An introduction to the multiname modelling in credit risk

Aurélien Alfonsi

Institut für Mathematik, MA 7-4, TU Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany.
e-mail : `alfonsi@cermics.enpc.fr`

April 25, 2007

Abstract

This paper is intended to be an introductory survey on credit risk models for multiname products. We first present the intensity models for single defaults which leads up to the copula model. We hint at its limits, especially concerning the dependence dynamics between defaults that it induces. As an alternative, we consider loss models and present several reduced form models that are designed to have known distributions through their Fourier transform. Last, we focus on two forward loss models whose principle is to model directly the future loss distributions rather than the loss itself. This simultaneous presentation makes appear similarities and differences between them.

Keywords: Credit Risk, Copula, Reduced-form Model, Forward Loss Model, CDO.

Introduction

Let us begin with a short introduction to the credit risk market. Credit derivatives are financial products that bring to their owner, under some pre-specified conditions, a cash protection when default events occur. A default event is either the bankruptcy of a financial entity or its inability to reimburse its debt. In general, we divide these derivatives into two categories. The first category consists in the single-name credit derivatives that are products that deal with only one default. The most widespread product that belongs to that category is the Credit Default Swap (CDS for short). Let us describe here briefly its mechanism. The owner of a CDS will receive a cash protection when a firm defaults, if this happens before the date of maturity. This maturity and the default entity are of course defined at the beginning of the CDS. In exchange, he will pay to the protection seller regular payments until the default or if it does not happen until the maturity. From a practical point of view, these products are well adapted if one has a strong exposure to an identified bankruptcy. However in general, financial actors have diversified investments and are naturally exposed to many defaults. They would prefer to buy directly a product that takes into consideration their whole aggregated default exposure, instead of buying many single-name products. This motivates the other category of products, the multiname credit derivatives whose payment structure depends on many defaults. The most representative products of this category are the Collateralized Debt Obligations (CDO). For the owner, it mainly consists of being reimbursed of its loss generated by bankruptcies when its total loss due to defaults is between two prespecified values. In exchange, he has to give regular payments that are proportional to the maximal cash protection that he could expect in the future from the CDO contract. Multiname products such as CDOs are thus financial tools that allow to be hedged according to a certain level of loss, without specifying a priori which entities will default. Therefore, they are really fitted to control one's exposure to defaults. All these products were originally dealt over the counter. This is still the case for bespoke products. Nonetheless, some of them such as CDS on large companies and CDO on type loss have been standardized in a recent past and are traded on a market. We will give below a precise description of these CDS and CDO.

Now, let us think about the two categories mentioned above in term of information. It is well-known that market prices reflect some information on the economy, so what kind of information could we grasp from the prices of single-name and multiname credit derivatives products? Single-name prices will inform on the default probability of one firm in the future and, through this way, on the health of that firm. Of course, the more prices we have on a default, the more we know about its distribution and its fluctuation. Multiname prices clearly include too that information for each underlying default. But they also bring information on the dependence between underlying defaults. This is related to another risk called credit risk contagion, that is the risk of interdependent failures. Though being very simple, it is interesting to keep in mind this interpretation of prices through information when modelling. This will indeed shed light on important issues for calibration and pricing. For example, it helps to understand which product can be used to calibrate a parameter or what are the products that can be priced within one model.

Now we would like to hint at the specificities of the credit risk market and especially their implication when modelling. First of all, it is intrinsically different from the other markets because the underlyings are unpredictable discrete events while usually underlyings are nonnegative paths. Obviously, the prices observed on the credit risk market are nonnegative paths that could each be modelled as a diffusion with jumps as on the other markets. Doing this, it would be however hard to catch the dependence between the prices of two products taking into account the same underlying default. Indeed, payoff structures are often directly constituted from the discrete default events. This is why modelling directly default events is an important issue to get consistency between prices. Another important feature of the credit risk market is that it is a rather young market. Therefore, there are still few products that are quoted on the market. It has to be taken into consideration because it means that we cannot fit a model with too many refinements. A last important feature is the similitude between the credit derivatives payoff and the fixed-income products. This is not really a blind chance because they are often complementary such as bonds and CDS (for a more detailed discussion on this, see Brigo and Mercurio [4]). As a consequence, many approaches coming from the fixed-income world have been adapted to credit risk as we will see here.

The goal of this paper is to give an overview of the modelling in credit risk when multiname products are considered. In a first part, we introduce the copula model that arises naturally once we start from intensity models for the single-name defaults. Then, we point out the drawbacks of this approach and motivate the introduction of loss models. In the second part we focus on loss models whose dynamics has a reduced-form. In the third part, we present two forward-loss models that, though being close in their conception, present nonetheless some differences. We hope that our parallel description will enlighten this and thus help to understand their mechanism. Last, we briefly point out the limits of the loss models presented here and state some main challenges that remain to address.

Through all these models, we will consider the same probabilistic background. We will indeed assume that all the market events are described by the probabilistic filtered space $(\Omega, (\mathcal{G}_t)_{t\geq 0}, \mathcal{G}, \mathbb{P})$. Here, as usual, $\mathcal{G}_t$ is the σ-field that describes all the future events that can occur before time t and $\mathcal{G}$ describes all the future events. We also assume that $\mathbb{P}$ is a martingale measure. It means that prices of future cash flows are $(\mathcal{G}_t)$-martingales under $\mathbb{P}$, and practically that prices can be computed through expectations. We will consider $m \geq 1$ defaultable entities and name $\tau^1, \dots, \tau^m$ their time of default. We assume they are positive $(\mathcal{G}_t)$-stopping times. We will suppose also that it exists a subfiltration $(\mathcal{F}_t)_{t\geq 0}$ that describes all default-risk free assets such as default-risk free zero-coupon bonds that pay with certainty a nominal value at a fixed maturity. These are products that give with certainty a unit payment at a fixed maturity. Let us introduce also some other standard notations. For $j \in \{1, \dots, m\}$, let us denote $(\mathcal{H}^j_t)_{t\geq 0}$ the filtration engendered by $(\tau^j \wedge t)_{t\geq 0}$ and $(\mathcal{F}^j_t)_{t\geq 0} = (\mathcal{F}_t \vee \mathcal{H}^j_t)_{t\geq 0}$. We also use the notation $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, $\mathcal{F}^j = \bigvee_{t\geq 0} \mathcal{F}^j_t$ and so on. Last, we will assume that

$$\mathcal{G}_t = \bigvee_{j=1}^{m} \mathcal{F}^j_t,$$

i.e. the market events only come from the default-free assets and the default times $\tau^1, \dots, \tau^m$.

1 The copula model

In that section, we will present the copula approach for multiname modelling. We will also introduce CDS and CDO payoffs and explain how in practice this model is fitted to market data. Let us first begin with the single default products.

1.1 Default intensity models and calibration to CDS

Intensity models (or reduced-form default models) assume that the default is explained by an exogenous process that is called "intensity of default".

Definition 1.1. *Let us consider $\tau : \Omega \to \mathbb{R}_+$ a positive random variable. We will say that τ follows a default intensity model with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if it exists a nonnegative càdlàg process $(\lambda_t, t \geq 0)$ that is $(\mathcal{F}_t)$-adapted and a random variable ξ that is independent from $\mathcal{F}$ and follows an exponential law of parameter 1 such that:*

$$\tau = \inf\{t \geq 0, \int_0^t \lambda_s ds \geq \xi\}.$$

The process $(\lambda_t, t \geq 0)$ is called the extended intensity of the default τ.

Usually, we assume moreover that the extended intensity satisfy the two following properties:

$$\forall t \geq 0, \mathbb{P}(\Lambda(t) < +\infty) = 1 \text{ and } \mathbb{P}(\lim_{t \to +\infty} \Lambda(t) = +\infty) = 1,$$

where $\Lambda(t) = \int_0^t \lambda_s ds$. They are in general satisfied by most of the intensity models proposed in the literature. If the first one did not hold, this would mean since $\{\Lambda(t) = +\infty\} \subset \{\tau \leq t\}$ that $\mathcal{F}_t$ can provide information on the default time for some $t > 0$. This is in contradiction with the modelling assumption that $(\mathcal{F}_t)_{t\geq 0}$ describes only the default-risk free world, and we prefer in general to avoid it. The second property just means that every entity will collapse someday.

In all the Section 1, we will assume that $\tau^1, \dots, \tau^m$ follow an intensity model with respect to $(\mathcal{F}_t)_{t\geq 0}$ and we will denote $\lambda_t^1, \dots, \lambda_t^m$ their respective intensities and $\xi^1, \dots, \xi^m$ the associated exponential random variables.

A Credit Default Swap is a product that brings a protection against the default of a firm during a fixed period $[T_0, T_n]$. The protection buyer must in exchange at some pre-defined maturities $T_1, \dots, T_n$ pay an amount that is usually proportional to the time elapsed since the last maturity. We denote by $\mathcal{T} = \{T_0, T_1, \dots, T_n\}$ the maturities that define a CDS contract, $\alpha_i = T_i - T_{i-1}$ and $D(t,T)$ the discount factor between t and T $(t < T)$ that is assumed $\mathcal{F}_T$-measurable. For $j \in \{1, \dots, m\}$, we also name $\text{L}\textsc{gd}^j \in [0,1]$ the loss fraction engendered by the default τ^j and assume it deterministic. Within this framework, the payoff of a CDS on τ^j at time T_0 reads (written for the protection seller):

$$\left(\sum_{i=1}^n D(T_0,T_i)\alpha_i R \mathbf{1}_{\tau^j > T_i}\right) + D(T_0,\tau^j)R(\tau^j - T_{\beta(\tau^j)-1})\mathbf{1}_{\tau^j \leq T_n} - \text{L}\textsc{gd}^j D(T_0,\tau^j)\mathbf{1}_{\tau^j \leq T_n}$$

where $\beta(t)$ is the index such that $t \in (T_{\beta(t)-1}, T_{\beta(t)}]$. Here, R denotes the CDS rate. In practice it is fixed such that the contract is fair for the protection buyer and the protection seller. The payments are usually made quarterly and the standard final maturities are one, three, five, seven and ten years. Here, we have neglected the counterparty risk, that is the risk that one of the two protagonists may default during the contract.

