
\]

Since \( 0 < f(u) \) for \( u \in (0, K^*) \), then by the definition of \( K_* \), we have that \( h(K_*) > 0 \). Therefore, there is a small \( \epsilon_0 \in (0, K_*) \) such that \( h(K_* - \epsilon) > 0 \) for every \( \epsilon \in [0, \epsilon_0] \). For any \( \epsilon \in [0, \epsilon_0] \), define two continuous functions as follows:

\[
f^*(u) = \begin{cases} \min[f'(0)u, h(K^*)] & \text{for} \quad u \in [0, K^*), \\ \max[h(K^*), f(u)] & \text{for} \quad u \in (K^*, +\infty) \end{cases}
\]

and

\[
f_\epsilon(u) = \begin{cases} \min\{ \inf_{v \in [u, K^*]} f(v), h(K_* - \epsilon) \} & \text{for} \quad u \in [0, K^*), \\ \min[h(K_* - \epsilon), f(u)] & \text{for} \quad u \in (K^*, +\infty). \end{cases}
\]

Then we have the following Lemma. Since the proof is similar to that of Lemma 3.1 in [10] we will omit it.

**Lemma 3.1.** Assume that \((H_0), (H_1'), (H_2')\) hold. Then the following statements are valid:

(i) \( f^* \) and \( f_\epsilon \) are continuous on \([0, +\infty)\) and nondecreasing on \([0, K^*)\);

(ii) \( f_\epsilon(u) \leq f(u) \leq f^*(u) \) for all \( u \geq 0 \);

(iii) \( 0 < f_\epsilon(u) \leq f'(0)u \) and \( 0 < f^*(u) \leq f'(0)u \) for all \( u \in (0, K^*) \);

(iv) \( f^*(0) = h(K^*) - f^*(K^*) = 0 \) and \( f_\epsilon(u) > h(u) \) for all \( u \in (0, K^*) \);

(v) \( f_\epsilon(0) = h(K_* - \epsilon) - f_\epsilon(K_* - \epsilon) = 0 \) and \( f_\epsilon(u) > h(u) \) for all \( u \in (0, K_* - \epsilon) \);

(vi) \( f^*(0) = f'(0) \) and \( \lim_{u \to 0^+} [f^*(0) - f_\epsilon(u)]u^{-\sigma} < +\infty \).

Now, we consider the following two auxiliary reaction diffusion equations:

\[
\frac{\partial u(t, x)}{\partial t} = D[f(u)(t, x) - u(t, x)] - h(u(t, x)) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s, x, y)k(s)f^*(u(t - s, y))dyds
\]

and

\[
\frac{\partial u(t, x)}{\partial t} = D[f(u)(t, x) - u(t, x)] - h(u(t, x)) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s, x, y)k(s)f_\epsilon(u(t - s, y))dyds.
\]

Obviously, their corresponding wave profile equations are

\[
c\varphi'(\xi) - D[f(\varphi(\xi)) - \varphi(\xi)] + h(\varphi(\xi)) - \int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f^*(\varphi(\xi - y - cs))dyds = 0,
\]

and

\[
c\varphi'(\xi) - D[f(\varphi(\xi)) - \varphi(\xi)] + h(\varphi(\xi)) - \int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f_\epsilon(\varphi(\xi - y - cs))dyds = 0,
\]

respectively.

Notice that Lemma 3.1 implies that the functions \( f^* \) and \( f_\epsilon \) satisfy all the conditions of Theorem 2.1. So, as a direct consequence of Theorem 2.1, we have the following result.

**Theorem 3.1.** Assume that \((H_0), (H_1'), (H_2')\), \((H_3)\) hold. If \( c^* > 0 \) is defined in Lemma 2.1, then for every \( c \geq c^* \), Eqs. (3.1) and (3.2) have traveling wavefronts \( \varphi^*(x + ct) \) and \( \varphi_\epsilon(x + ct) \), respectively, satisfying \( \varphi^*(+\infty) = K^*, \varphi^*(-\infty) = 0 \) and
Theorem

In order to show the main conclusion in this article, we need some lemmas. The first one is similar to Theorem 1.1 in [10], and we refer the readers to [10] without proof.