Let us assume now that $T_0 = 0$ (the current time) and that we have at our disposal market prices of the fair CDS rate on each default τ^j. Then, we can use these data to calibrate the intensities. More precisely, under the intensity model, the fair CDS rate is given by (see for example [3])

$$\begin{aligned} R^j(\mathcal{T}) &= \frac{\text{LGD}^j \mathbb{E}[D(0,\tau^j)\mathbf{1}_{\tau^j \le T_n}]}{\mathbb{E}[\sum_{i=1}^n D(0,T_i)\alpha_i \mathbf{1}_{\tau^j > T_i} + D(0,\tau^j)(\tau^j - T_{\beta(\tau^j)-1})\mathbf{1}_{\tau^j \le T_n}]} \\ &= \frac{\text{LGD}^j \mathbb{E}[\int_0^{T_n} D(0,t)\exp(-\Lambda^j(t))\lambda_t^j dt]}{\sum_{i=1}^n \mathbb{E}[D(0,T_i)\exp(-\Lambda^j(T_i))]\alpha_i + \mathbb{E}[\int_0^{T_n} D(0,t)(t - T_{\beta(t)-1})\lambda_t^j dt]} \end{aligned} \tag{1}$$

where $\Lambda^j(t) = \int_0^t \lambda_s^j ds$. Therefore, specifying a parameterized model for the extended intensities λ^j and a model for the discount factors, we are able to fit the prices given by the market. This is the case for example if one assumes that intensities are deterministic and piecewise linear or follow the SSRD model described in [3]. Thus, in a general manner, intensity models are often thought to be tractable when pricing the single-name products.

1.2 Dependence between defaults and calibration to CDO

The thresholds $\xi^1, \dots, \xi^m$ that trigger the defaults each follow an exponential law of parameter 1. Therefore, the random variables $\exp(-\xi^1), \dots, \exp(-\xi^m)$ follow an uniform distribution on $[0,1]$.

Definition 1.2. *Let $m \ge 2$. The cumulative distribution function of a random variable vector $(U_1, \dots, U_m)$*

$$(u_1, \dots, u_m) \in [0,1]^m \mapsto \mathbb{P}(U_1 \le u_1, \dots, U_m \le u_m)$$

each coordinate of which follows an uniform distribution on $[0,1]$ is called a (m-dimensional) copula.

Since we focus on credit risk issues, we will not go beyond this copula definition and we refer to the paper of Embrechts, Lindskog and Mc Neil [8] for a nice introduction to copula functions and dependence modelling. For a further reading on this topic, we mention the book of Nelsen [15] and hint at a recent development made by Alfonsi and Brigo [3] on a tractable copula family.

We will denote C the copula that is the cumulative distribution function of the vector $(\exp(-\xi^1), \dots, \exp(-\xi^m))$. Let us remark that for $t_1, \dots, t_m \ge 0$, we have

$$\begin{aligned} \mathbb{P}(\tau^1 \ge t_1, \dots, \tau^m \ge t_m | \mathcal{F}) &= \mathbb{P}(e^{-\xi^1} \le \exp(-\Lambda^1(t_1)), \dots, e^{-\xi^m} \le \exp(-\Lambda^m(t_m)) | \mathcal{F}) \\ &= C(\exp(-\Lambda^1(t_1)), \dots, \exp(-\Lambda^m(t_m))). \end{aligned}$$

Now, let us turn to the Collateralized Debt Obligation and describe a synthetic payoff. To understand its practical interest in terms of hedging, we suppose that we have sold protections against defaults $\tau^1, \dots, \tau^m$ through CDS contracts with the same schedule of payments $\mathcal{T} = \{T_0, \dots, T_n\}$. We assume that the number of CDS sold is the same for each underlying default, and our loss due to the bankruptcies at time t is thus proportional to the loss process

$$L(t) = \frac{1}{m}\sum_{j=1}^{m} \mathrm{L{\scriptstyle GD}}^j \mathbf{1}_{\tau^j \le t}. \tag{2}$$

Here, we have normalized the loss process $(L(t), t \ge 0)$ in order to have $L(t) = 1$ in the worst case (i.e. $\mathrm{L{\scriptstyle GD}}^j = 1$ for all j and every default before t). Let us assume for example that we do not want to undergo a loss that exceeds 3% of the largest loss that we may have. To do so, we conclude a contract with another firm that will reimburse the loss undergone above 3% (i.e. when $L(t) \ge 0.03$) in exchange of regular payments. Such contract is called the $[3\%, 100\%]$ tranche CDO on the basket loss process $(L(t), t \ge 0)$. Most often in practice, tranches are finer (for example $[3\%, 6\%]$, $[6\%, 9\%]$,...) and bring protection when the relative loss $(L(t), t \ge 0)$ is within two values. Let us now describe precisely the payoff structure of a CDO tranche $[a, b]$ for $0 \le a < b \le 1$. We introduce for that scope the function

$$\forall x \in [0,1],\ H_a^b(x) = \frac{(x-a)^+ - (x-b)^+}{b-a}.$$

At time T_0, the cash flow value of a tranche CDO $[a, b]$ with payment schedule $\mathcal{T}$ is for the protection seller:

$$\sum_{i=1}^{n} \alpha_i R[1 - H_a^b(L(T_i))] D(T_0, T_i) + \int_{T_0}^{T_n} R(t - T_{\beta(t)-1}) D(T_0, t) d[H_a^b(L(t))] \\ - \int_{T_0}^{T_n} D(T_0, t) d[H_a^b(L(t))]$$

Here, R is called the rate of the CDO tranche $[a, b]$. Let us observe here that the regular payments are, as for the CDS, proportional to the elapsed time since the last maturity. For the CDO tranche, they are also proportional to the maximal loss that remains to cover within the tranche. For few years, standard CDO tranches have been quoted on the market. As an example for the iTraxx index, there are $m = 125$ names and the quoted tranches are $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$ and $[12\%, 22\%]$. The final maturities are three, five, seven and ten years and the payments are quarterly. The riskiest tranche ($[0\%, 3\%]$ for the iTraxx) is usually called equity tranche, the intermediate ones are called mezzanine and the less risky ($[12\%, 22\%]$ for the iTraxx) is called senior tranche. Except for the equity tranche (for which an upfront payment value is quoted), the market quotes the rate $R_a^b(\mathcal{T})$ that makes the contract fair.

Now, we turn to the valuation of that rate and assume that $T_0 = 0$, which means that the tranche starts at the current time. Under our assumption, the fair rate value is given by

$$R_a^b(\mathcal{T}) = \frac{\mathbb{E}\left[\sum_{i=1}^{n} \alpha_i [1 - H_a^b(L(T_i))] D(0, T_i) + \int_0^{T_n} (t - T_{\beta(t)-1}) D(0, t) d[H_a^b(L(t))]\right]}{\mathbb{E}\left[\int_0^{T_n} D(0, t) d[H_a^b(L(t))]\right]}. \tag{3}$$

Let us explain how we can compute this price if we know the intensities dynamics $(\lambda_t^j, t \geq 0)$ and the copula C. We first calculate the above conditional expectations given the σ-field $\mathcal{F}$. An integration by parts on the integrals gives, for example on the denominator $\int_0^{T_n} D(0,t)d[H_a^b(L(t))] = D(0,T_n)H_a^b(L(T_n)) - \int_0^{T_n} H_a^b(L(t))d[D(0,t)]$ and thus

$$\mathbb{E}\left[\int_0^{T_n} D(0,t)d[H_a^b(L(t))]\middle|\mathcal{F}\right] = D(0,T_n)\mathbb{E}[H_a^b(L(T_n))|\mathcal{F}] - \int_0^{T_n} \mathbb{E}[H_a^b(L(t))|\mathcal{F}]d[D(0,t)]$$

since $D(0,t)$ is $\mathcal{F}$-measurable. Therefore, it is sufficient to know for any t the law of $L(t)$ conditioned to $\mathcal{F}$ to calculate the expectations that define each tranche. This is the case since we have

$$\forall x \in [0,1],\ \mathbb{P}(L(t) \leq x|\mathcal{F}) = \sum_{\substack{J \subset \{1,\dots,m\} \\ s.t. \frac{1}{m}\sum_{k \in J} \mathrm{LGD}^k \leq x}} \mathbb{P}(\forall j \in J, \tau^j \leq t, \forall j \notin J, \tau^j > t|\mathcal{F}) \quad (4)$$

and (sieve formula)

$$\begin{aligned}\mathbb{P}(\forall j \in J, \tau^j \leq t, \forall j \notin J, \tau^j > t|\mathcal{F}) &= \sum_{K \subset J} (-1)^{\#J-\#K}\mathbb{P}(\forall j \in K, \tau^j > 0, \forall j \notin K, \tau^j > t|\mathcal{F}) \\ &= \sum_{K \subset J} (-1)^{\#J-\#K} C(e^{-\mathbf{1}_{1 \notin K}\Lambda^1(t)}, \dots, e^{-\mathbf{1}_{m \notin K}\Lambda^m(t)}).\end{aligned}$$

Then, we can calculate the expectation $\mathbb{E}\left[\int_0^{T_n} D(0,t)d[H_a^b(L(t))]\right]$ through a Monte-Carlo method, and in the same manner the other expectations can be computed to deduce the fair price $R_a^b(\mathcal{T})$. However, for sake of simplicity, one assumes often deterministic extended intensities and discount factors when computing CDO prices to avoid this last step that might be time-consuming. This is not irrelevant because the main risk that determines the price of the CDO is the interdependence of the defaults. Fluctuations of default intensities and risk-free interest rates play a minor role.

The formula above implicitly requires that we are able to compute the value of a copula function although this is not always a trivial issue. For the copulas called factor copulas (such as the Gaussian copulas), Laurent and Gregory [13] and Hull and White [11] have proposed two rather efficient numerical integration methods. Both methods first compute the law of $L(t)$ conditioned to the factor(s) that define the copula C and then use a numerical integration with respect to the factor(s). This conditioned law is obtained through its Fourier transform in [13] and is approximated directly using a so-called "bucket method" in [11]. Let us finally give standard special cases for which calculations above are much simpler. First, if one assumes that the loss given defaults are identical ($\mathrm{LGD}^j = \mathrm{LGD}$), the loss is proportional to the number of defaults (and the sets J in (4) are simply $\#J \leq mx/\mathrm{LGD}$). Moreover, if the copula is symmetrical in the following sense: $C(u_1, \dots, u_m) = C(u_{\sigma(1)}, \dots, u_{\sigma(m)})$ for any permutation σ, and if the defaults have the same extended intensity $(\lambda_t, t \geq 0)$, we have a simpler formula to characterize the loss distribution:

$$\forall j \in \{0,\dots,m\},\ \mathbb{P}(L(t) = \frac{j}{m}\mathrm{LGD}|\mathcal{F}) = \binom{m}{j}\sum_{k=0}^{j}\binom{j}{k}(-1)^{j-k}C(\underbrace{1,\dots,1}_{k \text{ times}}, e^{-\Lambda(t)}, \dots, e^{-\Lambda(t)}).$$

1.3 Copula model and default intensity jumps

Here we would like to hint at the evolution of the rate of default when defaults occur. To that purpose, we need to introduce another definition of the intensity than the extended intensity introduced before.

Definition 1.3. *Let us consider $(\mathcal{G}'_t)_{t\geq 0}$ a subfiltration of $(\mathcal{G}_t)_{t\geq 0}$ and τ a $(\mathcal{G}'_t)$-stopping time. We will say that $(\lambda_t^{\tau|\mathcal{G}'}, t \geq 0)$ is the intensity of τ with respect to the filtration $(\mathcal{G}'_t)_{t\geq 0}$ if the two following properties hold:*

1. *$(\lambda_t^{\tau|\mathcal{G}'}, t \geq 0)$ is a $(\mathcal{G}'_t)$-adapted càdlàg process,*
2. *$\mathbf{1}_{\tau\leq t} - \int_0^t \lambda_s^{\tau|\mathcal{G}'} ds$ is a $(\mathcal{G}'_t)$-martingale.*

The intuitive meaning of the intensity is the following. Let us assume that we observe some evolution and have access to the information modelled by $(\mathcal{G}'_t)_{t\geq 0}$. From the martingale property, one has

$$\mathbb{P}(\tau \in (t, t+dt]|\mathcal{G}'_t) = \mathbb{E}\left[\int_t^{t+dt} \lambda_s^{\tau|\mathcal{G}'} ds \Big| \mathcal{G}'_t\right] \underset{dt\to 0}{\sim} \lambda_t^{\tau|\mathcal{G}'} dt$$

and thus $\lambda_t^{\tau|\mathcal{G}'}$ is the rate of probability that τ occurs in the immediate future knowing $\mathcal{G}'_t$. In particular, $\lambda_t^{\tau|\mathcal{G}'} = 0$ on $\{\tau \leq t\}$ and for $(\mathcal{G}'_t)_{t\geq 0} = (\mathcal{F}_t)_{t\geq 0} \vee \sigma(\tau \wedge t, t \geq 0)$, $\lambda_t^{\tau|\mathcal{G}'}$ coincides with the extended intensity λ_t for $t < \tau$.

Let us turn now to our model for the defaults $\tau^1, \dots, \tau^m$. It is not hard to see that for any $j \in \{1, \dots, m\}$, $\lambda_t^{\tau^j|\mathcal{F}^j} = \lambda_t^j \mathbf{1}_{\tau^j > t}$ is the intensity of τ^j with respect to $(\mathcal{F}_t^j)_{t\geq 0}$. It is equal to the extended intensity until the default occurs and then jumps to 0. This is the rate of default of someone that would have access only to the information modelled by $(\mathcal{F}_t^j)_{t\geq 0}$. However, one has in general access to the full information $\mathcal{G}_t = \bigvee_{j=1}^{m} \mathcal{F}_t^j$ and would like to know the intensity with respect to it. We have

$$\begin{aligned}
\lambda_t^{\tau^j|\mathcal{G}} &= \lim_{dt\to 0} \frac{\mathbb{P}(\tau^j \in (t, t+dt]|\mathcal{G}_t)}{dt} \\
&= \sum_{K\subset\{1,\dots,m\}-\{j\}} \mathbf{1}_{Def_t(K)} \lim_{dt\to 0} \frac{1}{dt} \mathbb{P}(\tau^j \in (t, t+dt]|\mathcal{F}_t \vee Def_t(K) \vee \forall k \in K, \tau^k)
\end{aligned}$$

where $Def_t(K) := \{\forall k \in K, \tau^k \leq t, \forall k \notin K, \tau^k > t\}$. Let us consider for example $j = 1$ and suppose that no default has occurred at time t ($K = \emptyset$). Then we have $\mathbb{P}(\tau^1 \in (t, t+dt]|\mathcal{F}_t \vee Def_t(\emptyset)) = \frac{\mathbb{E}(C(e^{-\Lambda^1(t)}, \dots, e^{-\Lambda^m(t)}) - C(e^{-\Lambda^1(t+dt)}, e^{-\Lambda^2(t)}, \dots, e^{-\Lambda^m(t)})|\mathcal{F}_t)}{C(e^{-\Lambda^1(t)}, \dots, e^{-\Lambda^m(t)})}$ and therefore

$$\lambda_t^{\tau^1|\mathcal{G}} = \lambda_t^1 e^{-\Lambda^1(t)} \frac{\partial_1 C(e^{-\Lambda^1(t)}, \dots, e^{-\Lambda^m(t)})}{C(e^{-\Lambda^1(t)}, \dots, e^{-\Lambda^m(t)})} \text{ on } Def_t(\emptyset).$$

Let us suppose now that τ^m is the only one default that has occurred before time t at time t_m ($K = \{m\}$). We have:

$\mathbb{P}(\tau^1 \in (t, t+dt] | \mathcal{F}_t \vee Def_t(\{m\}), \tau^m = t_m) = 1 - \mathbb{P}(\tau^1 > t + dt | \mathcal{F}_t \vee Def_t(\{m\}), \tau^m = t_m)$ $= \frac{\partial_m C(e^{-\Lambda^1(t)}, \ldots, e^{-\Lambda^{m-1}(t)}, e^{-\Lambda^m(t_m)}) - \partial_m C(e^{-\Lambda^1(t+dt)}, \ldots, e^{-\Lambda^{m-1}(t)}, e^{-\Lambda^m(t_m)})}{\partial_m C(e^{-\Lambda^1(t)}, \ldots, e^{-\Lambda^{m-1}(t)}, e^{-\Lambda^m(t_m)})}$ and therefore

$$\lambda_t^{\tau^1|\mathcal{G}} = \lambda_t^1 e^{-\Lambda^1(t)} \frac{\partial_1 \partial_m C(e^{-\Lambda^1(t)}, \ldots, e^{-\Lambda^{m-1}(t)}, e^{-\Lambda^m(t_m)})}{\partial_m C(e^{-\Lambda^1(t)}, \ldots, e^{-\Lambda^{m-1}(t)}, e^{-\Lambda^m(t_m)})} \text{ on } Def_t(\{m\}).$$

We can through this way calculate explicitly the intensity $\lambda_t^{\tau^1|\mathcal{G}}$ on the event $Def_t(K)$ for each K, if one assumes that the copula function is regular enough. We refer to the paper of Schönbucher and Schubert [17] for further details.

Let us make now some comments on these calculations. First, we have seen that even without default, the current intensity of each default time depends on the copula function. This means in practice that even single-name products such as a CDS that starts at a future date contain some risk of interdependence of defaults. Secondly, the formulas that define the intensity $\lambda_t^{\tau^1|\mathcal{G}}$ are not the same on $Def_t(\emptyset)$ and $Def_t(\{m\})$: there is therefore in general a jump of this intensity when the first default τ^m occurs. In a general manner, $\lambda_t^{\tau^1|\mathcal{G}}$ jumps at each default time until it vanishes when τ^1 occurs itself. Intuitively, jumps are all the more significant as the dependence between the default that happens and τ^1 is strong.

1.4 Strength and weakness of the copula model

The main strength of the copula approach is undoubtedly its ability to catch the information from the CDS market. Let us suppose that for each default that defines the loss process $L(t)$, one or more CDS are traded. We can then easily calibrate a deterministic intensity that fits exactly the prices observed (see [3] for details). If one assumes that the copula that describes the dependence between the defaults belong to a parametrized family (e.g. the Gaussian copula family), one can then fit its parameters from the CDO market prices since we are able to compute these prices in a quite efficient manner. With this calibrated model, we can then price every credit risk products that depend on the defaults that define the loss $L(t)$. Thus, one has a model that is at first sight coherent with the single risk and multiname risk markets and allow to price a rather broad set of credit risk products.

However, the situation is unfortunately not so idyllic. First of all we have in practice at most five CDO prices for each tranche to calibrate the copula function. It is then likely to have a copula family with few parameters, otherwise our calibration would not have a deep meaning. For a family parametrized by one real number, each tranche price corresponds “ideally” to a unique parameter. In general, this property does not hold, that is why in the one-factor Gaussian copula model, the base correlation mechanism has been introduced to retrieve this likely property. However in that case, the fitted parameter (the base correlation) depends in practice strongly on the tranche and the maturity. This skew observed shows that there is no one-factor Gaussian copula matching the prices on the market. The base correlation being a rather intuitive parametrization, the skew is used in practice to understand how the market anticipates the future losses. Many works have been done to identify a copula family or an extension to the copula model that depends on few parameters and could approximate well the tranche prices observed on the market. We mention here the work of Burtschell, Gregory and Laurent [7] and, less directly connected, the paper of Hull and White [12].

Nonetheless, the copula model presents some weaknesses that are deeper as only the base correlation skew and the calibration problem. The main one is that the copula model freezes the evolution of the dependence between the defaults. To illustrate this, let us assume that the defaults indeed follow the copula model with stochastic default intensities from time 0. Let us suppose for example that no default has been observed up to time t. Using the lack of memory property, the random variables $\xi^j - \Lambda^j(t)$ follow an exponential law of parameter 1 and we name C_t the copula defined as the cumulative distribution function of $(e^{-(\xi^1-\Lambda^1(t))}, \dots, e^{-(\xi^m-\Lambda^m(t))})$. Then, we have for $t_1, \dots, t_m > t$:

$$\begin{aligned}\mathbb{P}(\tau^1 > t_1, \dots, \tau^m > t_m | \mathcal{F} \vee \{\forall j, \tau^j > t\}) &= \frac{C(\exp(-\Lambda^1(t_1)), \dots, \exp(-\Lambda^m(t_m)))}{C(\exp(-\Lambda^1(t)), \dots, \exp(-\Lambda^m(t)))} \\ &= C_t(\exp(-(\Lambda^1(t_1) - \Lambda^1(t))), \dots, \exp(-(\Lambda^m(t_m) - \Lambda^m(t)))).\end{aligned}$$

The copula C_t is with this formula entirely determined by the initial copula C and the stochastic evolution up to time t of the intensities that describe the single defaults (more precisely $\Lambda^1(t), \dots, \Lambda^m(t)$). We can have analogous formulas when defaults occur before t. There is therefore no way through this model to get an autonomous evolution of the copula C_t and thus of the dependence between defaults. This does not seem to be realistic. Moreover, one would like to use (as for other financial markets) further option prices such as forward start contracts as a tool to catch the future evolution of the dependencies between defaults as it is seen today by the market. In a general manner, the main advantage of market calibration in comparison with the historical estimation is its ability to grasp the market's idea of the future trend: this is precisely what we cannot get within the copula model since it fixes the future interdependence between defaults from initial time. And there is no straightforward extension of this model that allows to get rid of this deficiency.