Lemma 3.2. If \( \varphi^*(x + ct) \) and \( \varphi_\epsilon(x + ct) \) are given in Theorem 3.1. Then there exists \( a_0 > 0 \) such that \( \varphi^*(\xi + a_0) > \varphi_\epsilon(\xi) \) for all \( \xi \in \mathbb{R} \).

Define a set
\[
\Omega^* = \left\{ \varphi \in C_{[0,K^*]}(\mathbb{R}, \mathbb{R}) \left| \begin{array}{l}
(1) \varphi_\epsilon(\xi) \leq \varphi(\xi) \leq \varphi^*(\xi + a_0) \quad \text{for all } \xi \in \mathbb{R}; \\
(2) \varphi(\xi_1) - \varphi(\xi_2) \leq \frac{2b^*K^*}{c} |\xi_1 - \xi_2| \quad \text{for } \xi_1, \xi_2 \in \mathbb{R}.
\end{array} \right. \right\}.
\]

Here and in what follows, \( b^* = L_{n^*}^* + D \) and \( L_n^* \) is the Lipschitz constant of \( h \) on \( [0,K^*] \). We can easily show that \( \Omega^* \) is nonempty, convex, and compact in \( X_{\mu^*} \), \( \mu^* \in (0, b^*) \), where
\[
X_{\mu^*} = \{ \psi \in C(\mathbb{R}, \mathbb{R}) \mid \sup_{\xi \in \mathbb{R}} |\psi(\xi)|e^{-\mu^*|\xi|} < \infty \}
\]
with a norm
\[
|\psi|_{\mu^*} = \sup_{\xi \in \mathbb{R}} |\psi(\xi)|e^{-\mu^*|\xi|}.
\]

Define operators \( Q^*, \tilde{Q} \) and \( Q_e : C_{[0,K^*]}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R}) \) by
\[
Q^*(\psi)(\xi) = L_n^*\psi(\xi) + DJ * \varphi(\xi) - h(\varphi(\xi)) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s,y)k(s)f^*(\varphi(\xi - y - cs)) dy ds,
\]
\[
\tilde{Q}(\varphi)(\xi) = L_n^*\varphi(\xi) + DJ * \varphi(\xi) - h(\varphi(\xi)) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s,y)k(s)f(\varphi(\xi - y - cs)) dy ds
\]
and
\[
Q_e(\varphi)(\xi) = L_n^*\varphi(\xi) + DJ * \varphi(\xi) - h(\varphi(\xi)) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(s,y)k(s)f_e(\varphi(\xi - y - cs)) dy ds.
\]

Then by (i) and (ii) of Lemma 3.1 we can easily show the following lemma.

Lemma 3.3. The following statements hold:

(i) \( Q_e(\psi)(\xi) \leq Q_e(\varphi)(\xi) \leq Q^*(\psi)(\xi) \leq Q^*(\varphi)(\xi) \) for any \( \psi, \varphi \in C_{[0,K^*]}(\mathbb{R}, \mathbb{R}) \) with \( 0 \leq \psi(\xi) \leq \varphi(\xi) \);

(ii) \( 0 \leq Q_e(\psi)(\xi) \leq Q_e(\varphi)(\xi) \leq Q^*(\psi)(\xi) \leq Q^*(\varphi)(\xi) \leq b^*K^* \) for any \( \psi \in C_{[0,K^*]}(\mathbb{R}, \mathbb{R}) \).

Define operator \( I^*: C_{[0,K^*]}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R}) \) by
\[
I^*(\varphi)(\xi) = \frac{1}{c} e^{-\frac{c}{r} \xi} \int_{-\infty}^{\xi} e^{\frac{c}{r} \eta} \tilde{Q}(\varphi)(\eta) d\eta.
\]

Clearly, a fixed point of \( I^* \) is a solution of Eq. (2.1). Similar toLemma 2.6, we have the following lemma.

Lemma 3.4. Assume that \( (H_0), (H_1'), (H_2') \) and \( (H_3) \) hold. Then the operator \( I^*: C_{[0,K^*]}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R}) \) is continuous with respect to the norm \( |\cdot|_{\mu^*} \) in \( X_{\mu^*} \).