2 Reduced form loss models

An alternative to the copula model is to try to model directly the loss process $(L(t), t \geq 0)$ without taking into consideration the individual default times. This is a rather natural idea since CDO are products that depend on the defaults only through this process. In this section we give examples of reduced form loss models coming from the works of Errais, Giesecke and Goldberg [9, 10] and Brigo, Pallavicini and Torresetti [5]. These examples, as we will see, belong to the general class of Affine models [9] for which the law of $L(t)$ is known analytically or semi-analytically for each time t through its Fourier transform. This ensures a rather time-efficient calibration to CDO prices. Last, we hint at the random thinning procedure introduced in [10] that provides a mean to get single-name intensities that are consistent with the loss model.

2.1 Generalized Poisson Loss model (Brigo and al. [5])

We begin our little tour of reduced form models with the Generalized Poisson Loss (GPL) model because it relies on Poisson processes that are simple and well-known processes. Let us first assume for sake of simplicity that the loss given the default of each name are all equal to $\text{LGD} \in (0, 1]$ and deterministic. Thus, the loss is proportional to the number of defaults $N(t) = \sum_{j=1}^{m} \mathbf{1}_{\tau^j \leq t}$:

$$L(t) = \frac{\text{LGD}}{m} N(t).$$

Let us assume that there are $k \leq m$ independent standard Poisson processes $M_1(t), \dots, M_k(t)$ with intensity 1 on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We suppose that we have k intensity processes $\lambda_t^{N_1}, \dots, \lambda_t^{N_k}$ that we suppose adapted to the riskless filtration $(\mathcal{F}_t)_{t \geq 0}$ and define the cumulative intensities $\Lambda^{N_l}(t) = \int_0^t \lambda_s^{N_l} ds$ for $l \in \{1, \dots, k\}$. Last, we define the time inhomogeneous Poisson processes

$$\forall l \in \{1, \dots, k\}, N_l(t) = M_l(\Lambda^{N_l}(t))$$

that we assume to be adapted with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$. Let us consider k integers such that $1 \leq j_1 < j_2 < \dots < j_k \leq m$. The GPL model assumes that the number of defaults at time t is given by

$$N(t) = \min(Z_t, m) \text{ where } Z_t = \sum_{l=1}^{k} j_l N_l(t).$$

Thus, it lets the possibility of simultaneous defaults: a jump of $N_l(t)$ induces exactly j_l defaults (if $Z_t \leq m$). That models in a rather extreme way the dependence between the defaults. One knows then exactly the distribution of Z_t given $\mathcal{F}$ through its Fourier transform using the independence of the Poisson processes M_l:

$$\forall u \in \mathbb{R},\ \mathbb{E}[e^{iuZ_t} | \mathcal{F}] = \prod_{l=1}^{k} \exp[\Lambda^{N_l}(t)(e^{ij_l u} - 1)] = \exp\left[\sum_{l=1}^{k} \Lambda^{N_l}(t)(e^{ij_l u} - 1)\right].$$

Let us now turn to the valuation of the CDO rate $R_a^b(\mathcal{T})$ defined in (3). We are exactly in the same situation as for the copula model since we know the distribution of the loss $L(t) = \frac{\text{LGD}}{m} \min(Z_t, m)$ given $\mathcal{F}$. Therefore, if one specifies a model for the discount factors and the intensities $\lambda^{N_l}(t)$ we can once again calculate the expectations in (3) using integrations by parts and conditioning first to $\mathcal{F}$. Typically, one assumes in a first implementation of this model that the discount factors and the intensities are deterministic: this allows to compute in a rather efficient way the CDO rate value using an inverse Fourier method. This is obviously an important point for calibration purpose to get a reasonable computation time. A calibration procedure of this model to real CDO tranche data is discussed in detail in [5]. It gives encouraging results: all tranche prices but the long-term maturity CDO prices are fitted correctly. A little modification of the definition of the loss process ($\frac{\text{LGD}}{m'}\min(Z_t, m')$ instead of $\frac{\text{LGD}}{m}\min(Z_t, m)$ for $m' > m$) is proposed in [5] to correct this problem. Though, we lose through this way the interpretation of $\min(Z_t, m)$ as the number of defaults before t.

Last, we want to mention an easy extension to this model. As we have seen, the main features of the GPL model are to allow multiple defaults and to have an analytically tractable loss distribution. The properties of the Poisson processes that are crucial here are that they are unit-jump increasing processes whose Fourier transform is known. Thus, one could extend the GPL model taking $\forall l \in \{1, \dots, k\}, N_l(t) = \tilde{M}_l(\Lambda^{N_l}(t))$ where the processes $\tilde{M}_l(t)$ are chosen to be independent unit-jump increasing processes with an analytical formula for their Fourier transform $u \mapsto \mathbb{E}[\exp(iu\tilde{M}_l(t))]$. A possible way is to take $\tilde{M}_l(t) = M_l(S_l(t))$, where $(S_l(t), t \geq 0)$ is an increasing process independent from M_l. In that case, the Fourier transform writes:

$$\forall u \in \mathbb{R},\ \mathbb{E}[\exp(iu\tilde{M}_l(t))] = \mathbb{E}[\exp(S_l(t)(e^{iu} - 1))].$$

If we take for S_l a Lévy subordinator such as an inverse Gaussian or a Gamma subordinator, we have an analytical formula for it (see [5] for the Gamma case). This is also the case if one assumes $S_l(t)$ to be a primitive of a Cox-Ingersoll-Ross process, or more generally if $S_l(t)$ is a positive linear combination of these (independent) processes.

2.2 Hawkes process and affine point process loss model (Errais, Giesecke and Goldberg [9, 10])

Contrary to the previous GPL model, the model proposed by Giesecke and Goldberg [10] and detailed in Errais and al. [9] excludes the possibility of simultaneous defaults. To take into consideration the contagion between defaults, they model directly the loss process as a pure jump process whose jump intensity increases when defaults occur. Namely, they model $L(t)$ as a $(\mathcal{G}_t)$-adapted Hawkes process. We consider here a particular case and assume that it has an intensity of jumps that solves

$$d\lambda_t^L = \kappa(\rho(t) - \lambda_t^L)dt + \delta dL(t) \text{ with } \delta, \kappa, \lambda_0^L \geq 0, \tag{5}$$

and a jump law distribution ν such that $\nu((0, 1/m]) = 1$. This means that the instantaneous rate of jumps is $\lim_{dt \to 0} \frac{\mathbb{P}(L(t+dt)-L(t)>0|\mathcal{G}_t)}{dt} = \lambda_t^L$ and thus $\lim_{dt \to 0} \frac{\mathbb{E}[f(L(t+dt))-f(L(t))|\mathcal{G}_t]}{dt} = \lambda_t^L \left(\int_0^{1/m} f(L(t) + x)\nu(dx) - f(L(t)) \right)$ for any bounded measurable function f. More precisely, the jump events (i.e. triggering and jump size) are supposed to be independent from the riskless filtration $(\mathcal{F}_t)_{t \geq 0}$ and we assume the stronger condition:

$$\lim_{dt \to 0} \frac{\mathbb{E}[f(L(t + dt)) - f(L(t))|\mathcal{G}_t \vee \mathcal{F}]}{dt} = \lambda_t^L \left(\int_0^{1/m} f(L(t) + x)\nu(dx) - f(L(t)) \right).$$

We also suppose that the process $\rho(t)$ is positive and adapted to $(\mathcal{F}_t)_{t \geq 0}$. In [9], $\rho(t)$ is assumed to be deterministic, but we want to show here that it is easy to nest dependence with the riskless filtration. In a more general way, the other fixed parameters could have been considered time-dependent and adapted to $(\mathcal{F}_t)_{t\geq 0}$ for what follows.

Through the choice of ν, this model let the possibility to have random recovery rate. Taking identical deterministic recovery rate would lead to $\nu(dx) = \delta_{\mathrm{LGD}/m}(dx)$. Let us remark also that the loss process is not bounded from above with this model. If it has undesirable effects, one can cap it as for the GPL model. With the parametrization above, the meaning of the parameters is rather clear: the intensity process has a mean-reversion toward $\rho(t)$ with a speed parametrized by κ. The parameter δ synthetizes the impact of a default on the further bankruptcies.

Under that model, Errais and al. [9] have stated that one can calculate the Fourier transform of the Loss distribution $\mathbb{E}[\exp(iuL(t))|\mathcal{F}]$ solving numerically ODEs. This allows to calculate as for the previous GPL or copula models the CDO rate value (3). More precisely, we have the following result.

Proposition 2.1. *Let us fix $s \geq t$. Under the above setting, the Fourier transform of the loss $L(s)$ conditioned to $\mathcal{G}_t \vee \mathcal{F}$ is given by*

$$\forall u \in \mathbb{R},\ \mathbb{E}[\exp(iuL(s))|\mathcal{G}_t \vee \mathcal{F}] = \exp(iuL(t) + a(u,t,s) + b(u,t,s)\lambda_t^L) \tag{6}$$

where the coefficient functions $a(u,t,s)$ and $b(u,t,s)$ are $\mathcal{F}_s$-measurable and solve the following ordinary differential equations:

$$\partial_t b(u,t,s) = \kappa b(u,t,s) + 1 - \int e^{(iu+\delta b(u,t,s))x}\nu(dx) \tag{7}$$

$$\partial_t a(u,t,s) = -\kappa\rho(t) b(u,t,s) \tag{8}$$

with the final condition $a(u,s,s) = 0$ and $b(u,s,s) = 0$. When ρ is a deterministic function, $a(u,t,s)$ and $b(u,t,s)$ are also deterministic.

This kind of result is standard and we just sketch the proof. The following process $(\mathbb{E}[\exp(iuL(s))|\mathcal{G}_t \vee \mathcal{F}], t \leq s)$ is a $(\mathcal{G}_t \vee \mathcal{F})$-martingale. If (6) holds, one has

$$\frac{d\exp(iuL(t) + a + b\lambda_t^L)}{\exp(iuL(t-) + a + b\lambda_{t-}^L)} = \exp((iu + b\delta)\Delta L(t)) - 1 + [\partial_t a + \partial_t b \lambda_t^L + b\kappa(\rho(t) - \lambda_t^L)]dt$$

where $\Delta L(t) = L(t) - L(t-)$. It is a martingale increment if and only if

$$\lambda_t^L \left(\int e^{(iu+\delta b)x}\nu(dx) - 1 + \partial_t b - b\kappa \right) + \partial_t a + b\kappa\rho(t) = 0 \ a.s.,$$

and one deduces (7) and (8) since λ_t^L takes a.s. an infinite number of values.

Through this way, one can even get the Fourier transform of the joint law of $J(t) := (L(t), N(t))$, where $N(t)$ denotes the number of jumps of the loss process up to time t. This is done explicitly in the paper of Errais, Giesecke and Goldberg [9]. They introduce in a more general manner d-dimensional affine point processes $(J(t), t \geq 0)$ for which the

Fourier transform can be calculated in the same way as for the Hawkes process. Of course, the Fourier transform of a sum of d-dimensional independent affine point process has also a similar form. This broaden considerably the possible models for the loss process. Within this framework, a time inhomogeneous Poisson process appears as a particular one-dimensional affine point process, and the GPL model as a sum of one-dimensional independent affine point processes. More generally, most of the extension of the GPL model we have considered in the previous section can also be seen a sum of one-dimensional independent affine point processes. They are interesting particular cases for which the Fourier transform is known analytically and does not require to solve ODEs.

2.3 Loss models and single defaults

Now, we would like to conclude this section on the reduced form loss models giving some clue on the natural question: once a loss model is given for $L(t)$, can we find a procedure to define the individual defaults $\tau^1, \dots, \tau^m$ in a coherent manner with the loss process? This might be of interest if one liked to price a credit risk product that relies on some of the defaults that define $L(t)$. This is the case for example of a CDO tranche whose underlying loss brings on the default times $\tau^1, \dots, \tau^j$ with $1 \leq j < m$. This approach that consists in first defining the aggregated loss and then the single defaults is called "top-down" in the literature. Instead, in the copula model, we have a bottom-up approach: we first define the single default events and then define the loss with the formula (2). We have not such a problem of coherency.

We explain here the random thinning procedure that has been introduced by Giesecke and Goldberg in [10]. Let us denote by $N(t) = \sum_{s \leq t} \mathbf{1}_{\Delta L(s) > 0}$ the number of loss jumps before time t. When $N(t) \leq m$, one would like to assign to each jump one of the default times $\tau^1, \dots, \tau^m$ and we exclude then simultaneous defaults. Ideally, it would be nice to have a direct relationship between $N(t)$ and the single default indicators $\mathbf{1}_{\tau^j \leq t}$ for $j \in \{1, \dots, m\}$. Giesecke and Goldberg [10] have investigated the following one called strong random thinning:

$$\mathbf{1}_{\tau^j \leq t} = \int_0^t \zeta_s^j dN(s)$$

where the $(\zeta_t^j, t \geq 0)$ are $(\mathcal{G}_t)$-predictable processes satisfying some technical conditions. However, as they show, this default representation is too strong and implies that one knows before the next jump which name will default. This is rather unrealistic and one has to find a weaker binding between the loss and the single defaults. The random thinning procedure they propose binds the $(\mathcal{G}_t)$-intensities of the single default to the $(\mathcal{G}_t)$-intensity λ_t^N of the counting process $N(t)$. The last intensity is defined as the process such that:

$$N(t) - \int_0^t \lambda_s^N ds \text{ is a } (\mathcal{G}_t)\text{-martingale.}$$

In the Hawkes process loss model, this is also the intensity of jumps defined by (5). They assume that there are $(\mathcal{G}_t)$-adapted processes $(\zeta_t^j, t \geq 0)$ such that $\zeta_t^j \lambda_t^N$ is the $(\mathcal{G}_t)$-intensity of the stopping time τ^j according to definition (1.3). Let us remind that single intensities vanish after their related default so that we may write $\zeta_t^j = \zeta_t^j \mathbf{1}_{\tau^j > t}$. Since $\min(N(t), m) = \sum_{j=1}^m \mathbf{1}_{\tau^j \leq t}$, the process $\int_0^t \sum_{j=1}^m \zeta_s^j \mathbf{1}_{\tau^j > s} \lambda_s^N ds - \int_0^t \lambda_s^N \mathbf{1}_{N(t) \leq m} ds$ is a $(\mathcal{G}_t)$-martingale and therefore we have necessarily:

$$\sum_{j=1}^m \zeta_t^j \mathbf{1}_{\tau^j > t} = \mathbf{1}_{N(t) \leq m}. \tag{9}$$

Reciprocally, if the processes $(N(t), t \geq 0)$ and $(\zeta_t^j, t \geq 0)$ are given and satisfy condition (9), it is easy to check that $\zeta_t^j \lambda_t^N$ is the $(\mathcal{G}_t)$-intensity of τ^j if one assumes at each jump that the default name is chosen independently according to the discrete probability law $(\zeta_t^1 \mathbf{1}_{\tau^1 > t}, \dots, \zeta_t^m \mathbf{1}_{\tau^m > t})$ on $\{1, \dots, m\}$.

Let us give an example of processes $(\zeta_t^j, t \geq 0)$ that satisfy (9) and consider m positive deterministic functions $z^j(t)$ for $j \in \{1, \dots, m\}$. Then, one can simply chose $\zeta_t^j = \frac{z^j(t) \mathbf{1}_{\tau^j > t}}{\sum_{j=1}^m z^j(t) \mathbf{1}_{\tau^j > t}}$ on $\{N(t) < m\}$ and $\zeta_t^j = 0$ on $\{N(t) \geq m\}$. Through the random thinning procedure, the law of τ^j at time 0 is simply given by:

$$\mathbb{P}(\tau^j \leq t) = \mathbb{E}\left[\int_0^t \zeta_s^j \lambda_s^N ds\right].$$

If one has a fast enough way to compute it, one can then calibrate the processes $(\zeta_t^j, t \geq 0)$ using the single CDS rate prices (1). Let us suppose for example that the loss process $L(t)$ comes from an Hawkes process. If we assume deterministic recovery rates ($\nu(dx) = \delta_{\mathrm{LGD}/m}(dx)$) and cap the loss by LGD, it has then exactly m jumps. One can then fit this capped loss to the CDO prices and use the random thinning procedure to get a model that is consistent with single-CDS and CDO data. If $L(t)$ may have more than m jumps (as for the general Hawkes process model) and is already calibrated to CDO prices, then it will be still consistent to CDO and CDS prices if the expected loss beyond m jumps at the larger final maturity $T_{\max}$ is negligible. This will be mainly the case if the event $\{N(T_{\max}) > m\}$ is very rare. Otherwise, assigning a default name to each of the m first jumps is not relevant.

Getting coherent single default events starting from the loss process is a current topic of research. As an alternative to the random thinning, we cite here the recent work of Brigo, Pallavicini and Torresetti [6] that consider another approach based on the Common Poisson model [14] and extend their GPL model to take into account the single defaults.

3 Forward loss models

After having presented the copula model and some reduced form loss models, we conclude our introduction to multiname credit risk modelling with the presentation of two forward loss models proposed contemporaneously by Schönbucher [16] and Sidenius, Piterbarg and Andersen [18]. Forward loss models are to reduced form models what Heath-Jarrow-Morton and Brace-Gatarek-Musiela models are to short interest rate models. In the fixed income case, forward (short or LIBOR) rates are thus directly specified: these rates are mainly the fair interest rates one should apply at a current time t for a future contract at time $T > t$. The forward loss models that we consider here target to model how the future loss distributions $L(T)$ is seen by the market at time t. Contrary to the previous models, the loss itself is not directly modelled. We have decided to present here both models at the same time with analogous notations to point out similarities and differences. We focus only on the main construction steps of these models and refer to the original papers [16, 18] for technical proofs and further developments. We keep the same filtered probability space as for the previous model and make the models stick to it. Last, we will assume in this section that the loss process is independent from the the interest rates so that the computation of the CDO rates (3) is straightforward once we know the future loss distributions (and the expected discount factors).

3.1 Schönbucher's forward model [16]

3.1.1 Description of the loss

Here, we will assume that each default has a null recovery rate (LGD $= 1$), so that the loss process is given by

$$t \geq 0,\ L(t) = \frac{1}{m}\sum_{j=1}^{m} \mathbf{1}_{\tau^j \leq t} \tag{10}$$

and is thus normalised to 1. The law of $L(T)$ seen at time $t \leq T$ is described by $\pi(t,T) \in \mathbb{R}^{m+1}$ where $\pi_k(t,T) = \mathbb{P}(L(T) = k/m|\mathcal{G}_t)$ for $k = 0,\dots,m$. We have the following straightforward properties:

(i) $\pi_k(t,T) \geq 0$,

(ii) $\sum_{k=0}^{m} \pi_k(t,T) = 1$,

(iii) For each $n \in \{0,\dots,m\}$, $T \mapsto \mathbb{P}(L(T) \leq n/m|\mathcal{G}_t) = \sum_{k=0}^{n} \pi_k(t,T)$ is non-increasing,

(iv) $\pi_k(t,t) = \mathbf{1}_{\{L(t)=k/m\}}$.

These properties come immediately from the fact that $(L(t), t \geq 0)$ is an increasing $(\mathcal{G}_t)$-adapted process taking values in $\{0, 1/m, \dots, 1\}$. They are thus necessary arbitrage-free conditions on $\pi(t,T)$.