Lemma 3.5. Assume that \( (H_0), (H_1'), (H_2') \) and \( (H_3) \) hold. Then \( I^*(\Omega^*) \subset \Omega^* \).

Proof. For any \( \varphi \in \Omega^* \), we have \( \varphi_\epsilon(\xi) \leq \varphi^*(\xi + a_0) \leq K^* \) for all \( \xi \in \mathbb{R} \). Notice that \( \varphi_\epsilon(\xi) \) is a solution of (3.4) and \( \varphi_\epsilon(\xi) = \frac{1}{c} e^{-\frac{c}{r} \xi} \int_{-\infty}^{\xi} e^{\frac{c}{r} \eta} Q_e(\varphi_\epsilon)(\eta) dy \), then by Lemma 3.3, we have
\[
I^*(\varphi)(\xi) = \frac{1}{c} e^{-\frac{c}{r} \xi} \int_{-\infty}^{\xi} e^{\frac{c}{r} \eta} \tilde{Q}(\varphi)(\eta) dy \geq \frac{1}{c} e^{-\frac{c}{r} \xi} \int_{-\infty}^{\xi} e^{\frac{c}{r} \eta} Q_e(\varphi)(\eta) dy \geq \frac{1}{c} e^{-\frac{c}{r} \xi} \int_{-\infty}^{\xi} e^{\frac{c}{r} \eta} Q_e(\varphi_\epsilon)(\eta) dy = \varphi_\epsilon(\xi).
\]

Similarly, we can get that \( I^*(\varphi)(\xi) \) is bounded for all \( \xi \in \mathbb{R} \).
For any \( \varphi \in \Omega^* \) and \( u, v \in \mathbb{R} \), assuming that \( u \leq v \). By a similar derivation procedure as for (2.10), we can easily show that

\[
|l^*(\varphi)(u) - l^*(\varphi)(v)| \leq \frac{2b^*K^*}{c}|u - v|.
\]

Therefore, \( l^*(\Omega^*) \subseteq \Omega^* \). The proof is complete. \( \square \)

Now, we are in a position to state and prove our main result.

**Theorem 3.2.** Assume that (H_0), (H_1'), (H_2') and (H_3) hold. Then there exists \( c^* > 0 \) (which is defined in Lemma 2.1), such that for every \( c \geq c^* \), Eq. (1.5) admits a traveling wave solution \( u(t, x) = \varphi(x + ct) \) satisfying \( \varphi(-\infty) = 0 \).

\[
0 < K_* \leq \liminf_{\xi \to -\infty} \varphi(\xi) \leq \limsup_{\xi \to -\infty} \varphi(\xi) \leq K^*
\]

and

\[
A \leq \liminf_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_\alpha(c)\xi} \leq \limsup_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_\alpha(c)\xi} \leq \Lambda e^{a_0\lambda_\alpha(c)},
\]

where \( a_0 \) is defined in Lemma 3.2, \( \lambda_\alpha(c) > 0 \) is the smallest root of the equation \( \triangle(\lambda, c) = 0 \). Furthermore, if (H_3') also holds, then \( \varphi(+\infty) = K \).

**Proof.** By Lemmas 3.4 and 3.5 and Schauder's fixed point theorem, we know that \( l^* \) admits a fixed point \( \varphi(\xi) \in \Omega^* \) which satisfies

\[
\varphi_\epsilon(\xi) \leq \varphi(\xi) \leq \varphi^*(\xi + a_0), \tag{3.9}
\]

and

\[
\varphi(\xi) = \frac{1}{c}e^{-\frac{\nu^*_\epsilon}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\nu^*_\epsilon}{c}y}Q(\varphi)(y)dy. \tag{3.10}
\]

By (3.9), we have

\[
\varphi_\epsilon(\xi)e^{-\lambda_\alpha(c)\xi} \leq \varphi(\xi)e^{-\lambda_\alpha(c)\xi} \leq \varphi^*(\xi + a_0)e^{-\lambda_\alpha(c)\xi}. \tag{3.11}
\]