Assumptions on the loss process (A.1). *We assume that the loss process is defined by* (10)*, and therefore* $\pi(t,T)$ *satisfies properties (i-iv). Moreover, we assume that*

$$T \mapsto \pi(t,T) \text{ is } \mathcal{C}^1 \text{ and } \forall k,\ \pi_k(t,T) > 0 \implies \forall T' \geq T, \pi_k(t,T') > 0. \tag{11}$$

Under these assumptions **(A.1)**, it is shown in [16] that there is a nondecreasing time-inhomogeneous Markov chain $(\tilde{L}_t(T), T \geq t)$ on states $\{0, 1/m, \dots, 1\}$ with transition matrices $(a(t,T), T \geq t)$ such that $a(t,T)$ is $\mathcal{G}_t$-measurable and

$$\forall T \geq t, \forall k \in \{0,\dots,m\},\ \pi_k(t,T) = \mathbb{P}(\tilde{L}_t(T) = k/m|\mathcal{G}_t). \tag{12}$$

The generator of $(\tilde{L}_t(T), T \geq t)$ writes for a bounded function f:

$$\lim_{dT \to 0} \frac{\mathbb{E}\left[f(\tilde{L}_t(T+dT)) - f(\tilde{L}_t(T))|\mathcal{G}_T\right]}{dT} = \sum_{k=0}^{m-1} \mathbf{1}_{\{\tilde{L}_t(T)=k/m\}} \sum_{l=k+1}^{m} a_{k,l}(t,T)\left(f(\frac{l}{m}) - f(\frac{k}{m})\right). \tag{13}$$

Mainly, this result says us that the Markov chain model with transition rates that are fixed at time t ($a(t,T) \in \mathcal{G}_t$) is rich enough to represent the future marginal laws $(\pi(t,T), T \geq t)$ when the loss satisfies **(A.1)**.

Reciprocally, if one starts from $\mathcal{G}_t$-measurable transition rates $a(t,T)$, can we find a loss process satisfying **(A.1)** that is consistent with (12)? I.e. such that

$$\forall t \geq 0, \forall T \geq t, \forall k \in \{0,\dots,m\},\ \pi_k(t,T) = \mathbb{P}(\tilde{L}_t(T) = k/m|\mathcal{G}_t) \tag{14}$$

where $(\tilde{L}_t(T), T \geq t)$ is defined by $\tilde{L}_t(t) = L(t)$ and (13). If this holds, $L(t+dt)$ and $\tilde{L}_t(t+dt)$ have the same law conditioned to $\mathcal{G}_t$, and we get from (13) taken at $T = t$

$$\lim_{dt \to 0} \frac{\mathbb{E}\left[f(L(t+dt)) - f(L(t))|\mathcal{G}_t\right]}{dt} = \sum_{k=0}^{m-1} \mathbf{1}_{\{L(t)=k/m\}} \sum_{l=k+1}^{m} a_{k,l}(t,t)\left(f(\frac{l}{m}) - f(\frac{k}{m})\right). \tag{15}$$

Therefore the loss dynamics is fully characterized by the transition rates if the consistency equality (14) holds. The idea of the Schönbucher's model is to specify a dynamics for the transition rates and identify conditions under which it exists a loss process that is consistent with. The loss dynamics is then given by (15).

We will restrict the transition rate matrices that we consider. Indeed, it has been shown in [16] that under additional assumptions on $\pi(t,T)$ that we do not specify here, the Markov chain $(\tilde{L}_t(T), T \geq t)$ can be chosen to have only jumps of size $1/m$. This amounts to exclude simultaneous defaults after t. If these additional assumptions are not satisfied, a result stated in [16] shows that one can however approximate closely the forward loss distributions by a nondecreasing Markov chain with $1/m$ jumps. As a consequence, it does not seem so much restrictive to focus on the transition matrices that allow only $1/m$ jumps ($a_{k,l}(t,T) = 0$ for $l \geq k+2$). We use then the shorter notation $a_k(t,T) = a_{k,k+1}(t,T)$ for $k < m$ and set $a_m(t,T) = 0$.

Model principle: We model the dynamics for the transition rates $a_k(t,T)$, $k \in \{0,\dots m-1\}$ and look for conditions under which it exists a loss process satisfying **(A.1)** that is consistent in the sense of (14). The loss dynamics is then fully characterized by

$$\lim_{dt \to 0} \frac{\mathbb{E}\left[f(L(t+dt)) - f(L(t))|\mathcal{G}_t\right]}{dt} = \sum_{k=0}^{m-1} \mathbf{1}_{\{L(t)=k/m\}} a_k(t,t)\left(f(\frac{k+1}{m}) - f(\frac{k}{m})\right). \tag{16}$$

3.1.2 Consistency and option valuation

The model being precised, we would like to find forward rate dynamics for $a_k(t,T)$ that are consistent with the loss distribution in the sense of (14). To do so, one has first to compute $\mathbb{P}(\tilde{L}_t(T) = k/m|\mathcal{G}_t)$ in function of the transition rates. We introduce as in [16] $P_{k,l}(t,T) = \mathbb{P}(\tilde{L}_t(T) = l/m|\mathcal{G}_t, \tilde{L}_t(t) = k/m)$. From (15), one gets the Kolmogorov equation $\partial_T P_{k,l}(t,T) = P_{k,l-1}(t,T)a_{l-1}(t,T) - P_{k,l}(t,T)a_l(t,T)$. Since $P_{k,l}(t,t) = \mathbf{1}_{l=k}$, we deduce

$$\begin{aligned}
l < k, &\quad P_{k,l}(t,T) = 0 \\
l = k, &\quad P_{k,k}(t,T) = \exp\left(-\int_t^T a_k(t,s)ds\right) \\
l > k, &\quad P_{k,l}(t,T) = \int_t^T P_{k,l-1}(t,s)a_{l-1}(t,s)\exp\left(-\int_s^T a_l(t,u)du\right)ds.
\end{aligned} \tag{17}$$

Let us emphasise here that these are the transition probabilities of our representation of the loss $\tilde{L}_t$, not of the true loss process. The following key result stated in [16] gives necessary and sufficient conditions to get a loss representation that is consistent with the real loss in the sense of (14).

Proposition 3.1. *Let us assume that the loss process $(L(t), t \geq 0)$ starts from $L(0) = 0$ and is such that $\pi(t,T)$ satisfy properties (i-iv) and condition (11). Let us assume that for each $k \in \{0, \dots, m-1\}$, the transition rate $a_k(t,T)$ is nonnegative and $(\mathcal{G}_t)$-adapted and satisfies $\mathbb{E}[\sup_{t\leq T} a_k(t,T)] < \infty$. Then the following conditions are equivalent:*

1. *$\forall T \geq t \geq 0, \forall k \in \{0, \dots, m\}$, $\pi_k(t,T) = \mathbb{P}(\tilde{L}_t(T) = k/m|\mathcal{G}_t)$. (consistency)*
2. *$\forall T > 0, \forall l \in \{0, \dots, m\}, (P_{L(t),l}(t,T))_{t\in[0,T]}$ is a $(\mathcal{G}_t)$-martingale.*
3. *$\forall T > 0, \forall l \in \{0, \dots, m\}, (a_l(t,T)P_{L(t),l}(t,T))_{t\in[0,T]}$ is a $(\mathcal{G}_t)$-martingale and $\lim_{dt\to 0} \frac{\mathbb{P}(L(t+dt)-L(t)=1/m|\mathcal{G}_t)}{dt} = a_{mL(t)}(t,t)$.*

To give an example of option valuation, we assume null interest rate and consider a toy CDO product between two maturities $0 < T_1 < T_2$ that pays $\mathbf{1}_{L(T_1)<a, L(T_2)>b}$ at time T_2 with $0 \leq a < b \leq 1$. Its fair price at time T_1 is simply given by $\sum_{k<ma} \mathbf{1}_{L(T_1)=k} \sum_{l>mb} P_{k,l}(T_1,T_2)$. A call option on this CDO with strike R values at time 0: $\mathbb{E}[\sum_{k<ma} \mathbf{1}_{L(T_1)=k}(\sum_{l>mb} P_{k,l}(T_1,T_2) - R)^+]$. Simulating paths until T_1, this can be calculated by the Monte-Carlo method.

3.1.3 Consistent transition rates dynamics and simulation

Our aim is now to specify transition rates that are consistent with the loss process. Following [16], we assume that there is a d-dimensional $(\mathcal{G}_t)$-Brownian motion and $(\mathcal{G}_t)$-predictable coefficients $\mu_k(t,T) \in \mathbb{R}$ and $\sigma_k(t,T) \in \mathbb{R}^d$ such that

$$\forall k \in \{0, \dots, m-1\},\ a_k(t,T) = a_k(0,T) + \int_0^t \mu_k(s,T)ds + \int_0^t \sigma_k(s,T)dW_s. \tag{18}$$

In preparation for applying Proposition 3.1 and find a consistency condition, one has to know the transition probability dynamics. It is shown in [16] that

$$\forall k, l \in \{0, \dots, m\}, t \leq T,\ dP_{k,l}(t,T) = u_{k,l}(t,T)dt + v_{k,l}(t,T)dW_t,$$

and we refer to the original paper for the recursive explicit formulas of $u_{k,l}(t,T)$ and $v_{k,l}(t,T)$ in function of the transition rate dynamics. Applying Proposition 3.1, it is then shown that the transition rates (18) are consistent with the loss if and only if

$$\forall l \in \{0,\dots,m\},\ P_{L(t),l}(t,T)\mu_l(t,T) = -\sigma_l(t,T)v_{L(t),l}(t,T) \text{ and}$$
$$\lim_{dt\to 0}\frac{\mathbb{P}(L(t+dt)-L(t)=1/m|\mathcal{G}_t)}{dt} = a_{mL(t)}(t,t).$$

Existence of coefficients $\mu_k(t,T)$ and $\sigma_k(t,T)$ that satisfy this condition is not directly addressed. It is however given a simulation scheme for the loss $(L(t), t \geq 0)$ that satisfies asymptotically this condition once $a(0,T)$ and $\sigma(t,T)$ have been fixed for $0 \leq t \leq T$. At time 0, $a(0,T)$ and $\sigma(t,T)$ have to be parametrized and calibrated to market data. Typically, since $a(0,T)$ fully determines $\pi(0,T)$, it is chosen to fit exactly the prices of CDO tranches that start immediately and the volatilities $\sigma(t,T)$ should be fitted to other products. We set $\hat{L}(0)=0$ and $\hat{a}(0,T)=a(0,T)$, and consider a time step Δt. Let us assume that we have simulated $\hat{L}$ and $\hat{a}$ up to time $t=q\Delta t$ so that $\hat{L}(t)$ and $(\hat{a}(t,T), T\geq t)$ are known. Now, one should remark from (17) (resp. from $v_{k,l}$'s formula in [16]) that the transition probabilities $P_{k,l}(t,T)$ and the coefficients $v_{k,l}(t,T)$ are entirely determined by the transition rates $(a(t,T), T\geq t)$ and $(\sigma(t,T), T\geq t)$. Therefore, we can estimate these with $(\hat{a}(t,T), T\geq t)$ and $(\sigma(t,T), T\geq t)$, and we set

$$\forall l \geq \hat{L}(t), \forall T\geq t,\ \hat{\mu}_l(t,T) = -\frac{\hat{v}_{\hat{L}(t),l}(t,T)}{\hat{P}_{\hat{L}(t),l}(t,T)}\sigma_l(t,T)$$

and $\hat{\mu}_l(t,T)=0$ for $l<\hat{L}(t)$. Then, we set $\hat{L}(t+\Delta t) = \hat{L}(t) + B_q/m$ where B_q is an independent Bernoulli variable of parameter $\hat{a}_{m\hat{L}(t)}(t,t)\Delta t$ and:

$$\forall l, \forall T \geq t+\Delta t,\ a_l(t+\Delta t,T) = a_l(t,T) + \hat{\mu}_l(t,T)\Delta t + \sigma(t,T)(W_{t+\Delta t}-W_t).$$

In fact, only transition rates above $\hat{L}(t+\Delta t)$ will be useful next, and we can instead simply take $a_l(t+\Delta t,T)=0$ for $l<\hat{L}(t+\Delta t)$. We can then continue the iteration.

To conclude this section, one should mention that more sophisticated dynamics are also considered in [16]. In particular, extensions are proposed to take into account possible dependence of the loss to single defaults. In connection with that, let us mention also that the random thinning procedure introduced in section 2.3 can be applied here to assign a default name to each loss jump.

3.2 The SPA model [18]

3.2.1 Description of the loss

Assumptions on the loss process (A.2). *Following Sidenius, Piterbarg and Andersen, we consider a general loss process $(L(t), t\geq 0)$ valued in $[0,1]$ that is a non-decreasing $(\mathcal{G}_t)$-adapted Markov process. More precisely, we assume it satisfies $L(0)=0$ and*

$$\lim_{dt\to 0}\frac{\mathbb{E}\left[f(L(t+dt))-f(L(t))|\mathcal{G}_t\vee\mathcal{F}\right]}{dt} = \mathcal{A}_t f(L(t))$$

so that jump events (triggering and jumps) are independent from $\mathcal{F}$. The generator $\mathcal{A}_t$ is assumed to be defined on a domain $\mathcal{D}$ that is dense in the set of bounded measurable functions, and to satisfy: $\forall f \in \mathcal{D}$, $\forall l \in [0,1], \mathcal{A}_t f(l)$ is $\mathcal{F}_t$-measurable.

We assume also that filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$ are such that for each $\mathcal{G}_t$ integrable variable X, $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X|\mathcal{F}_t]$. In comparison to the Schönbucher's model, the jumps may have different sizes which can take into account different recoveries. Their number may also exceed m. The loss dynamics depending only on the riskless filtration and the loss itself, there is no possible particular dependence to some single default.

Instead of dealing directly with the forward loss distributions $\pi(t,T) = (\pi_x(t,T), x \in [0,1])$ where $\pi_x(t,T) = \mathbb{P}(L(T) \leq x|\mathcal{G}_t)$ the SPA model introduces for $t, T \geq 0$ (not only $T \geq t$), $p(t,T) = (p_x(t,T), x \in [0,1])$ where $p_x(t,T) = \mathbb{P}(L(T) \leq x|\mathcal{F}_t)$. It satisfies the following properties:

(i) $x \mapsto p_x(t,T)$ is non-decreasing,

(ii) $p_1(t,T) = 1$,

(iii) for each $x \in [0,1]$, $T \mapsto p_x(t,T)$ is non-increasing,

(iv) $(p_x(t,T))_{t\geq 0}$ is a $(\mathcal{F}_t)$-martingale.

They reflect (except the last one) that $(L(t), t \geq 0)$ is a non-decreasing process valued in $[0,1]$ and are thus arbitrage-free conditions. For $t \geq T$, $\mathbf{1}_{L(T)\leq x}$ is both $\mathcal{G}_t$ and $\mathcal{G}_T$ measurable, and from $\forall X \in L^1(\mathcal{G}_t), \mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X|\mathcal{F}_t]$ we deduce

(v) $p_x(t,T) = p_x(T,T)$ for $t \geq T$.

The SPA model proposes to model the forward loss distributions $\pi(t,T)$ through $p(t,T) = \mathbb{E}[\pi(t,T)|\mathcal{F}_t]$. Clearly, $\pi(t,T)$ cannot in general be fully characterized by $p(t,T)$ since we may have $\pi(t,T) \neq \pi'(t,T)$ and $\mathbb{E}[\pi(t,T)|\mathcal{F}_t] = \mathbb{E}[\pi'(t,T)|\mathcal{F}_t]$. Thus, one needs to specify a model to determine in an univocal and consistent manner a loss process that satisfies **(A.2)** from $p(t,T)$. Let us suppose that a family $(p(t,T); t, T \geq 0)$ satisfying properties (i-v) is given. A loss process $(L(t), t \geq 0)$ satisfying **(A.2)** will be consistent with it if one has

$$\forall t, T \geq 0, \forall x \in [0,1],\ \mathbb{P}(L(T) \leq x|\mathcal{F}_t) = p_x(t,T). \tag{19}$$

This condition is analogous to (14). In the Schönbucher's model, it characterizes the loss dynamics and leads to overlapped conditions between the loss and the transition rates (Proposition 3.1). Here we have two levels: the probabilities $p(t,T)$ are first fixed and we look for a loss process that is consistent with them. In [18], several constructions are proposed for the loss process and we deal in the next section with the simplest one.

Model principle: The loss process is described through $p(t,T)$ dynamics. To do so, one needs to have a univocal construction of a loss process satisfying **(A.2)** from $(p(t,T); t, T \geq 0)$ such that consistency condition (19) holds. The choice of this construction is free, but it induces, beyond conditions (i-v), necessarily restrictions on $p(t,T)$ that should be satisfied. Last, this choice is not neutral for option valuation and is thus of importance.

3.2.2 Consistency and option valuation

Let us consider a family $(p(t,T); t,T \geq 0)$ that satisfies properties (i-v). Moreover we assume that $p_x(t,T)$ is continuously differentiable w.r.t. T and also that there is a finite grid $x_0 = 0 < x_1 < ... < x_q = 1$ such that $\forall t \geq 0, p_{x_i}(t,t) < p_{x_{i+1}}(t,t)$. We will construct a loss process $(L(t), t \geq 0)$ that is consistent in a slightly weaker sense as (19), i.e.

$$\forall t, T \geq 0, \forall i \in \{0,\dots,q\},\ \mathbb{P}(L(T) \leq x_i | \mathcal{F}_t) = p_{x_i}(t,T). \tag{20}$$

Let us suppose that the loss process $(L(t), t \geq 0)$ takes values on the grid and has the generator $\lim_{dt\to 0} \frac{\mathbb{E}[f(L(t+dt))-f(L(t))|\mathcal{G}_t\vee\mathcal{F}]}{dt} = \sum_{i=0}^{q-1} \mathbf{1}_{\{L(t)=x_i\}} a_{x_i}(t)\left(f(x_{i+1}) - f(x_i)\right)$ for any bounded function f. Defining $p^L_x(t,T) = \mathbb{P}(L(T) \leq x|\mathcal{F}_t)$, one deduces that $\partial_T p^L_{x_i}(t,t) = -a_{x_i}(t)(p^L_{x_i}(t,t) - p^L_{x_{i-1}}(t,t))$ (with the convention $p^L_{x_{-1}}(t,T) = 0$). Therefore, if L is consistent, one has necessarily $a_{x_i}(t) = \frac{-\partial_T p_{x_i}(t,t)}{p_{x_i}(t,t)-p_{x_{i-1}}(t,t)}$ (where $p_{x_{-1}}(t,T) = 0$).

We then define the loss process $(L(t), t \geq 0)$ with the generator

$$\lim_{dt\to 0} \frac{\mathbb{E}\left[f(L(t+dt)) - f(L(t))|\mathcal{G}_t \vee \mathcal{F}\right]}{dt} = \sum_{i=0}^{q-1} \mathbf{1}_{\{L(t)=x_i\}} \frac{-\partial_T p_{x_i}(t,t)\left(f(x_{i+1}) - f(x_i)\right)}{p_{x_i}(t,t) - p_{x_{i-1}}(t,t)}$$

and the initial condition $L(0) = 0$. One has to check that it is consistent (i.e. $\forall i, p^L_{x_i}(t,T) = p_{x_i}(t,T)$). Let us fix $t \geq 0$. One has $p^L_{x_i}(t,0) = p_{x_i}(t,0) = 1$, and for $T \leq t$, $\partial_T p^L_{x_i}(t,T) = \frac{\partial_T p_{x_i}(T,T)}{p_{x_i}(T,T)-p_{x_{i-1}}(T,T)}(p^L_{x_i}(t,T) - p^L_{x_{i-1}}(t,T))$. Since $p_{x_i}(t,T) = p_{x_i}(T,T)$ for $t \geq T$, one has also $\partial_T p_{x_i}(t,T) = \partial_T p_{x_i}(T,T)$ because $(p_{x_i}(t,T) - p_{x_i}(t,T-\varepsilon))/\varepsilon = (p_{x_i}(T,T) - p_{x_i}(T,T-\varepsilon))/\varepsilon$ and thus

$$\partial_T p^L_{x_i}(t,T) = \frac{\partial_T p_{x_i}(t,T)}{p_{x_i}(t,T) - p_{x_{i-1}}(t,T)}(p^L_{x_i}(t,T) - p^L_{x_{i-1}}(t,T)). \tag{21}$$

The vector $(p_{x_i}(t,T), i = 0,\dots,q)$ solving the same linear ODE (21) as $(p^L_{x_i}(t,T), i = 0,\dots,q)$ with the same initial value, one deduces $p_{x_i}(t,T) = p^L_{x_i}(t,T)$ for $T \leq t$. In particular, we have $p_{x_i}(T,T) = p^L_{x_i}(T,T)$ for any $T \geq 0$ and then $p_{x_i}(t,T) = \mathbb{E}[p_{x_i}(T,T)|\mathcal{F}_t] = \mathbb{E}[p^L_{x_i}(T,T)|\mathcal{F}_t] = p^L_{x_i}(t,T)$ for $t \leq T$. We have therefore shown that there is only one Markov chain that jumps from x_i to x_{i+1} that is consistent in the sense of (20).