Then we obtain from Theorem 3.1 that

\[
A \leq \liminf_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_\alpha(c)\xi} \leq \limsup_{\xi \to -\infty} \varphi(\xi)e^{-\lambda_\alpha(c)\xi} \leq \Lambda e^{a_0\lambda_\alpha(c)}
\]

by letting \( \xi \to +\infty \) in (3.11). Since \( \varphi^*(+\infty) = K^*, \varphi^*(-\infty) = 0 \) and \( \varphi(\epsilon) = K_* - \epsilon, \varphi(-\infty) = 0 \), letting \( \xi \to -\infty \) and \( \xi \to +\infty \) in (3.9), respectively, we obtain

\[
\varphi(-\infty) = 0 \quad \text{and} \quad K_* - \epsilon \leq \liminf_{\xi \to +\infty} \varphi(\xi) \leq \limsup_{\xi \to +\infty} \varphi(\xi) \leq K^*. \tag{3.12}
\]

Furthermore, since \( \varphi(\xi) \) is independent of \( \epsilon \), we can get that

\[
K_* \leq \liminf_{\xi \to +\infty} \varphi(\xi) \leq \limsup_{\xi \to +\infty} \varphi(\xi) \leq K^*. \tag{3.13}
\]

Under the assumption (H_3'), if \( \lim_{\xi \to +\infty} \varphi(\xi) \) exists, then by a similar derivation procedure as for (2.11), we can easily get that \( \varphi(+\infty) = K \).

In what follows, we shall prove the existence of \( \lim_{\xi \to +\infty} \varphi(\xi) \). In fact, there are two possibilities for \( \varphi(\xi) \):

(i) \( \varphi(\xi) \) is monotone eventually;

(ii) \( \varphi(\xi) \) is oscillatory eventually.

For case (i), it is clear that \( \lim_{\xi \to +\infty} \varphi(\xi) \) exists, since \( \varphi(\xi) \) is monotone eventually and bounded.

Next, we study case (ii). Set \( a_* = \liminf_{\xi \to +\infty} \varphi(\xi) \) and \( a^* = \limsup_{\xi \to +\infty} \varphi(\xi) \), then \( K_* \leq a_* \leq a^* \leq K \). We claim that \( a_* = a^* \). Otherwise, notice that there exists a sequence \( \{\xi_i\}_{i \in \mathbb{N}} \), with \( \xi_i \to +\infty \) as \( i \to +\infty \), such that \( \varphi(\xi_i) = 0 \) and \( \varphi(\xi_i) \to a^* \) as \( i \to +\infty \). Therefore, by (2.1), we have

\[
h(\varphi(\xi_i)) = D[J(\varphi(\xi_i)) - \varphi(\xi_i)] + \int_{0}^{\infty} \int_{-\infty}^{\infty} G(s, y)k(s)f(\varphi(\xi_i - y - cs)dyds. \tag{3.14}
\]

Since \( \int_{-\infty}^{\infty} |J(y)dy = 1, \int_{-\infty}^{\infty} k(s)ds = 1 \) and \( \int_{-\infty}^{\infty} G(s, y)dy = 1 \) for any \( s \in (0, +\infty) \), for any \( \epsilon > 0 \), there is a large enough number \( N > 0 \), such that

\[
K^* \int_{|y|>N} f(y)dy < \epsilon, \quad h(K^*) \int_{|y|>N} G(s, y)dy < \epsilon, \quad h(K^*) \int_{N}^{\infty} k(s)ds < \epsilon. \tag{3.15}
\]
Therefore, we must have a contradiction. Similarly, we know that
\[ \varphi(\xi) \in [a_\ast - \delta, a^\ast + \delta] \quad \text{for all } \xi \geq N_0. \] (3.17)
Since \( \xi_i \to +\infty \) as \( i \to +\infty \), there exists a positive integer \( i_0 \) such that
\[ \xi_i \geq N_0 + N + cN \quad \text{for all } i \geq i_0. \] (3.18)