Now let us turn to the valuation of our call option on the toy CDO $\mathbf{1}_{L(T_1)<a, L(T_2)>b}$ introduced in section 3.1.2. Having also here a Markov chain loss process, it is very similar and the price is given by $\mathbb{E}[\sum_{k,x_k<a} \mathbf{1}_{L(T_1)=x_k} (\sum_{l,x_l>b} P_{x_k,x_l}(T_1,T_2) - R)^+]$ where $P_{x_k,x_l}(t,T) = \mathbb{P}(L(T) = x_l|\mathcal{F}_t, L(t) = x_k)$. In the Schönbucher's model, these quantities could be directly calculated from the transition rates. Here, there is a priori no formula that expresses $P_{x_k,x_l}(t,T)$ in function of $(p(s,u); u \geq 0, s \leq t)$ which is undesirable because the Monte-Carlo valuation requires then two interlocked mean valuations. However, using Kolmogorov equations, one has similarly to (17) explicit formulas for $(\mathbb{P}(L(T) = x_l|\mathcal{F}, L(t) = x_k), T \geq t)$ in function of $(p_{x_i}(u,u), u \in [t,T])$ with $l-1 \leq i \leq k$. For the Monte-Carlo valuation of $\mathbb{P}(L(T) = x_l|\mathcal{F}_t, L(t) = x_k)$, it is thus sufficient to simulate $(p_{x_i}(u,u), u \in [t,T])$, not the whole loss process.

Other constructions of loss process are investigated in [18]. Let us stress that each construction may require additional assumptions on $p(t,T)$ beyond properties (i-v). In the case above, these are the continuous differentiation w.r.t. T and $\forall t \geq 0, p_{x_i}(t,t) < p_{x_{i+1}}(t,t)$. When specifying dynamics for the probabilities $p_x(t,T)$, one has then to be careful that these assumptions are also satisfied.

3.2.3 Probability rates dynamics and simulation

In this section, we are concerned in specifying dynamics for the probabilities $(p_x(t,T), t \leq T)$ that satisfy the conditions (i-v). In [18], two parametrizations of these probabilities are considered. The first one (HJM-like) models $p_x(t,T)$ through the forward short rate process $f_x(t,T) = -\frac{\partial_T p_x(t,T)}{p_x(t,T)}$, while the second one (BGM-like) describes them using the "Libor" rates $F_x(t,n) = \frac{p_x(t,T_n) - p_x(t,T_{n+1})}{p_x(t,T_{n+1})(T_{n+1}-T_n)}$ where $T_0 < T_1 < \dots$ are fixed maturities.

We focus here on the HJM-like parametrisation and consider a countable family $(W^\nu, \nu \geq 1)$ of independent $(\mathcal{F}_t)$-Brownian motion. To satisfy property (iv) and (v), we assume

$$\forall t \leq T, dp_x(t,T) = p_x(t,T) \sum_{\nu=1}^{\infty} \Sigma_x^\nu(t,T) dW_t^\nu$$

with $p_x(0,T) = 1$ and $p_x(t,T) = p_x(T,T)$ for $t \geq T$. Coefficients $\Sigma_x^\nu(t,T)$ are $\mathcal{F}_t$ measurable and supposed regular enough for what follows. We also assume $\Sigma_x^\nu(T,T) = 0$. We then easily get the forward rate dynamics:

$$t \leq T, \; df_x(t,T) = \sum_{\nu=1}^{\infty} \left(\sigma_x^\nu(t,T) \left(\int_t^T \sigma_x^\nu(t,u) du \right) dt + \sigma_x^\nu(t,T) dW_t^\nu \right)$$

where $\sigma_x^\nu(t,T) = -\partial_T \Sigma_x^\nu(t,T)$. From $f_x(t,T) = -\frac{\partial_T p_x(t,T)}{p_x(t,T)}$, we get for $T \geq t$, $p_x(t,T) = p_x(t,t)\exp(-\int_t^T f_x(t,u)du)$. For $u \leq t$, one has $p_x(t,u) = p_x(u,u)$ and then $f_x(u,u) = -\partial_T \ln(p_x(t,u))$ so that $p_x(t,t) = \exp(-\int_0^t f_x(u,u)du)$. Therefore, we get

$$\forall t, T \geq 0, p_x(t,T) = \exp\left(-\int_0^{t \wedge T} f_x(u,u)du - \int_{t \wedge T}^T f_x(t,u)du \right).$$

Therefore we can rewrite properties (i-iii) with the forward rates $(f_x(t,T), 0 \leq t \leq T)$:

(i) $x \mapsto \int_0^t f_x(u,u)du + \int_t^T f_x(t,u)du$ is non-increasing,

(ii) $f_1(t,T) = 0$,

(iii) for each $x \in [0,1]$, $f_x(t,T) \geq 0$.

The property (i) is clearly satisfied if $x \mapsto f_x(t,T)$ is non-increasing. In [18], the authors look for coefficients that write $\sigma_x^\nu(t,T) = \varphi^\nu(t,T,f_x(t,T))$ and give sufficient conditions on φ^ν such that these three conditions hold. Some practical examples are also presented.

Let us now address quickly to the simulation issue. Once a forward rate dynamics has been selected to calibrate market data, one can simulate the forward rates up to a final maturity $T_{\max}$ using a Euler scheme. Typically, since $p(0,T) = \pi(0,T)$ $(\mathcal{F}_0 = \mathcal{G}_0)$, $f(0,T)$ is chosen to fit exactly the prices of CDO tranches that start at the current time and the dynamics of $f(t,T)$ should be fitted to other products. Then, $p(t,T)$ can be deduced easily for $t \geq 0$ and $T \leq T_{\max}$. Finally, one has to simulate the chosen loss process. If it is the one described in the previous section, this is a simple Markov chain that jumps from x_i to x_{i+1} with the intensity $\frac{-\partial_T p_{x_i}(t,t)}{p_{x_i}(t,t)-p_{x_{i-1}}(t,t)}$.

Further issues in credit modelling

Though not being exhaustive, we have presented here several main research directions in credit risk. We have focused on the default modelling and have skipped some topics related to credit risk such as the recovery modelling. All the models presented here allow, at least theoretically, to price any option derivative on the loss process since they are designed for. All of them can be calibrated quite easily to the prices of CDO that start at the current time. However, the calibration to other prices such as options on forward-start CDO often requires Monte-Carlo simulation and is thus more time consuming. Research on loss model is nowadays very active. The diversity of traded products being in expansion, it will go on to create models that fit easily the market data. The robustness of the calibration procedure is also an important issue to investigate. Last, we have not treated here the hedging problem in credit risk. This is of course another main topic of research. We mention here the connected work of Bielecki, Jeanblanc and Rutkowski [2].

The models that we have introduced here are related mainly to only one aggregated loss, but they do not cover all the credit products that are dealt over the counter. Indeed, all these models need to be calibrated with some market data and are thus linked to a specific loss process $L(t)$. However in practice, one has to price products such as bespoke CDO whose associated loss is not traded on the markets. This loss may come from a sub-basket of $L(t)$, may have only some common names with $L(t)$ or may have no name in common. In each case, how can we make use of the information on $L(t)$ to price our product? This is a tough question, and it is theoretically treated here only in the sub-basket case because one can have a model for single defaults consistent with $L(t)$ (e.g. copula model or random thinning procedure). Other credit risk products such as CDO squared bring on many loss processes that may or not have names in common. Each of them may have common names with the calibrated loss $L(t)$. Once again, how can we price such products using for the best the market information? Currently, heuristic arguments (such as classification of the firms by size or sector), economic analysis and historical data are used to fill the lack of market prices and price these products. Nonetheless, beyond the problem of the information available on the market, designing an efficient and coherent model for many loss processes is certainly an important issue for the next coming year.

Acknowledgments: The author would like to thank Damiano Brigo for the many illuminating discussions that he had with him during the last years. He also thanks Benjamin Jourdain and Andrea Pallavicini for their comments.

References

[1] Alfonsi, A. and Brigo, D., (2005) New Families of Copulas Based on Periodic Functions. *Communications in Statistics: Theory and Methods, Vol. 34, No. 7, pp 1437-1447.*

[2] Bielecki, T.R., Jeanblanc, M. and Rutkowski, M. (2006) Hedging of Basket Credit Derivatives in CDS Market. *working paper.*

[3] Brigo, D. and Alfonsi, A. (2005). Credit default swap calibration and derivatives pricing with the SSRD stochastic intensity model, *Finance and Stoch., Vol. 9, No. 1, pp 29-42.*

[4] Brigo, D. and Mercurio, F. (2006). Interest Rate Models - Theory and Practice With Smile, Inflation and Credit, second edition. *Series: Springer Finance*

[5] Brigo, D., Pallavicini, A. and Torresetti, R. (2006). Calibration of CDO Tranches with the dynamical Generalized-Poisson Loss model, *working paper.*

[6] Brigo, D., Pallavicini, A. and Torresetti, R. (2006). Default correlation, cluster dynamics and single names: The GPCL dynamical loss model, *working paper.*

[7] Burtschell, X., Gregory, J., and Laurent, J.-P. (2005). A Comparative Analysis of CDO Pricing Models, *working paper.*

[8] Embrechts P., Lindskog F., McNeil A. (2001). Modelling Dependence with Copulas and Applications to Risk Management.

[9] Errais, E., Giesecke, K. and Goldberg L. R. (2006). Pricing credit from the top down with affine point processes, *working paper.*

[10] Giesecke, K. and Goldberg L. R. (2005). A top down approach to multi-name credit, *working paper.*

[11] Hull, J. and White, A. (2004). Valuation of a CDO and an n-th to Default CDS Without Monte Carlo Simulation, *Journal of Derivatives, Vol. 12, No. 2, (Winter 2004), pp. 8-23.*

[12] Hull, J. and White, A. (2006). Valuing Credit Derivatives Using an Implied Copula Approach, *working paper.*

[13] Laurent, J.-P. and Gregory, J. (2003). Basket Default Swaps, CDO's and Factor Copulas, *working paper.*

[14] Lindskog F., McNeil A. (2001). Common Poisson shock models: applications to insurance and credit risk modelling. *ASTIN Bulletin, Vol 33, No. 2, pp 209-238.*

[15] Nelsen, R. (1999). An Introduction to Copulas. *Springer*, New York.

[16] Schönbucher, P. (2005). Portfolio Losses and the Term Structure of Loss Transition Rates: A New Methodology for the Pricing of Portfolio Credit Derivatives, *working paper.*

[17] Schönbucher, P., and Schubert, D. (2001). Copula-Dependent Default Risk in Intensity Models, *working paper.*

[18] Sidenius, J., Piterbarg, V. and Andersen, L. (2005). A new framework for dynamic credit portfolio loss modelling, *working paper.*