Therefore, by (3.14)-(3.18), for \( i \geq i_0 \), we have
\[
h(\varphi(\xi_i)) = D \left[ \int_{|y|<N} f(y)\varphi(\xi_i - y)dy + \int_{|y|\geq N} f(y)\varphi(\xi_i - y)dy - \varphi(\xi_i) \right] \\
+ \int_0^N \int_{|y|\geq N} G(s, y)k(s)f(\varphi(\xi_i - y - cs))dyds + \int_0^N \int_{|y|<N} G(s, y)k(s)f(\varphi(\xi_i - y - cs))dyds \\
+ \int_N^\infty \int_{|y|\geq N} G(s, y)k(s)f(\varphi(\xi_i - y - cs))dyds + \int_N^\infty \int_{|y|<N} G(s, y)k(s)f(\varphi(\xi_i - y - cs))dyds \\
\leq D[\varphi + (a^\ast + \delta) - \varphi(\xi_i)] + \max(f(u)|u \in [a_\ast - \delta, a^\ast + \delta]) + \varepsilon + \varepsilon + \varepsilon \\
\leq D[\varphi + (a^\ast + \delta) - \varphi(\xi_i)] + \max(f(u)|u \in [a_\ast, a^\ast]) + 4\varepsilon.
\]

Letting \( i \to +\infty \) in the last inequality, we have
\[
h(a^\ast) \leq D[\varphi + (a^\ast + \delta) - a^\ast] + \max(f(u)|u \in [a_\ast, a^\ast]) + 4\varepsilon \\
\leq (2D + 4)\varepsilon + \max(f(u)|u \in [a_\ast, a^\ast]).
\]

It follows
\[
h(a^\ast) \leq \max(f(u)|u \in [a_\ast, a^\ast]) \tag{3.19}
\]
as \( \varepsilon \to 0^+ \).

Similar to the above argument, we can also get that
\[
h(a_\ast) \geq \min(f(u)|u \in [a_\ast, a^\ast]). \tag{3.20}
\]

Therefore, we have \( a_\ast < K < a^\ast \). In fact, if \( a_\ast < a^\ast \leq K \), by (H3'), we get that \( h(a_\ast) < \min(f(u)|u \in [a_\ast, a^\ast]) \), which is a contradiction. Similarly, we know that \( K \leq a_\ast < a^\ast \) is impossible. Since \( f \) is continuous, there exists \( u_1, u_2 \in [a_\ast, a^\ast] \) such that
\[
f(u_1) = \max(f(u)|u \in [a_\ast, a^\ast]) \quad \text{and} \quad f(u_2) = \min(f(u)|u \in [a_\ast, a^\ast]). \tag{3.21}
\]

We assert that \( u_1 < K < u_2 \). In fact, \( a_\ast \leq u_2 \leq K \) is impossible. Otherwise, if \( u_2 = K \), then \( h(a_\ast) \geq \min(f(u)|u \in [a_\ast, a^\ast]) = f(u_2) = f(K) = h(K) \), this is a contradiction. Similarly, if \( u_2 = a_\ast \), then \( h(a_\ast) \geq \min(f(u)|u \in [a_\ast, a^\ast]) = f(u_2) = f(a_\ast) \), which is a contradiction according to (H3'). If \( a_\ast < u_2 < K \), then \( h(a_\ast) \geq \min(f(u)|u \in [a_\ast, a^\ast]) = f(u_2) > h(u_2) > h(a_\ast) \), which is impossible. Similarly, we can show that \( K \leq a_\ast \leq a^\ast \) is also impossible.

Next, we prove that \( a_\ast = u_1 \) and \( a^\ast = u_2 \). Otherwise, by (3.19)-(3.21) and (H3') we have
\[
h(a^\ast) - h(a_\ast) \leq f(u_1) - f(u_2) \\
= 2h(K) - h(u_1) - [2h(K) - h(u_2)] \\
= h(u_2) - h(u_1) \\
< h(a^\ast) - h(a_\ast),
\]
which is impossible. Therefore, \( a_\ast = u_1, a^\ast = u_2 \). But we also have the following contradiction:
\[
h(a^\ast) - h(a_\ast) \leq f(a_\ast) - f(a^\ast) \\
< 2h(K) - h(a_\ast) - [2h(K) - h(a^\ast)] \\
= h(a^\ast) - h(a_\ast).
\]

Therefore, we must have \( a_\ast = a^\ast \), which implies that \( \lim_{\xi \to +\infty} \varphi(\xi) \) exist. The proof is complete. \( \square \)
References


